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On Dagum-Exponential mixture distribution: properties, simulation and application in insurance

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Abstact

In actuarial science, one of the most significant subjects is the modeling of claims amount. We refer to two distinct distribution types—heavy-tailed and light-tailed distributions—that are associated with two risk categories. Weibull, Dagum, Pareto, and Log-Normal are some of the distributions of heavy tail claim amounts. Conversely, the light tail claim amounts are distributed as exponential, gamma, and exponential mixing.

This article presents the Dagum-Exponential mixture distribution, a new distribution created by combining the dagum distribution (α, θ, r) and the exponential distribution (θ). It is possible to obtain the quantile function, skewness, kurtosis, reliability function, and hazard function of the Dagum-Exponential distribution. Additionally, we estimate the model parameters using the maximum likelihood method. Using an actual data set in the application highlights the significance of the new model.

Keywords: Dagum distribution, Maximum likelihood estimation, Exponential distribution, Simulation.

1. Introduction

There are two kinds of distributions connected to two kinds of risk in actuarial science. In contrast to heavy-tailed distributions (Log-normal, Pareto, etc.) that are defined by the non-definition of the generating function of the moments, we are discussing light-tailed distributions (Exponential, Phase-type, etc.).

The modeling of claims amount is one of the main areas of actuarial theory and application.

It has been demonstrated that in some circumstances, combining distributions that are more closely matched to claims results in different distributions.

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Many contemporary distributions are the result of combining two distinct distributions. This study's primary objective is to present a novel lifetime distribution that is produced by fusing the exponential and Dagum distributions. The other new and classic distributions are represented by this novel model as both a general and a particular case. It provides a crucial framework for actuarial and biological science modeling of lifetime data. For more details see [1-14].

This paper defines the Dagum-Exponential mixture distribution, a new distribution. Each of those distributions has a unique characteristic that makes it unique, and numerous researchers have introduced this process.

The exponential distribution's probability density function (pdf) and cumulative distribution function (cdf) are as follows:

$$f(x) = \theta exp^{-\theta x}; x > 0, \theta > 0. \quad (1)$$

And

$$F(x) = 1 - exp^{-\theta x}; x > 0, \ \ \theta > 0. \quad (2)$$

A random variable X is from the Dagum distribution with three parameters as follow

$$f(x) = \frac{\alpha \, r \, \theta^{\alpha} \, x^{r-1}}{(\lambda + x^{r})^{\alpha+1}}; \alpha > 0, \theta, r > 0. \quad (3)$$

Where $\theta > 0$ is scale parameter, α and r are positive shape parameters.

$$F(x) = 1 - (\frac{\theta}{\theta + x^{r}})^{\alpha}; \alpha > 0, \theta, r > 0. \quad (4)$$

Creating a new distribution with more flexible behavior than the based distributions is the goal of mixture distributions.

This study has the following organizational structure. We ascertain the new distribution in section 2. The statistical properties of the Dagum-Exponential mixture distribution are examined in section 3. The maximum likelihood approach to estimating the unknown model parameters is discussed in section 4. Section 5 examines the estimation of the stress-strength parameter, while Section 6 provides actuarial measures and Section 7 offers simulations and real data application. The conclusion brings the last section to a close.

2. The mixture of Dagum-Exponential distribution

Let $f_1(x)$,$f_2(x)$ be the pdfs and $F_1(x)$, $F_2(x)$ be the CDFs of Dagum and Exponential respectively of a random variable X, then the pdf and CDF of the proposed distribution of random variable X are respectively of the form:

$$f(x) = pf_1(x) + (1-p)f_2(x) \quad (5)$$

And

$$F(x) = pF_1(x) + (1-p)F_2(x) \tag{6}$$

Where $0 \leq p \leq 1$ such that $p = \frac{1}{1+\theta}$, is called the mixing proportion.

2.1. Density function

Using (2) and (4) in (6), the pdf of the proposed mixture distribution is obtained as follows

$$f(x, \alpha, \theta, r) = \frac{1}{1+\theta} \quad \theta exp^{-\theta x} + \frac{\theta}{1+\theta} \frac{\alpha\, r\, \theta^{\alpha}\, x^{r-1}}{(\theta + x^r)^{\alpha+1}}$$

$$= \frac{\theta}{1+\theta}\left[exp^{-\theta x} + \frac{\alpha r\, \theta^{\alpha} x^{r-1}}{(\theta + x^r)^{\alpha+1}}\right]; x > 0, \alpha > 0, \theta > 0, r > 0. \tag{7}$$

We plot the pdf of the Dagum-Exponential mixture distribution for the selected values of the parameters.

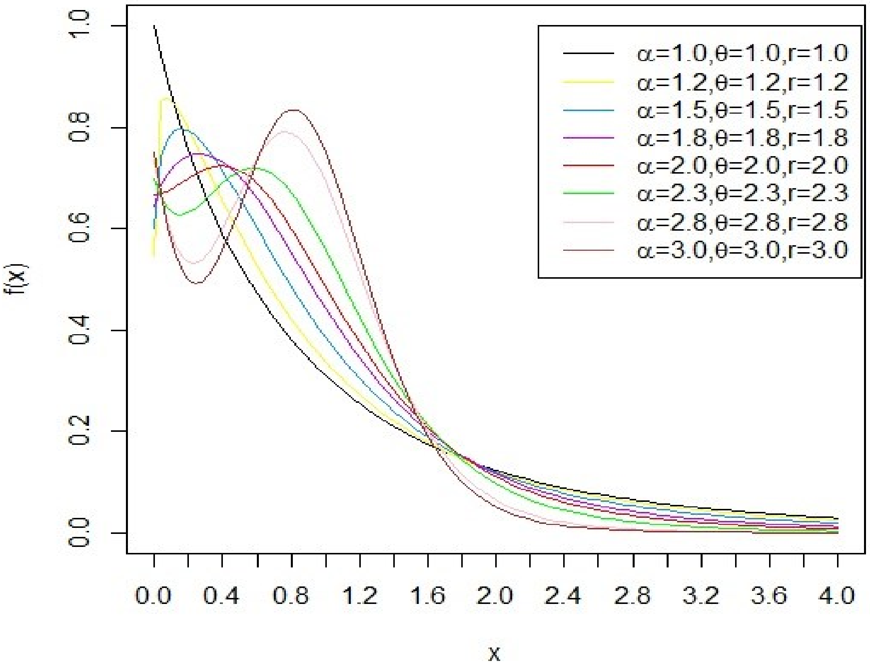

Figure 1. The pdf of Dagum-Exponential mixture distribution for various values of parameters.

2.2. Cumulative Distribution Function

And the corresponding CDF is obtained by

$$F(x, \alpha, \theta, r) = \int_{-\infty}^{x} f(y, \alpha, \theta, r) dy \; = \int_{0}^{x} \frac{\theta}{1+\theta}\left[exp^{-\theta y} + \frac{\alpha r \theta^{\alpha} y^{r-1}}{(\theta + y^r)^{\alpha+1}}\right] dy$$

$$= \frac{\theta}{1+\theta}\left[\int_0^x exp^{-\theta y} dy + \int_0^x \frac{\alpha\, r\, \theta^{\alpha}\, y^{r-1}}{(\theta + y^r)^{\alpha+1}} dy\right]$$

$$= \frac{\theta}{1+\theta}\left[-\frac{exp^{-\theta y}}{\theta}\Big|_0^x - \frac{\theta^{\alpha}}{(\theta + y^r)^{\alpha}}\Big|_0^x\right]$$

$$F(x,\alpha,\lambda,r) = \frac{1}{1+\theta}\left[1+\theta - exp^{-\theta x} - \frac{\theta^{\alpha+1}}{(\theta + x^r)^{\alpha}}\right]; x > 0, \alpha > 0, \theta > 0, r > 0 \qquad (8)$$

We plot the CDF of the Dagum-Exponential mixture distribution.

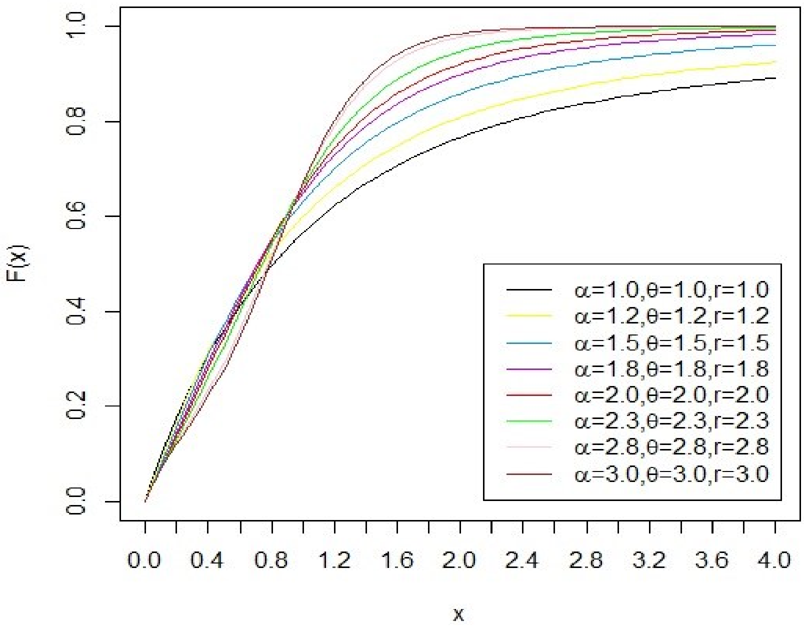

Figure 2. The cdf of Dagum-Exponential mixture distribution for various values of parameters.

2.3. Reliability function

The mathematical expression for reliability (survival) function is

$$S(x) = \overline{F(x)} = 1 - F(x) = 1 - \left[\frac{1}{1+\theta}(1+\theta - exp^{-\theta x} + \frac{\theta^{\alpha+1}}{(\theta + x^r)^{\alpha}})\right]$$

$$S(x) = \frac{1}{1+\theta}\left[exp^{-\theta x} - \frac{\theta^{\alpha+1}}{(\theta + x^r)^{\alpha}}\right]; x > 0, \alpha > 0, \theta > 0, r > 0 \qquad (9)$$

We plot reliability function of the Dagum-Exponential mixture distribution.

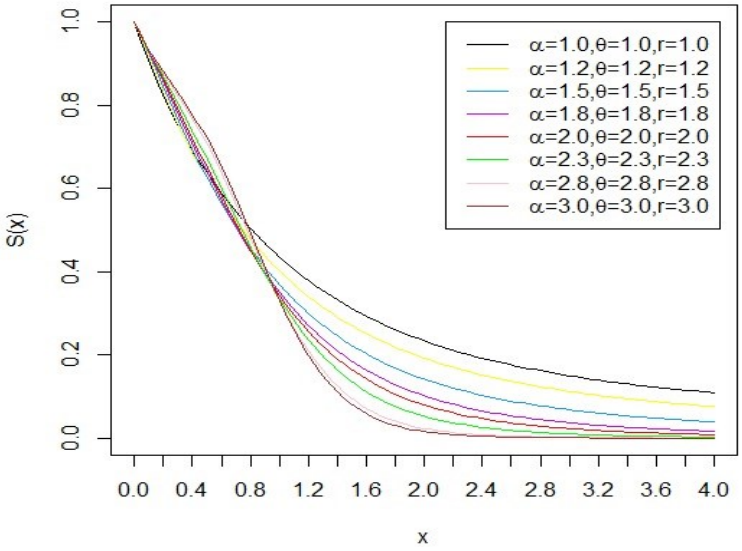

Figure 3. The reliability function(rf) of Dagum-Exponential mixture distribution.

2.4. Hazard function

The mathematical expression for hazard function is

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} = \frac{\frac{\theta}{1+\theta}\left[exp^{-\theta x} + \frac{\alpha r \theta^{\alpha} x^{r-1}}{(\theta + x^r)^{\alpha+1}}\right]}{\frac{1}{1+\theta}\left[exp^{-\theta x} - \frac{\theta^{\alpha+1}}{(\theta + x^r)^{\alpha}}\right]}$$

$$h(x) = \frac{\theta}{1+\theta^{\alpha+1}(\theta + x^r)^{-\alpha}} + \frac{\alpha r x^{r-1}(\theta + x^r)^{-1}}{1 + exp^{-\theta x}} ; x > 0, \alpha > 0, \theta > 0, r > 0 \qquad (10)$$

We plot hazard rate function of the Dagum-Exponential mixture distribution

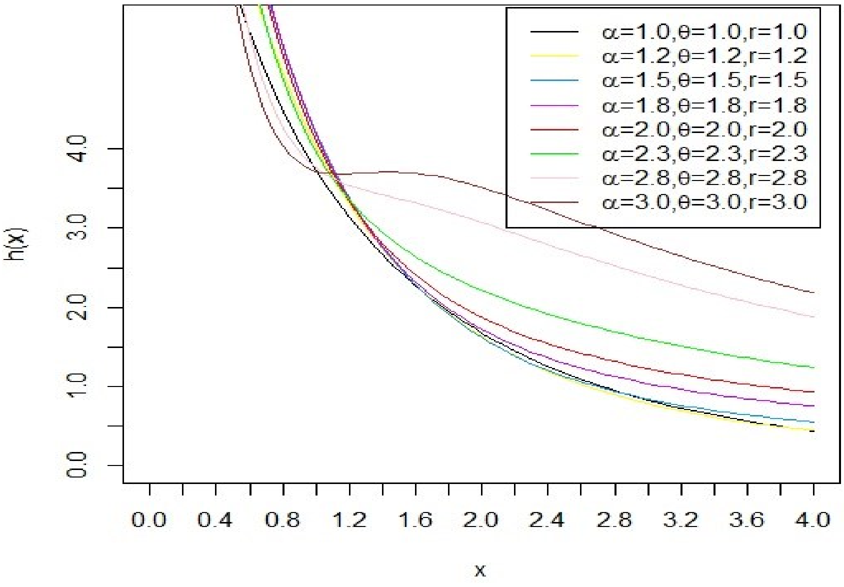

Figure 4. The hazard rate function of Dagum-Exponential mixture distribution.

3. Statistical properties of Dagum-Pareto mixture distribution :

3.1. k^{th} moments

The $\mathbf{k^{th}}$ raw moment of a continuous random variable X denoted by $\boldsymbol{\mu_k}$ is defined as

$$\mu_k = E(X^k) = \int_{-\infty}^{+\infty} x^k f(x)dx$$

$$= \int_{0}^{+\infty} x^k \left(\frac{\theta}{1+\theta}\left[exp^{-\theta x} + \frac{\alpha r \theta^{\alpha} x^{r-1}}{(\theta + x^r)^{\alpha+1}}\right]\right)dx$$

$$= \frac{\theta}{1+\theta}\left[\int_{0}^{+\infty} x^k exp^{-\theta x}\, dx + \int_{0}^{+\infty} \frac{\alpha r \theta^{\alpha} x^{k+r-1}}{(\theta + x^r)^{\alpha+1}} dx\right]$$

$$= \frac{\theta}{1+\theta}\left[\frac{k!}{\theta^{k+1}} + \frac{\theta^{\frac{k}{r}}}{\Gamma(\alpha)}\Gamma(\frac{k}{r}+1)\Gamma(\alpha-\frac{k}{r})\right]$$

$$\mu_k = E(X^k) = \frac{1}{1+\theta}\left[\frac{k!}{\theta^{k}} + \frac{\theta^{\frac{k+1}{r}}}{\Gamma(\alpha)}\Gamma(\frac{k}{r}+1)\Gamma(\alpha-\frac{k}{r})\right] \quad (11)$$

Where Γ (.) is the gamma function.

It becomes easy to obtain the first four raw moment of the Dagum-Exponential mixture distribution.

We obtain the mean the Dagum-Exponential mixture distribution as follows.

$$\mu_1 = E(X) = \frac{1}{1+\theta}\left[\frac{1}{\theta} + \frac{\theta^{\frac{1}{r}+1}\Gamma\left(\frac{1}{r}+1\right)\Gamma\left(\alpha-\frac{1}{r}\right)}{\Gamma(\alpha)}\right]$$

$$= \frac{\theta}{1+\theta}\left[\frac{1}{\theta^2} + \frac{\theta^{\frac{1}{r}}\Gamma\left(\frac{1}{r}+1\right)\Gamma\left(\alpha-\frac{1}{r}\right)}{\Gamma(\alpha)}\right] \tag{12}$$

$$\mu_2 = E(X^2) = \frac{1}{1+\theta}\left[\frac{2!}{\theta^2} + \frac{\theta^{\frac{2}{r}+1}\Gamma\left(\frac{2}{r}+1\right)\Gamma\left(\alpha-\frac{2}{r}\right)}{\Gamma(\alpha)}\right]$$

$$= \frac{\theta}{1+\theta}\left[\frac{2}{\theta^3} + \frac{\theta^{\frac{2}{r}}\Gamma\left(\frac{2}{r}+1\right)\Gamma\left(\alpha-\frac{2}{r}\right)}{\Gamma(\alpha)}\right] \tag{13}$$

$$\mu_3 = E(X^3) = \frac{1}{1+\theta}\left[\frac{3!}{\theta^3} + \frac{\theta^{\frac{3}{r}+1}\Gamma\left(\frac{3}{r}+1\right)\Gamma\left(\alpha-\frac{3}{r}\right)}{\Gamma(\alpha)}\right]$$

$$= \frac{\theta}{1+\theta}\left[\frac{6}{\theta^4} + \frac{\theta^{\frac{3}{r}}\Gamma\left(\frac{3}{r}+1\right)\Gamma\left(\alpha-\frac{3}{r}\right)}{\Gamma(\alpha)}\right] \tag{14}$$

$$\mu_4 = E(X^4) = \frac{1}{1+\theta}\left[\frac{4!}{\theta^4} + \frac{\theta^{\frac{4}{r}+1}\Gamma\left(\frac{4}{r}+1\right)\Gamma\left(\alpha-\frac{4}{r}\right)}{\Gamma(\alpha)}\right]$$

$$= \frac{\theta}{1+\theta}\left[\frac{24}{\theta^5} + \frac{\theta^{\frac{4}{r}}\Gamma\left(\frac{4}{r}+1\right)\Gamma\left(\alpha-\frac{4}{r}\right)}{\Gamma(\alpha)}\right] \tag{15}$$

Also, the central moment is given by the expression.

$$\mu_r' = E(X-\mu)^k = E\left\{\sum_{i=1}^{k}\binom{k}{i}X^{k-i}(-\mu)^i\right\}$$

$$= \sum_{i=1}^{k}\binom{k}{i}(-1)^i\mu_{k-i}\mu^i \tag{16}$$

The 2nd, 3rd and 4th central moments can be obtained a

$\mu_2' = \mu_2 - \mu_1^2$

$\mu_3' = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$

And,

$\mu_4' = \mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1 - 3\mu_1^4$

3.1. Skewness and Kurtosis

The Skewness is defined by

$$S_k = \frac{\mu_3'}{(\mu_2')^{\frac{3}{2}}} = \frac{\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{\frac{3}{2}}} \tag{17}$$

The Kurtosis is defined by

$$K_s = \frac{\mu_4'}{(\mu_2')^2} = \frac{\mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2} \tag{18}$$

The numerical values for mean (μ_1), variance (σ^2), coefficient of variation (ε), skewness (S_k) and kurtosis (K_s) for the Dagum-Pareto mixture distribution for selected values of α, λ, r are listed in Table 1.

α	θ	r	μ_1	σ^2	ε	Sk	K_s
0.1	0.3	0.3	2.234545	12.11642	1.557751	1.862607	6.170821
0.5			3.252231	6.583494	0.7889453	4.280486	1.262418
1.2			3.666999	3.560506	0.5145707	12.2754	-31.28079
1.7			2.149001	13.75354	1.725721	1.418681	5.194416
3.3			2.583389	10.21779	1.237339	2.283026	6.134248
0.3	0.1	0.3	9.063421	99.67259	1.101528	2.009764	1.014444
	0.2		3.65872	28.28166	1.453528	1.765916	4.663778
0.3	0.3	0.1	2.562834	10.5259	1.265928	2.142873	5.99421
		0.4	3.279311	6.453348	0.7746579	4.402207	0.6918331
		0.6	2.027931	10.26434	1.579837	3.049043	9.642333
		0.9	3.295122	4.341839	0.6323614	9.89535	0.4820928
		1.5	2.809848	4.807161	0.7802999	10.21272	16.97129
0.5	0.5	0.5	-0.6666667	2.222222	-2.236068	10.55424	65.69874
8			1.342262	3.533154	1.400374	2.312207	9.406382
9			1.340278	3.537816	1.403372	2.309462	9.400535
0.5	0.5	3	1.691475	2.971228	1.019066	2.576232	6.124965
		4	1.677818	2.924721	1.019289	2.624937	8.476268
		5	1.672253	2.910544	1.020199	2.653701	8.658063
		8	1.667181	2.898324	1.021152	2.682739	8.83027
0.5	0.1	0.5	0.6885714	7.518212	3.982065	2.459821	9.173544
	0.2		2.065641	12.58788	1.717598	1.882239	6.446879
	0.3		4.006667	25.57916	1.262292	1.957307	4.143562
	0.4		9.069091	99.56861	1.100265	2.012646	0.9995889

Table 1: Quantities (μ_1), (σ^2), (ε), (S_k) and (K_s) of Dagum-Exponential mixture distribution.

3.2. Quantile Function

The quantile function of a probability distribution with CDF, F(x) is defined by $q = F^{-1}(x_q)$ where 0 < q < 1. Hence, the quantile function of Dagum-Exponential mixture distribution is derived as follows

$$q = F^{-1}(x_q), \qquad 0 < q < 1$$

$$q = \frac{1}{1+\theta}\left[1 - exp^{-\theta x_q} + \theta - \frac{\theta^{\alpha+1}}{\left(\theta + x_q^r\right)^{\alpha}}\right]$$

$$q(1+\theta) = \frac{\theta\left(\theta + x_q^r\right)^{\alpha} - \theta^{\alpha+1}}{\left(\theta + x_q\right)^{\alpha}} + G(x_q)$$

Where G(y) is the CDF of exponential distribution which is one of the mixture density of the Dagum-Exponential mixture distribution.

$$q(1+\theta) = \theta - \frac{\theta^{\alpha+1}}{\left(\theta + x_q^r\right)^{\alpha}} + G(x_q)$$

$$x_q = \left(\left[\theta^{\alpha} - \frac{\theta^{\alpha-1}}{q(1+\theta)} + \theta^{\alpha-1}G^{-1}(x_q)\right]^{\frac{1}{\alpha}} - \theta\right)^{\frac{1}{r}}$$

But $G^{-1}(x_q) = \frac{-\log(1-q)}{\theta}$, Which is the quantile function of the exponential distribution. Then the quantile function of the Dagum-Exponential mixture distribution becomes

$$x_q = \left[\left[\theta^{\alpha} - \frac{\theta^{\alpha-1}}{q(1+\theta)} - \theta^{\alpha-2}\log(1-q)\right]^{\frac{1}{\alpha}} - \theta\right]^{\frac{1}{r}} \tag{19}$$

$$Q_q = F^{-1}(x_q)$$

The 1st, 2nd and 3rd quartiles of the Dagum-Exponential mixture distribution are given by

$$Q_1 = F^{-1}\left(\frac{1}{4}\right) = \left[\left[\theta^{\alpha} - \frac{\theta^{\alpha-1}}{\frac{1}{4}(1+\theta)} - \theta^{\alpha-2}\log\left(\frac{3}{4}\right)\right]^{\frac{1}{\alpha}} - \theta\right]^{\frac{1}{r}}$$

$$Q_2 = F^{-1}\left(\frac{1}{2}\right) = \left[\left[\theta^{\alpha} - \frac{\theta^{\alpha-1}}{\frac{1}{2}(1+\theta)} - \theta^{\alpha-2}\log\left(\frac{1}{2}\right)\right]^{\frac{1}{\alpha}} - \theta\right]^{\frac{1}{r}}$$

$$Q_3 = F^{-1}\left(\frac{3}{4}\right) = \left[\left[\theta^{\alpha} - \frac{\theta^{\alpha-1}}{\frac{3}{4}(1+\theta)} - \theta^{\alpha-2}\log\left(\frac{1}{4}\right)\right]^{\frac{1}{\alpha}} - \theta\right]^{\frac{1}{r}}$$

4. Maximum likelihood estimates of the parameters :

Let X_1, X_2, ..., X_n be a random sample of size n from the Dagum-Exponential mixture distribution with parameters (α, θ, r). The likelihood function is defined by:

$$L(x_1, x_2, .., x_n; \alpha, \theta, r) = \prod_{i=1}^{n} f(x_i; \alpha, \theta, r)$$

$$= \prod_{i=1}^{n} \frac{\theta}{1+\theta}\left[exp^{-\theta x_i} + \frac{\alpha r \theta^{\alpha} x_i^{r-1}}{(\theta + x_i^r)^{\alpha+1}}\right] \tag{20}$$

The log-likelihood function is

$$l = \ln L\,(x_1, x_2, .., x_n; \alpha, \theta, r) = \frac{\theta^n}{(1+\theta)^n}\left[exp^{-\theta \sum_{i=1}^{n} x_i} + \frac{\alpha^n \theta^n r^n \prod_{i=1}^{n} {x_i}^{r-1}}{\prod_{i=1}^{n}(\theta + x_i^r)^{\alpha+1}}\right]$$

$$= n\,ln\,\theta - n\,ln(1+\theta) - \theta \sum_{i=1}^{n} x_i + n\,ln\,\alpha + n\,ln\,r + nr\,ln\,\theta + (r-1)\sum_{i=1}^{n} x_i$$
$$- (\alpha+1)\sum_{i=1}^{n} ln(\theta + x_i^r) \tag{21}$$

Differentiate (21) with respect to α, and r, respectively to have the following results

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + n\ln\theta \sum_{i=1}^{n} \ln(\theta + x_i^r) \tag{22}$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n} \ln x_i + \frac{n\alpha}{\theta} - (\alpha+1)\sum_{i=1}^{n} \ln(\theta + x_i^r)^{-1} \tag{23}$$

$$\frac{\partial l}{\partial r} = \frac{n}{r} + \sum_{i=1}^{n} \ln x_i - (\alpha+1)\sum_{i=1}^{n} \frac{x_i^r \ln x_i}{(\theta + x_i^r)} \tag{24}$$

Equate the above equations from (22) to (24) to zero and then solve these equations simultaneously for x will yield the maximum likelihood estimates ($\widehat{\alpha}$, $\widehat{\theta}$, $\widehat{r}$)of parameters (α, θ, r).It is much easier to solve these equations using algorithms in statistical software like R and so on when data sets are available.

5. Estimation of the stress-strength parameter

The stress-strength model in dependability characterizes the life of a component with random strength X under random stress Y. When X > Y, the component will work well. The component fails when the stress applied to it surpasses the strength. We examine the issue of estimating R = P(X > Y) in this section.

The stress-strength reliability R of distribution can be obtained as

$$R(x) = P(X,Y) = \int_{0}^{+\infty} P(X > Y|Y = y)\, f_Y(y)dy$$

$$= \int_{0}^{+\infty} [1 - F_x(y)]\, f_Y(y)dy = \int_{0}^{+\infty} S_X(y)\, f_Y(y)dy$$

$$= \int_{0}^{+\infty} \frac{1}{1+\theta_1}\left[exp^{-\theta_1 y} + \frac{{\theta_1}^{\alpha_1+1}}{(\theta_1 + y^{r_1})^{\alpha_1}}\right] \times \frac{\theta_2}{1+\theta_2}\left[exp^{-\theta_2 y} + \frac{\alpha_2 r_2 {\theta_2}^{\alpha_2} y^{r_2-1}}{(\theta_2 + y^{r_2})^{\alpha_2+1}}\right] dy$$

$$= \frac{\theta_2}{(1+\theta_1)(1+\theta_2)} \int_{0}^{+\infty} exp^{-(\theta_1+\theta_2)y} + \frac{{\theta_1}^{\alpha_1+1} exp^{-\theta_2 y}}{(\theta_1 + y^{r_1})^{\alpha_1}}$$
$$+ \left[exp^{-\theta_1 y} + \frac{{\theta_1}^{\alpha_1+1}}{(\theta_1 + y^{r_1})^{\alpha_1}}\right] \frac{\alpha_2 r_2 {\theta_2}^{\alpha_2} y^{r_2-1}}{(\theta_2 + y^{r_2})^{\alpha_2+1}} dy$$

$$R(x) = \frac{\theta_2}{(1+\theta_1)(1+\theta_2)} \left[\int_{0}^{+\infty} exp^{-(\theta_1+\theta_2)y} dy + \int_{0}^{+\infty} \frac{{\theta_1}^{\alpha_1+1} exp^{-\theta_2 y}}{(\theta_1 + y^{r_1})^{\alpha_1}} dy \right.$$
$$\left. + \int_{0}^{+\infty} \left[exp^{-\theta_1 y} + \frac{{\theta_1}^{\alpha_1+1}}{(\theta_1 + y^{r_1})^{\alpha_1}}\right] \frac{\alpha_2 r_2 {\theta_2}^{\alpha_2} y^{r_2-1}}{(\theta_2 + y^{r_2})^{\alpha_2+1}} dy \right] \quad (25)$$

$$= H(\alpha_1, \theta_1, r_1, \alpha_2, \theta_2, r_2) \quad (26)$$

Let's first get the maximal likelihood estimators (MLEs) of $\alpha_1, \theta_1, r_1, \alpha_2, \theta_2,$ and r_2 before calculating the MLE of R. Consequently, $\alpha_1, \theta_1, r_1, \alpha_2, \theta_2,$ and r_2's log-likelihood function can be found by

$$L(x_i, y_j, \alpha_1, \theta_1, r_1, \alpha_2, \theta_2, r_2) = \prod_{i=1}^{n_1} f_X(x_i) \prod_{j=1}^{n_2} f_X(y_j)$$

$$= \prod_{i=1}^{n_1} \frac{\theta_1}{1+\theta_1}\left[exp^{-\theta_1 x_i} + \frac{\alpha_1 r_1\, {\theta_1}^{\alpha_1} {x_i}^{r_1-1}}{(\theta_1 + {x_i}^{r_1})^{\alpha_1+1}}\right] \prod_{j=1}^{n_2} \frac{\theta_2}{1+\theta_2}\left[exp^{-\theta_2 y_j} + \frac{\alpha_2 r_2\, {\theta_2}^{\alpha_2} {y_j}^{r_2-1}}{\left(\theta_2 + {y_j}^{r_2}\right)^{\alpha_2+1}}\right]$$

$$= \frac{{\theta_1}^{n_1}}{(1+\theta_1)^{n_1}}\left[exp^{-\theta_1 \sum_{i=1}^{n_1} \ln x_i} + \frac{{\alpha_1}^{n_1}{r_1}^{n_1}{\theta_1}^{n_1\alpha_1} \prod_{i=1}^{n_1} x_i^{r_1-1}}{\prod_{i=1}^{n_1}\left(\theta_1 + x_i^{r_1}\right)^{\alpha_1+1}}\right] + \frac{{\theta_2}^{n_2}}{(1+\theta_2)^{n_2}}\left[exp^{-\theta_2 \sum_{j=1}^{n_2} \ln y_j} + \frac{{\alpha_2}^{n_2}{r_2}^{n_2}{\theta_2}^{n_2\alpha_2} \prod_{j=1}^{n_2} y_j^{r_2-1}}{\prod_{j=1}^{n_2}\left(\theta_2 + y_j^{r_2}\right)^{\alpha_2+1}}\right] \qquad (27)$$

$$\ln L = \sum_{i=1}^{n_1} f_X(x_i) \sum_{j=1}^{n_2} f_X(y_j)$$

$$l = n_1 \ln\theta_1 - n_1 \ln(1+\theta_1) - \theta_1 \sum_{i=1}^{n_1} \ln x_i + n_1 \ln\alpha_1 + n_1 \ln r_1 + n_1\alpha_1 \ln\theta_1 + (r_1 - 1)\sum_{i=1}^{n_1} \ln x_i - (\alpha_1+1)\sum_{i=1}^{n_1} \ln\left(\theta_1 + x_i^{r_1}\right) + n_2 \ln\theta_2 - n_2 \ln(1+\theta_2) - \theta_2 \sum_{j=1}^{n_2} \ln y_j + n_2 \ln\alpha_2 + n_2 \ln r_2 + n_2\alpha_2 \ln\theta_2 + (r_2 - 1)\sum_{j=1}^{n_2} \ln y_j - (\alpha_2 + 1)\sum_{j=1}^{n_2} \ln\left(\theta_2 + y_j^{r_2}\right) = 0 \qquad (28)$$

Differentiate (28) with respect to $\alpha_1, \theta_1, r_1, \alpha_2, \theta_2$, and r_2, respectively to have the following results

$$\frac{\partial l}{\partial \alpha_1} = \frac{n_1}{\alpha_1} + n_1 \ln\theta_1 - \sum_{i=1}^{n_1} \ln\left(\theta_1 + x_i^{r_1}\right) = 0 \qquad (29)$$

$$\frac{\partial l}{\partial \theta_1} = \frac{n_1}{\theta_1} - \frac{n_1}{1+\theta_1} - \sum_{i=1}^{n_1} \ln x_i + \frac{n_1\alpha_1}{\theta_1} - (\alpha_1+1)\sum_{i=1}^{n_1} \ln(\theta_1 + x_i^{r})^{-1} = 0 \qquad (30)$$

$$\frac{\partial l}{\partial r_1} = \frac{n_1}{r_1} + \sum_{i=1}^{n_1} \ln x_i - (\alpha_1+1)\sum_{i=1}^{n_1} \frac{x_i^{r_1} \ln x_i}{\left(\theta_1 + x_i^{r_1}\right)} = 0 \qquad (31)$$

$$\frac{\partial l}{\partial \alpha_2} = \frac{n_2}{\alpha_2} + n_2 \ln\theta_2 - \sum_{j=1}^{n_2} \ln\left(\theta_2 + y_j^{r_2}\right) = 0 \qquad (32)$$

$$\frac{\partial l}{\partial \theta_2} = \frac{n_2}{\theta_2} - \frac{n_2}{1+\theta_2} - \sum_{j=1}^{n_2} \ln y_j + + \frac{n_2\alpha_2}{\theta_2} - (\alpha_2+1)\sum_{j=1}^{n_2} \ln\left(\theta_2 + y_j^{r_2}\right)^{-1} = 0 \quad (33)$$

$$\frac{\partial l}{\partial r_2} = \frac{n_2}{r_2} + \sum_{j=1}^{n_2} \ln y_j - (\alpha_2+1)\sum_{j=1}^{n_2} \frac{y_j^{r_2} \ln y_j}{\left(\theta_2 + y_j^{r_2}\right)} = 0 \qquad (34)$$

After obtaining $\widehat{\alpha_1}, \widehat{\theta_1}, \widehat{r_1}, \widehat{\alpha_2}, \widehat{\theta_2}$, and $\widehat{r_2}$, we proceed to determine the MLE of R.

$$\hat{R} = H\left(\widehat{\alpha_1}, \widehat{\theta_1}, \widehat{r_1}, \widehat{\alpha_2}, \widehat{\theta_2}, \widehat{r_2}\right) \tag{35}$$

In this case, the maximum likelihood method fails to provide an explicit estimate for the MLEs of R or the parameters. The MLEs must be found numerically in practice; however, these equations are significantly simpler to solve when we use the algorithms in statistical software such as R.

6. Actuarial measures

Several actuarial characteristics of the Dagum-Exponential mixture distribution are covered and numerically determined in this section.

6.1. Value at risk or the quantile function

The value at risk of the Dagum- Exponential mixture distribution is defined as follows

$$VaR = Q_p = x_p = \left[\left[\theta^{\alpha} - \frac{\theta^{\alpha-1}}{(1-p)(1+\theta)} - \theta^{\alpha-2}\log(p)\right]^{\frac{1}{\alpha}} - \theta\right]^{\frac{1}{r}} \tag{36}$$

6.2. Mean excess function

The mean excess or residual life function for a claim amount random variable X is the predicted payment per claim on a policy with a set amount deductible of x, where claims with amounts less than or equal to x are ignored. . It is defined for the Dagum- Exponential mixture distribution as follows

$$e(x) = E(X - x/X > x) = \frac{1}{1 - F(x)} \int_x^{\infty} 1 - F(u)du$$

$$= \frac{1}{\overline{F(x)}} \int_x^{\infty} [\overline{F(u)}]\, du$$

$$= \frac{1}{\frac{1}{1+\theta}\left[exp^{-\theta x} - \frac{\theta^{\alpha+1}}{(\theta + x^r)^{\alpha}}\right]} \times \frac{1}{1+\theta} \int_x^{\infty} \left[exp^{-\theta u} - \frac{\theta^{\alpha+1}}{(\theta + u^r)^{\alpha}}\right] du$$

$$= \frac{1}{exp^{-\theta x} - \frac{\theta^{\alpha+1}}{(\theta + x^r)^{\alpha}}}\left[\frac{exp^{-\theta x}}{\theta} + \frac{(\theta + x^r)^{2\alpha}\Gamma\left(1 + \frac{1}{r}\right)\Gamma\left(\alpha - \frac{1}{r}\right)}{\theta^{2\alpha - \frac{1}{r} - 1}\Gamma(\alpha)}\bar{B}\left(\frac{x^r}{\lambda + x^r}; 1 + \frac{1}{r}; \alpha - \frac{1}{r}\right)\right.$$
$$\left. - \frac{x(\theta + x^r)^{\alpha}}{\theta^{\alpha-1}}\right] \tag{37}$$

Where $\bar{B}\left(\frac{x^r}{\lambda+x^r};1+\frac{1}{r};\alpha-\frac{1}{r}\right)=1-B\left(\frac{x^r}{\lambda+x^r};1+\frac{1}{r};\alpha-\frac{1}{r}\right)$, and $B\left(\frac{x^r}{\lambda+x^r};1+\frac{1}{r};\alpha-\frac{1}{r}\right)$ is the incomplete beta function.

6.3. Limited expected value function

The following is the definition of the limited expected value function L of a claim size variable X, or of the associated cdf F(x).

$$L(u)=E\{min(X,u)\}=\int_0^u xdF(x)+u[1-F(u)], \qquad where\; u>0.$$

$$=\int_0^u xf(x)dx+u\overline{F(u)}$$

$$=m_1(u)+u\overline{F(u)}$$

Where

$$m_1(u)=\int_0^u xf(x)dx=\int_0^u x\frac{\theta}{1+\theta}\left[exp^{-\theta x}+\frac{\alpha r\,\theta^\alpha x^{r-1}}{(\theta+x^r)^{\alpha+1}}\right]dx$$

$$=\frac{\theta}{1+\theta}\left[\frac{exp^{-\theta u}(u-1)}{\theta}+\frac{1}{\theta^2}+\frac{\theta^{\frac{1}{r}}\Gamma\left(1+\frac{1}{r}\right)\Gamma\left(\alpha-\frac{1}{r}\right)\bar{B}\left(\frac{u^r}{\lambda+u^r};1+\frac{1}{r};\alpha-\frac{1}{r}\right)}{\Gamma(\alpha)}\right]$$

And,

$$u\overline{F(u)}=\frac{u}{1+\theta}\left[exp^{-\theta u}-\frac{\theta^{\alpha+1}}{(\theta+u^r)^\alpha}\right]$$

So,

$$L(u)=\frac{\theta}{1+\theta}\left[\frac{exp^{-\theta u}(u-1)}{\theta}+\frac{1}{\theta^2}+\frac{\theta^{\frac{1}{r}}\Gamma\left(1+\frac{1}{r}\right)\Gamma\left(\alpha-\frac{1}{r}\right)\bar{B}\left(\frac{u^r}{\lambda+u^r};1+\frac{1}{r};\alpha-\frac{1}{r}\right)}{\Gamma(\alpha)}\right]$$
$$+\frac{u}{1+\theta}\left[exp^{-\theta u}-\frac{\theta^{\alpha+1}}{(\theta+u^r)^\alpha}\right] \qquad (38)$$

6.4. Tail value at risk

A risk measure linked to the overall value at risk is the tail value at risk (TVaR), sometimes referred to as the tail conditional expectation. The expectation of losses beyond VaR is measured by TVaR. For the Dagum- Exponential mixture distribution, the TVaR is defined as follows

$$TVaR = E(X|X > VaR) = \frac{1}{1-p}\int_{VaR}^{\infty} xf(x)dx$$

$$= \frac{1}{(1-p)(1+\theta)}\int_{VaR}^{\infty}\left[\theta x exp^{-\theta x} + \frac{\alpha r\,\theta^{\alpha+1}x^{r}}{(\theta+x^{r})^{\alpha+1}}\right]dx$$

$$TVaR = \frac{1}{(1-p)(1+\theta)}\left[\frac{1}{\theta} - \frac{\ln(1-p)}{\theta} + \frac{\lambda^{\frac{1}{r}+1}\Gamma\left(1+\frac{1}{r}\right)\Gamma\left(\alpha-\frac{1}{r}\right)}{(1-p)\Gamma(\alpha)}\bar{B}\left(\frac{VaR_p(X)^r}{\lambda+VaR_p(X)^r}; 1 + \frac{1}{r}; \alpha - \frac{1}{r}\right)\right] \qquad (39)$$

7. Numerical computations

7.1. Simulation Study

In this Section, we perform the simulation studies by using the DagumPareto mixture distribution to see the performance of the above estimators corresponding to this distribution.The generation of the Dagum-Exponential mixture distribution can be easily obtained by its quantile function.We generate N = 1000 samples of size n = 30, 50,... , 300,and (α, θ, r) = (0.3, 0.5, 0.5),(0.7, 0.3, 1.9),(0.5, 1.4, 0.8),(1.2, 1.2, 1.2).

For each sample, the MLEs are obtained, these are used to compute values of the following quantities with the help of R package.

1) Average bias of the MLE $\hat{\lambda}$ of the parameter $\lambda = \alpha, \theta$, r :

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}-\lambda)$$

2) Mean Square Error (MSE) $\widehat{\lambda}$ of the parameter $\lambda = \alpha, \theta$, r :

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}-\lambda)^2$$

parameters α, θ, r	n	Av Bias $(\hat{\alpha})$	MSE $(\hat{\alpha})$	Av Bias $(\widehat{\theta})$	MSE $(\widehat{\theta})$	Av Bias $(\hat{r})$	MSE $(\hat{r})$
0.3, 0.5, 0.5	30	0.00141627	0.00200583	0.00502043	0.0252047	0.01288778	0.166095
	50	0.00105241	0.00110758	0.00423498	0.0179351	0.01082644	0.1172117
	100	0.00092550	0.00085656	0.00344188	0.0118466	0.00887452	0.0787571
	150	0.00072632	0.00052755	0.00263798	0.0069589	0.00696255	0.0484772
	200	0.00046960	0.00022052	0.00175304	0.0030731	0.00447595	0.0200342
	300	0.00042262	0.00017860	0.00110075	0.0012116	0.00267971	0.0071807
0.7,0.3,1.9	30	0.00131534	0.00173012	0.04580596	2.098186	0.08356383	6.982913
	50	0.00074201	0.00055058	0.04250207	1.806426	0.07772872	6.041754
	100	0.00057333	0.00032871	0.03606934	1.300997	0.06573282	4.320804
	150	0.00042025	0.00017661	0.02720505	0.7401149	0.04934979	2.435402
	200	0.00014794	0.00011887	0.00689226	0.0475033	0.01127154	0.1270477
	300	0.00009620	0.00009254	0.00528147	0.0278939	0.00846321	0.071626
0.5,1.4,0.8	30	0.00154194	0.00237759	0.05305677	2.815021	0.0961573	9.246225
	50	0.00123384	0.00152237	0.04186695	1.752841	0.07532416	5.67373
	100	0.00082067	0.00067351	0.04111728	1.69063	0.07415485	5.498941
	150	0.00055826	0.00031166	0.03426607	1.174163	0.06156011	3.789647
	200	0.00012827	0.00016455	0.01099087	0.1207992	0.01770881	0.313602
	300	0.00012317	0.00015170	0.00691773	0.0478550	0.01063283	0.1130572
1.2,1.2,1.2	30	0.00683663	0.00767395	0.08278980	0.7983013	0.05384835	2.0289964
	50	0.00526521	0.00403378	0.07181435	0.7720555	0.05380267	2.0289472
	100	0.00397107	0.00388511	0.067821183	0. 7559072	0.05298244	2.0280714
	150	0.00107997	0.00116633	0.02931559	0.4843542	0.0390933	1.528286
	200	0.00092171	0.00084957	0.02447384	0.4711257	0.03893505	1.515938
	300	0.00078030	0.00060882	0.02161525	0.4672192	0.03879364	1.504946

Table 2: The Bias and MSE of the parameters λ=α, θ, r.

7.2. Real Application

Data Set

Conducting an analysis on a mandatory auto insurance recording dataset seems like an exciting research project. Dataset contributed by the Traffic Insurances Information and Monitoring Center is the subject of this research analysis.

The dataset includes claim data, with the policy expiring in December 2018 and starting in January 2016 (see Annex).

To confirm whether our model fits this data, we consider a number of goodness-of-fit tests with particular known distributions that are frequently used for fitting data sets. The Dagum-Exponential distribution, Weibull distribution, Exponential distribution, Log-Normal distribution, and Dagum distribution are among these tests. The iterative process of the algorithm is implemented using R code.

Distribution	$-2logL$	AIC	BIC	CAIC
Dagum-Exponential	**3489, 82**	**3500, 64**	**3505, 15**	**3501,34**
Dagum	3892,78	3887,58	3892,52	3888, 45
Exponential	68588.8	68624,22	68612.44	68611,77
Log-Normal	65599.5	65604,54	65608.98	65604,81
Weibull	3825,08	3828,24	3831, 46	3828,45

Table 3: Goodness of fit statistics of Dagum-Exponential model.

The values of $AIC, CAIC, BIC, -2logL, K\text{-}S$ statistics in Table 3, indicate that Dagum-Exponential model is a strong competitor to the other distributions commonly used in literature for fitting lifetime data, moreover the best fit measured the previous goodness of fit statistics.

8. Conclusion

This study presents a new three-parameter life time distribution called the Dagum-Exponential model, which is derived from the combination of the exponential and Dagum distributions. This new distribution has properties that are useful in a variety of situations, including decreasing hazard rate and reversed hazard rate. We have studied its shape, properties of the probability density function (pdf), hazard rate function (hrf), reliability function (rf), moments, and quantile function, among other relevant mathematical properties.

The maximum likelihood method is used to estimate the parameter of the Dagum-Exponential model. A simulation approach has been used to assess the suggested estimator's performance under the described estimation technique. Using goodness-of-fit statistics, the new model's effectiveness in comparison to the most recent iterations of the Dagum, Exponential, Log-Normal, and exponential distributions is illustrated using a real-world dataset pertaining to claim data, where the policy expires in two years. In order to solve the infinite moments problem, future research efforts might concentrate on investigating the Bayesian estimation of the Dagum-Exponential model parameters and presenting a shortened version of the Dagum-Exponential model.

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Annexe

Amounts claimed between 2016 and 2018.

Month-Year	Claim amounts / DA
January, 2016	4477181800.5
February, 2016	5006102503.1
March, 2016	4121727487.9
April, 2016	4288391254.7
May, 2016	2344716296.1
June, 2016	33892121.2
July, 2016	877209622.3
August, 2016	4757585745.8
September, 2016	68463532.2
October, 2016	1250647142.5
November, 2016	8544193298.1
December, 2016	4967526209.9
January, 2017	4808504924.4
February, 2017	4278134665.8
March, 2017	1238844907.3
April, 2017	4244801650.2
May, 2017	4946063021.3
June, 2017	4766793446.4
July, 2017	4705322787.4
August, 2017	4804505758.3
September, 2017	4856654476.3
October, 2017	8463219700.8
November, 2017	2288724944.2
December, 2017	6287162470.5
January, 2018	61299958.4
February, 2018	84208270.1
March, 2018	8646708121.8
April, 2018	6423014566.1
May, 2018	5129211945.7
June, 2018	18453338.4
July, 2018	74026741.9
August, 2018	5283458942.7
September, 2018	1254577870.6
October, 2018	10735988.2
November, 2018	5164514853.5
December, 2018	61615231.0