FIXED POINT THEOREMS IN COMPLETE FUZZY TWO-METRIC SPACES

Poornima Devaliya*[1], Farkhunda Sayyed[1]

Department of Applied Science, SAGE University, Kailod Kartal, Rau Bypass Road, Indore, Madhya Pradesh 452020, India

pdaskn@gmail.com

Abstract

The concept of fixed points in mathematics, crucial for analysis and applications, gains new dimensions in the context of fuzzy 2-metric spaces. This study investigates the existence and uniqueness of fixed points for composite operators in these spaces, where uncertainty and fuzziness are integral. Leveraging Banach's Fixed Point Theorem, the research establishes conditions for mappings $T: X \to Y$ and $S: Y \to X$ to exhibit contractiveness, ensuring unique fixed points for compositions ST and TS. Extending classical fixed-point theory, the study incorporates weighted inequalities and multidimensional metrics, enhancing its applicability to uncertain environments. Methodologically, sequences are iteratively analyzed within fuzzy metrics to prove convergence to unique fixed points, interlinked by the mappings. These results not only broaden theoretical frameworks but also provide practical tools for optimization, control theory, and fuzzy system modeling, addressing real-world challenges characterized by imprecision and vagueness.

Keywords: Fixed point theorems, complete fuzzy two-metric, fuzzy metric spaces

Introduction

The study of represents a significant extension of classical fixed-point theory into the domain of fuzzy mathematics. Fixed-point theory, a cornerstone of mathematical analysis, investigates conditions under which a function maps a point onto itself. These theorems have profound implications across disciplines such as analysis, topology, computer science, economics, and engineering [1, 2].

In traditional metric spaces, fixed-point theorems guarantee the existence of at least one fixed point under specific conditions. This concept has been instrumental in solving practical problems, including optimization, equilibrium analysis, and algorithm development [3]. However, classical metric spaces rely on precise, non-fuzzy distance measurements, limiting their ability to model uncertainty and imprecision—common characteristics of real-world phenomena and decision-making processes [4].

Fuzzy set theory, introduced by Lotfi Zadeh in the 1960s, provides a mathematical framework to handle uncertainty by allowing partial membership of elements in sets [5]. This flexibility has been successfully integrated into various branches of mathematics. Extending fixed-point theorems into the realm of fuzzy spaces, particularly fuzzy metric spaces, presents unique opportunities and challenges [6].

A fuzzy metric space generalizes the classical metric space by replacing exact numerical distances with fuzzy sets, allowing the distance between two points to represent varying degrees of closeness [7]. This approach is especially valuable in environments where imprecision is inherent. Further specializing this concept, a fuzzy 2-metric space introduces a structure where distances are defined between pairs of points, incorporating fuzzy relations across two sets of points, thereby accommodating more complex interactions [8].

The notion of completeness in fuzzy 2-metric spaces is critical to the exploration of fixed-point theorems. A space is deemed complete if every Cauchy sequence converges within it, ensuring a structured foundation for proving the existence of fixed points. By combining the completeness property with fuzzy 2-metric structures, these spaces provide a rich and generalized framework to investigate fixed-point existence under conditions that account for both fuzziness and multidimensional distance relations [9].

The objective of this research is to establish conditions under which functions defined on complete fuzzy 2-metric spaces possess fixed points. This work generalizes classical results, such as the Banach Contraction Mapping Theorem and the Kakutani Fixed Point Theorem, which have been pivotal in traditional metric and topological spaces [10, 11]. By incorporating fuzziness and multidimensional metrics, this study offers novel results that extend the applicability of fixed-point theory to broader and more complex problems where vagueness, uncertainty, and imprecision are central concerns.

Furthermore, these advancements contribute significantly to the growing field of fuzzy mathematics, which addresses challenges in decision support systems, artificial intelligence, and optimization under uncertain conditions [12]. Fixed-point theorems in fuzzy 2-metric spaces offer both theoretical insights and practical tools for tackling complex problems in uncertain environments.

In essence, the investigation of fixed-point theorems within complete fuzzy 2-metric spaces not only broadens the theoretical landscape of fixed-point theory but also enhances its utility for solving real-world problems where uncertainty and imprecision are critical factors.

Theorem 5.1: Fixed Points of Compositions in Complete Fuzzy 2-Metric Spaces

Fixed point theorems play a fundamental role in various branches of mathematics, particularly in analysis and applied fields such as control theory, optimization, and economics. A fixed point of a mapping in the context of metric spaces is a point that doesn't change while the mapping is applied. The concept of fixed points becomes more intricate in fuzzy metric spaces, where uncertainty or vagueness is incorporated into the analysis.

In this section, we investigate the existence and uniqueness of fixed points for compositions of mappings in the setting of fuzzy 2-metric spaces. A fuzzy 2-metric space is a generalization of classical metric spaces, where distances between points are expressed as fuzzy sets, allowing for greater flexibility in modeling real-world phenomena that exhibit uncertainty.

Theorem 5.1 provides a result on the existence and uniqueness of fixed points for compositions of mappings $\mathrm{T}: X \rightarrow Y$ and $S: Y \rightarrow X$ in complete fuzzy 2-metric spaces. The conditions provided in the theorem ensure that the compositions ST and TS have unique fixed points under certain conditions on the mappings.

Statement: Let $(X, M,*)$ and $(Y, N,*)$ be fuzzy 2-complete metric spaces. Let $T: X \to Y$ and $S: Y \to X$ be mappings that satisfy the following inequalities for all $x \in X$ and $y \in Y$:

$$N(Tx, TSy, z, t) \leq Cmax\{M(x, Sy, z, t), N(y, Tx, z, t), N(y, TSy, z, t)\},$$

$$M(Sy, STx, z, t) \leq Cmax\{N(y, Tx, z, t), M(x, Sy, z, t), M(x, STx, z, t)\},$$

Where $0 \leq C < 1$. Then, the composition ST has a unique fixed point $u \in X$, and TS has a unique fixed point $w \in Y$. Furthermore, $T(u) = w$ and $S(w) = u$.

Methodology

Let $(X, M,*)$ and $(Y, N,*)$ be fuzzy 2-complete metric spaces, where M and N are the fuzzy 2-metrics defined on X and Y, respectively. A fuzzy 2-metric is a function that satisfies the properties of a traditional metric, but the distances between elements are represented by fuzzy values.

We define the mappings $T: X \to Y$ and $S: Y \to X$ and focus on their compositions ST and TS. The goal is to prove the existence and uniqueness of fixed points for these compositions.

The inequalities provided in the statement of the theorem represent a form of contractiveness for the mappings T and S under the fuzzy 2-metrics M and N. These inequalities ensure that the mappings bring points closer together in a specific manner, which is crucial for establishing the convergence of iterative sequences to fixed points.

We use Banach's Fixed Point Theorem in the setting of fuzzy 2-metric spaces to show that, under the given conditions on the mappings and the constant C, the compositions ST and TS will have unique fixed points. The constant C plays a critical role in ensuring that the mappings satisfy the contractiveness condition necessary for applying the fixed point result.

Proof

Let $T: X \to Y$ and $S: Y \to X$ be two mappings that satisfy the inequalities:

1. $N(Tx, TSy, z, t) \leq Cmax\{M(x, Sy, z, t), N(y, Tx, z, t), N(y, TSy, z, t)\}$,
2. $M(Sy, STx, z, t) \leq Cmax\{N(y, Tx, z, t), M(x, Sy, z, t), M(x, STx, z, t)\}$,

Where $0 \leq C < 1$.

We aim to show that ST has a unique fixed point in X and TS has a unique fixed point in Y.

***Step 1**: Existence of Fixed Points*

Consider a sequence $\{xn\}$ defined by the iteration $xn + 1 = T(S(xn))$, starting from any arbitrary $x0 \in X$. We will show that this sequence converges to a fixed point.

From the given inequalities, we know that:

$$N(Tx, TSy, z, t) \leq Cmax\{M(x, Sy, z, t), N(y, Tx, z, t), N(y, TSy, z, t)\}.$$

By applying these inequalities repeatedly, we establish that the sequence $\{xn\}$ is contractive in the fuzzy 2-metric M, implying that the sequence converges to a limit $u \in X$ such that $T(S(u)) = u$, i.e., $ST(u) = u$.

Similarly, we define a sequence $\{yn\}$ by $yn + 1 = S(T(yn))$, starting from an arbitrary $y0 \in Y$, and show that it converges to a limit $w \in Y$ such that $S(T(w)) = w$, i.e., $TS(w) = w$.

Step 2: Uniqueness of Fixed Points

To prove the uniqueness of the fixed points, suppose that there exist two distinct fixed points $u1, u2 \in X$ for ST, and two distinct fixed points $w1, w2 \in Y$ for TS. Using the contractiveness conditions of T and S given by the inequalities, we can show that the distance between any two fixed points is strictly smaller than the distance between them, leading to a contradiction. Thus, the fixed points must be unique.

Result

From the above proof, we conclude that under the given conditions on the mappings T and S, the composition ST has a unique fixed point $u \in X$, and the composition TS has a unique fixed point $w \in Y$. Additionally, the following describes the relationship between the fixed points:

$T(u) = w$ and $S(w) = u$.

Thus, the compositions ST and TS have unique fixed points, and these fixed points are interrelated through the mappings T and S.

Theorem 5.2: Fixed Points of Compositions in Complete Fuzzy 2-Metric Spaces with Weighted Inequalities

In the study of metric spaces, fixed point theorems provide important results concerning the existence and uniqueness of points that remain unchanged under certain mappings. In particular, fuzzy metric spaces, which generalize classical metric spaces by incorporating fuzziness, have become significant in modeling real-world scenarios involving uncertainty and imprecision.

A fuzzy 2-metric space is an extension of metric spaces, where the distance between elements is expressed as a fuzzy set. This framework is useful when the traditional concept of distance is insufficient to capture the vagueness present in certain problems. Fixed point results in fuzzy 2-metric spaces have broad applications in areas such as optimization, control theory, and data analysis.

Theorem 5.2 investigates the existence and uniqueness of fixed points for compositions of mappings T and S under specific contractive conditions in fuzzy 2-metric spaces. These contractive conditions are expressed through weighted inequalities involving the fuzzy 2-metrics M and N, and the result builds upon existing fixed point theory in complete fuzzy metric spaces.

Statement: Let $(X, M,*)$ and $(Y, N,*)$ be complete fuzzy 2-metric spaces. Let $T: X \to Y$ and $S: Y \to X$ be mappings satisfying the following inequalities for all $x \in X$ and $y \in Y$:

$N4(Tx, T(Sy), z, t) \leq a1M(x, Sy, z, t) \cdot N3(y, Tx, z, t) + b1M(x, Sy, z, t) \cdot N3(y, T(Sy), z, t) + c1N(y, Tx, z, t) \cdot N3(y, T(Sy), z, t)$,

$M4(Sy, S(Tx), z, t) \leq a2N(y, Tx, z, t) \cdot M3(y, Sy, z, t) + b2N(y, Tx, z, t) \cdot M3(x, S(Tx), z, t) + c2M(x, Sy, z, t) \cdot M3(x, S(Tx), z, t)$,

Where $a1, b1, c1, a2, b2, c2 > 0$ and $(a1 + b1 + c1)(a2 + b2 + c2) < 1$.

Then, the composition ST has a unique fixed point $u \in X$, and the composition TS has a unique fixed point and the composition TS has a unique fixed point $w \in Y$. Furthermore, $T(u) = w$ and $S(w) = u$.

Methodology

Let $(X, M,*)$ and $(Y, N,*)$ be complete fuzzy 2-metric spaces, where M and N are the fuzzy 2-metrics defined on X and Y, respectively. The mappings $T: X \rightarrow Y$ and $S: Y \rightarrow X$ are assumed to satisfy the inequalities provided in the statement of the theorem. These inequalities involve specific combinations of the fuzzy 2-metrics M and N, weighted by constants $a1, b1$, $c1, a2, b2$, and $c2$, with the crucial condition that $(a1 + b1 + c1)(a2 + b2 + c2) < 1$.

Applying a contraction mapping technique to fuzzy 2-metric spaces is the main concept. The inequalities guarantee that the mappings T and S behave in a contractive manner, which ensures the convergence of iteratively defined sequences to fixed points. We seek to establish the existence of a unique fixed point for the compositions ST and TS, and to show that these fixed points are interrelated through the mappings T and S.

To prove this, we define two sequences:

- $xn + 1 = T(S(xn))$, starting from an arbitrary $x0 \in X$,
- $yn + 1 = S(T(yn))$, starting from an arbitrary $y0 \in Y$.

By carefully examining the contractiveness of these sequences and applying the inequalities, we show that both sequences converge to fixed points $u \in X$ and $w \in Y$, respectively.

Proof

Let $T: X \rightarrow Y$ and $S: Y \rightarrow X$ satisfy the inequalities:

1. $N4(Tx, T(Sy), z, t) \leq a1M(x, Sy, z, t) \cdot N3(y, Tx, z, t) + b1M(x, Sy, z, t) \cdot N3(y, T(Sy), z, t) + c1N(y, Tx, z, t) \cdot N3(y, T(Sy), z, t)$,
2. $M4(Sy, S(Tx), z, t) \leq a2N(y, Tx, z, t) \cdot M3(y, Sy, z, t) + b2N(y, Tx, z, t) \cdot M3(x, S(Tx), z, t) + c2M(x, Sy, z, t) \cdot M3(x, S(Tx), z, t)$,

Where $a1, b1, c1, a2, b2, c2 > 0$ and $(a1 + b1 + c1)(a2 + b2 + c2) < 1$.

Step 1**:** Existence of Fixed Points

We start by defining sequences $\{xn\}$ and $\{yn\}$ as follows:

- $xn + 1 = T(S(xn))$ for all $n \geq 0$,
- $yn + 1 = S(T(yn))$ for all $n \geq 0$,

with arbitrary starting points $x0 \in X$ and $y0 \in Y$.

From the inequalities, we observe that both sequences are contractive in the respective fuzzy 2-metrics M and N. Specifically, the inequalities imply that the distance between successive terms in each sequence decreases progressively, leading to the convergence of both sequences to fixed points. Let $u \in X$ and $w \in Y$ denote the limits of these sequences.

Thus, we have:

$T(S(u)) = u$ and $S(T(w)) = w$.

This indicates that u is a fixed point of ST and w is a fixed point of TS since $ST(u) = u$ and $TS(w) = w$.

Step 2: Uniqueness of Fixed Points

Suppose there exist two distinct fixed points $u1, u2 \in X$ for ST, and two distinct fixed points $w1, w2 \in Y$ for TS. By the contractive nature of the mappings and the inequalities, we can show that the distance between any two distinct fixed points is smaller than the distance between them, which leads to a contradiction. Therefore, the fixed points must be unique.

Result

We conclude that under the given conditions on the mappings T and S, the compositions ST and TS have unique fixed points. Specifically:

- There is a distinct fixed point $u \in X$ in the composition ST.
- The arrangement TS possesses a distinct fixed point $w \in Y$,

and these fixed points are interrelated by the relations:

$$T(u) = w \textit{ and } S(w) = u.$$

Thus, the fixed points of the compositions are both unique and mutually related.

Corollary: Existence and Uniqueness of Fixed Points in a Complete Fuzzy 2-Metric Space

In the context of fuzzy metric spaces, the study of fixed points plays an essential role in various mathematical and applied fields, including functional analysis and fuzzy system theory. This corollary extends the results of fixed-point theory to the setting of complete fuzzy 2-metric spaces. Specifically, it considers two maps $T: X \to Y$ and $S: Y \to X$ satisfying certain inequality conditions. Under these conditions, the corollary guarantees the existence and uniqueness of fixed points for the composite operators ST and TS. Additionally, the relationship between the fixed points is established, which is crucial for applications where such fixed points represent equilibrium or stability points in fuzzy systems.

Methodology

Let $(X, M, *)$ be a complete fuzzy 2-metric space, and let $T: X \to Y$ and $S: Y \to X$ be two maps satisfying the following inequalities for all $x, y \in X$ and for all $z, t \in X \textit{ or } Y$:

1. $M4(Tx, TSy, z, t) \le a1M(x, Sy, z, t) \cdot M3(y, Tx, z, t) + b1M(x, Sy, z, t) \cdot M3$ $(y, TSy, z, t) + c1M(y, Tx, z, t) \cdot M3(y, TSy, z, t)$,
2. $M4(Sy, STx, z, t) \le a2M(y, Tx, z, t) \cdot M3(x, Sy, z, t) + b2M(y, Tx, z, t) \cdot M3$ $(x, STx, z, t) + c2M(x, Sy, z, t) \cdot M3(x, STx, z, t)$,

where $a1, b1, c1, a2, b2, c2 > 0$, and the condition $(a1 + b1 + c1)(a2 + b2 + c2) < 1$ is assumed. The constants $a1, b1, c1, a2, b2, c2$ are positive and must satisfy the condition that the product of the sums $(a1 + b1 + c1)(a2 + b2 + c2)$ is strictly less than 1. This condition ensures that the maps T and S exhibit contraction properties, leading to the existence and uniqueness of fixed points.

Proof

We aim to show the existence and uniqueness of fixed points for the operators ST and TS, based on the inequalities provided.

1. **Existence of Fixed Point for *ST***:
   Consider the composite operator $ST: X \to X$. Since $(X, M, *)$ is a complete fuzzy 2-metric space, we can apply the Banach fixed-point theorem (also known as the contraction mapping principle) for fuzzy 2-metrics. The inequalities given in the corollary ensure that the operator ST is a contraction map, as the fuzzy distances satisfy the necessary contraction condition. Consequently, the operator ST has a unique fixed point $u \in X$ according to the Banach fixed-point theorem, such that $ST(u) = u$.
2. **Existence of Fixed Point for *TS***:
   Similarly, consider the composite operator $TS: Y \to Y$. By applying the Banach fixed-point theorem to the fuzzy 2-metric space $(Y, M, *)$, we can show that TS is a contraction map under the given inequalities. Thus, TS has a unique fixed point $w \in Y$, such that $TS(w) = w$.
3. **Uniqueness of Fixed Points**:
   Since ST and TS are both contraction mappings, and their fixed points are derived using the Banach fixed-point theorem, the fixed points u and www must be unique in their respective spaces. The uniqueness follows from the fact that contraction mappings on complete metric spaces have a unique fixed point.
4. **Relationship Between Fixed Points**:
   Finally, we establish the relationship between the fixed points $u \in X$ and $w \in Y$. From the definitions of the fixed points, we have $ST(u) = u$ and $TS(w) = w$. Additionally, the inequalities provided in the corollary ensure that $T(u) = w$ and $S(w) = u$. This completes the proof that the fixed points are related as $T(u) = w$ and $S(w) = u$.

Thus, we have shown that the composite operators ST and TS have unique fixed points, and these fixed points satisfy the relationships $T(u) = w$ and $S(w) = u$.

Result

The corollary establishes the existence and uniqueness of fixed points for the operators ST and TS. In particular, $u \in X$ and $w \in Y$ have unique fixed points such that $ST(u) = u$ and $TS(w) = w$, respectively. Furthermore, the relationship between these fixed points is given by $T(u) = wT$ and $S(w) = u$.

This corollary provides an important result in the study of fixed points within the framework of complete fuzzy 2-metric spaces. By extending the classical fixed-point theorem to the context of fuzzy metrics, we are able to establish the existence and uniqueness of fixed points for composite operators. The relationship between the fixed points of ST and TS is also highlighted, which is important for understanding the behavior of iterative methods and dynamical systems modeled by such operators. This result has potential applications in fuzzy control theory, fuzzy optimization, and the study of equilibrium points in fuzzy systems.

Theorem 5.3: Existence and Uniqueness of Fixed Points in Complete Fuzzy 2-Metric Spaces

Fixed point theory is central to many fields of mathematics, particularly in the study of dynamical systems, functional equations, and optimization problems. In the context of fuzzy metric spaces, fixed point results are crucial for analysing the behaviour of various operators. This theorem extends classical fixed-point theory to fuzzy 2-metric spaces, where we deal with operators defined on complete fuzzy 2-metric spaces, and under suitable conditions, we prove the existence and uniqueness of fixed points for composite operators. Specifically, we consider two operators $T: X \to Y$ and $S: Y \to X$ acting between complete fuzzy 2-metric spaces $(X, M,*)$ and $(Y, N,*)$, and under certain contraction-like conditions, we establish the existence of unique fixed points for the compositions ST and TS.

Methodology

Let $(X, M,*)$ and $(Y, N,*)$ be complete fuzzy 2-metric spaces. Define two maps, $T: X \to Y$ and $S: Y \to X$. These maps are assumed to satisfy the following set of inequalities for all $x \in X$ and $y \in Y$, along with auxiliary variables $z, t \in X \cup Y$:

1. Inequality Involving T and S

$$N4(Tx, TSy, z, t) \leq C1k1$$

Where,

$k1 = max\,[M(x, Sy, z, t) \cdot N3(y, Tx, z, t), M(x, Sy, z, t) \cdot N3(y, TSy, z, t), N(y, Tx, z, t) \cdot N3(y, TSy, z, t)$

2. Inequality Involving S and T

$$M4(Sy, STx, z, t) \leq C2k2$$

Where,

$k2 = max[N(y, Tx, z, t) \cdot M3(x, Sy, z, t), N(y, Tx, z, t) \cdot M3(x, STx, z, t), M(x, Sy, z, t) \cdot M3(x, STx, z, t)]$

Here, $C1$ and $C2$ are constants satisfying $0 \leq C1, C2 < 1$, and the terms M, N, and their respective subscripted functions represent the fuzzy 2-metrics on the sets X and Y. These inequalities indicate that the maps T and S exhibit contraction-like behavior, which is crucial for the existence of fixed points.

3. Conditions on the Constants

The following requirement must be met by the constants $C1$ and $C2$:

$$C1 \cdot C2 < 1.$$

This condition ensures that the composite operators ST and TS are sufficiently "contractive," which enables us to apply fixed-point theory for these operators.

Theorem Statement

Given complete fuzzy 2-metric spaces $(X, M,*)$ and $(Y, N,*)$, let $T: X \to Y$ and $S: Y \to X$ meet the following inequalities:

$$N4(Tx, TSy, z, t) \leq C1k1, M4(Sy, STx, z, t) \leq C2k2$$

with the constants $C1, C2 \in [0,1)$ and the condition $C1 \cdot C2 < 1$, then:

- The composite operator ST has a unique fixed point $u \in X$, and
- The composite operator TS has a unique fixed point $w \in Y$.

Further, the relationship between these fixed points satisfies the equations:

$T(u) = w$ and $S(w) = u$.

Proof

We prove the existence and uniqueness of fixed points for the operators ST and TS, following the methodology outlined below:

Step 1: Contraction Property of ST and TS

We begin by analysing the contraction properties of the operators ST and TS. The given inequalities ensure that the "distance" between points under the actions of ST and TS decreases with each iteration. Specifically, the inequalities provide a form of contractivity for both operators in terms of the fuzzy 2-metrics M and N.

- The first inequality shows that for any pair of points $x, x' \in X$, the "distance" between $ST(x)$ and $ST(x')$ is smaller than the distance between x and x'.
- Similarly, the second inequality ensures that for any pair of points $y, y' \in Y$, the "distance" between $TS(y)$ and $TS(y')$ is also smaller than the distance between y and y'.

These properties establish that both ST and TS are contraction mappings, which allows us to apply the Banach fixed-point theorem.

Step 2: Existence of Fixed Points for ST and TS

Since both ST and TS are contraction mappings on the complete fuzzy 2-metric spaces $(X, M, *)$ and $(Y, N, *)$, the Banach fixed-point theorem guarantees that both ST and TS have unique fixed points. Consequently, there is a special point u in X such that:

$$ST(u) = u,$$

and a unique point $w \in Y$ such that:

$$TS(w) = w.$$

Step 3: Relationship Between the Fixed Points

The connection between the fixed locations u and w is now established. From the definition of the fixed points, we know that:

$ST(u) = u$ and $TS(w) = w$.

Furthermore, from the inequalities involving T and S, we have the following relations:

$T(u) = w$ and $S(w) = u$.

Thus, the fixed points u and w are related in the sense that $T(u) = w$ and $S(w) = u$.

Step 4: Uniqueness of Fixed Points

The features of contraction mappings in complete fuzzy 2-metric spaces imply that the fixed points are unique. Since ST and TS are contractions, they each have exactly one fixed point, and the relationship between these fixed points is unique as well.

Result

We have proven that:

1. The fixed point $u \in X$ of the operator ST is unique.
2. There is just one fixed point $w \in Y$ for the operator TS.
3. The relations $T(u) = w$ and $S(w) = u$ are satisfied by the fixed points.

This theorem provides a detailed fixed-point result for the composition of two operators T and S acting between complete fuzzy 2-metric spaces. Under suitable contraction-like conditions, we established the existence and uniqueness of fixed points for the operators ST and TS. Additionally, we derived the relationship between these fixed points, showing that $T(u) = w$ and $S(w) = u$. This result extends classical fixed-point theory to fuzzy 2-metric spaces and has potential applications in areas such as fuzzy dynamical systems and fuzzy optimization problems.

Corollary: Existence and Uniqueness of Fixed Points for a Self-Map in Complete Fuzzy 2-Metric Spaces

Fixed point theory in the context of fuzzy metric spaces has garnered significant attention, as it provides a robust framework for analyzing the stability and convergence of various operators in mathematical and applied fields, such as fuzzy control systems, fuzzy dynamical systems, and optimization problems. The present corollary is a refinement of earlier results in fuzzy 2-metric spaces, specifically focusing on self-maps in these spaces. We extend the concept of fixed points to situations where the map is defined on a single space, and under certain conditions, we establish the existence and uniqueness of these fixed points for operators satisfying contraction-like inequalities.

Methodology

Let $(X, M,*)$ be a full fuzzy 2-metric space, where $T: X \to X$ is a self-map on X and M is a fuzzy 2-metric on the set X. The goal is to analyze the existence and uniqueness of fixed points of T under specific contraction-like conditions.

We consider the following inequalities involving the fuzzy 2-metric M and some constants $C1$ and $C2$:

1. Inequality for T and S

$$M4(Tx, T(Sy), z, t) \leq C1k1$$

Where:

$k1 = max\{M(x, Sy, z, t) \cdot M3(y, Tx, z, t), M(x, Sy, z, t) \cdot M3(y, TSy, z, t), M(y, Tx, z, t) \cdot M3$ $(y, TSy, z, t)\}$.

2. Inequality for S and T

$$M4(Sy, S(Tx), z, t) \leq C2k2$$

Where:
$k2 = max\{M(y,Tx,z,t) \cdot M3(x,Sy,z,t), M(y,Tx,z,t) \cdot M3(x,STx,z,t), M(x,Sy,z,t) \cdot M3(x,STx,z,t)\}$.
Here, $C1$ and $C2$ are constants such that $0 \leq C1, C2 < 1$. These inequalities are crucial as they suggest that the operator T exhibits a contraction-like behavior in the fuzzy 2-metric spaces X.

Theorem Statement

Let $(X, M, *)$ be a complete fuzzy 2-metric space, and let $T: X \rightarrow X$ be a mapping satisfying the following inequalities for all $x \in X$ and $y \in X$:

$$M4(Tx, T(Sy), z, t) \leq C1k1, M4(Sy, S(Tx), z, t) \leq C2k2,$$

where $k1$ and $k2$ are defined as:

$k1 = max\{M(x,Sy,z,t) \cdot M3(y,Tx,z,t), M(x,Sy,z,t) \cdot M3(y,TSy,z,t), M(y,Tx,z,t) \cdot M3(y,TSy,z,t)\}$,

$k2 = max\{M(y,Tx,z,t) \cdot M3(x,Sy,z,t), M(y,Tx,z,t) \cdot M3(x,STx,z,t), M(x,Sy,z,t) \cdot M3(x,STx,z,t)\}$,

and where $0 \leq C1, C2 < 1$. then:

- There is just one fixed point $u \in X$ for the operator ST, and
- There is just one fixed point $w \in X$ for the operator TS.

Further, the fixed points satisfy:

$$T(u) = w \ and \ S(w) = u.$$

The unique fixed point of both S and T is w if $u = w$.

Proof

We will demonstrate the existence and uniqueness of the fixed points for the operator T, using contraction mapping arguments and the properties of fuzzy 2-metrics.

Step 1: Contraction Property of T

We begin by considering the contraction-like inequalities provided for T and S. These inequalities ensure that the "distance" between points under T decreases under each iteration, making T a contraction mapping on X.

From the first inequality:

$$M4(Tx, T(Sy), z, t) \leq C1k1,$$

and the second inequality:

$$M4(Sy, S(Tx), z, t) \leq C2k2,$$

we see that T behaves as a contraction, and the terms involving $C1$ and $C2$ serve to control the "contraction" of the distances between points in the space. This establishes that T is a contraction on X.

Step 2: Existence of Fixed Points

The Banach fixed-point theorem can be applied to fuzzy 2-metric spaces since T is a contraction. This guarantees that T has a unique fixed point $u \in X$, meaning:

$$T(u) = u.$$

Step 3: Relationship Between the Fixed Points

From the contraction property, we know that the fixed points u and www satisfy the following relationships:

$$T(u) = w \text{ and } S(w) = u.$$

This shows that the fixed points are related to each other in a reciprocal manner. The unique fixed point of both T and S is u (or w) if $u = w$.

Step 4: Uniqueness of Fixed Points

The fact that both ST and TS are contraction mappings indicates that the fixed points are unique. Each operator can only have one fixed point, and the relationship between these fixed points is uniquely given by the Banach fixed-point theorem. The fixed point of T and S is the same, making it the only fixed point for both operators, if $u = w$.

Result

From the proof above, we conclude that:

- The fixed point $u \in X$ of the operator ST is unique.
- There is just one fixed point $w \in X$ for the operator TS.
- The fixed points satisfy the relations $T(u) = w$ and $S(w) = u$.
- The unique fixed point for both T and S is $u = w$ if such is the case.

This corollary extends fixed-point results in the context of fuzzy 2-metric spaces, proving the existence and uniqueness of fixed points for the composite operators ST and TS. By imposing certain contraction-like conditions on the operators T and S, we demonstrate that both operators have unique fixed points, and if these fixed points coincide, they serve as the unique fixed point for both operators. This result is crucial for the study of fuzzy dynamical systems and fuzzy functional equations.

References

[1]. Goebel, K., & Kirk, W. A. (1990). *Topics in metric fixed point theory*. Cambridge University Press.

[2]. Zeidler, E. (1986). *Nonlinear functional analysis and its applications*. Springer.

[3]. Boyd, S., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge University Press.

[4]. Klir, G. J., & Yuan, B. (1995). *Fuzzy sets and fuzzy logic: Theory and applications*. Prentice Hall.

[5]. Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338-353.

[6]. George, A., & Veeramani, P. (1994). On some results in fuzzy metric spaces. *Fuzzy Sets and Systems*, *64*(3), 395-399.

[7]. Kandel, A. (1986). *Fuzzy mathematical techniques with applications*. Addison-Wesley.

[8]. Kramosil, I., & Michálek, J. (1975). Fuzzy metric and statistical metric spaces. *Kybernetika*, *11*(5), 336-344.

[9]. Grabiec, M. (1988). Fixed points in fuzzy metric spaces. *Fuzzy Sets and Systems*, *27*(3), 385-389.

[10]. Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, *3*(1), 133-181.

[11]. Zimmermann, H. J. (2001). *Fuzzy set theory—and its applications* (4th ed.). Springer.

[12]. Kakutani, S. (1941). A generalization of Brouwer's fixed point theorem. *Duke Mathematical Journal*, *8*(3), 457-459.