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2 – BLOCK DUPLICATION OF PATH GRAPHS

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Abstract:

In this paper, we present the 2 *- block duplication of path graphs. In this process of* 2 *– block duplication, we use two distinct edges for duplicating the* 2*- blocks* B_i *and* B_j*. The resulting graph is called the* 2*- block duplicated graph, denoted by* $D_G^2(B_iB_j)$*. The main focus is on path graphs and we present* 2*- block duplication on path graphs* P_3, P_4 , P_5 , P_6 *and* P_n.

Keywords: AUM Block Duplication, l *– block duplication,* 2 – block duplication, path graph, neighborhood of a block.

AMS classification: 05C78.

1. Introduction

Graph theory is an interesting branch of Mathematics with widespread applications across disciplines such as computer science, biology, network theory, operations research and so on. One of the fundamental types of graphs studied is the path graph, which consists of vertices connected in a linear sequence by edges. Due to their simplicity and versatility, path graphs serve as a key structure in various theoretical and applied studies, including the modelling of transportation systems, molecular structures, and communication networks.

In graph theory, various transformations and operations can be performed on graphs for an in depth understanding of the characterization. One such transformation is block duplication, where a set of any two blocks within the graph is duplicated. In this paper, we focus on the operation of two block duplication within path graphs.

The block labelling technique was formulated by A. Uma Maheswari et.al. in [4] [5] [6] [7] [8] [9] [10] [11] [12] and [13]. AUM Block colouring was studied in [7] and [9]. Moreover, AUM Block Sum Prime Distance Labelling was introduced by the authors for snake families of graphs in [14]. The authors introduced the concept of block duplication in [15]. In this paper 2- block duplication of path graphs P_3, P_4 , P_5 , P_6 *and* P_n is studied with suitable examples.

2. Preliminaries

Block of a graph G[2]: The graph G is said to be separable if it has at least one cut-vertex. Otherwise, G is non-separable. A maximal non separable connected subgraph of graph G is called a **block of graph G.**

Path Graph[1]: A **Path graph P_n** is a graph whose vertices can be listed in the order v_1, v_2, …, v_n such that the edges are $\{v_i v_{i+1}\}$ where $i = 1,2, \dots, n-1$.

Definition: Neighbourhood of a block[15] :

Let G be any Graph. The neighborhood of a block B is the set of all blocks that have a common vertex with B and it is denoted by N(B).

Definition: AUM Block Duplication [15]: Let G be any Graph. **Duplication of a block B_i by an edge e** is the graph, which is obtained by adding the new edge $e = uv$ to G and joining the vertices u and v with the vertex common to B_i and its neighboring blocks B_j .

The graph obtained after the duplication of a graph G is called block duplicated graph and it is denoted by $D_G\ (B)$.

l - block duplication [15]: Let n be the number of blocks in a graph. Let l be any positive integer.

Duplication of l blocks by edges is called l – block duplication. After the duplication the resulting graph is called l - block duplicated graph and it is denoted by $D_G^l(B)$.

Different types of duplication of l - blocks by edges [15]:

Type 1: Duplication of l - blocks by same edge.

Type 2: Duplication of l - blocks by distinct edges.

Type 3: Duplication of l - blocks by allowing repeated edges for some blocks.

Duplication of l - blocks by same edge ($l \leq n$)[15]:

Let G be any Graph with n blocks. In this type l blocks are duplicated with same edge e.

Duplication of l - blocks by distinct edges ($l \leq n$)[15]:

Let G be any Graph with n blocks. In this type l blocks are duplicated by l distinct edges e_1, e_2, e_3, …, e_l.

Duplication of l -blocks by allowing repeated edges for some blocks ($l \leq n$)[15]:

Let G be any Graph with n blocks. In this type some blocks are duplicated with distinct edges and some blocks are duplicated with edges that has been already used.

3. Main results

2 - block duplication of path graphs by distinct edges

In this section, consider the path graphs P_3, P_4 , P_5 , P_6 specifically and general graph P_n for $n > 6$. We apply 2 – block duplication using distinct edges.

PROPOSITION 3.1: 2 - block duplication of P_3 by distinct edges results in Duplicated graph $D^2_{P_3}$ (B_1B_2) (as in Figure 2).

Proof:

Let u_1 , u_2 , u_3 be the vertices and B_1 , B_2 be the blocks of P_3.

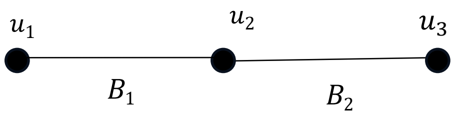

Figure 1: Path Graph P_3

Here the neighbor sets are N(B_1) = { B_2} and N(B_2) = { B_1}

Let us consider the edge v_1w_1 and v_2w_2 for duplication of B_1 and B_2 respectively.

For duplication of B_1, Let us join the vertices v_1 and w_1 with u_2 which is common to B_1 and its neighbor B_2.

For duplication of B_2, Let us join the vertices v_2 and w_2 with u_2, which is common to B_2 and its neighbor B_1.

Thus, the duplicated graph is obtained as follows

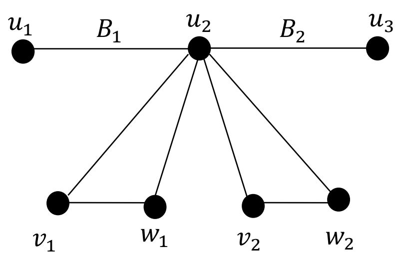

Figure 2: Duplicated graph $D^2_{P_3}$ (B_1B_2)

PROPOSITION 3.2: 2 - Block duplication of P_4 by distinct edges results in Duplicated graphs $D^2_{P_4}$ (B_1B_2), $D^2_{P_4}$ (B_1B_3) and $D^2_{P_4}$ (B_2B_3) as in figure 4, 5 and 6 respectively.

Proof:

Let u_1, u_2, u_3, u_4 be the vertices and B_1 , B_2 , B_3 be the blocks of P_4.

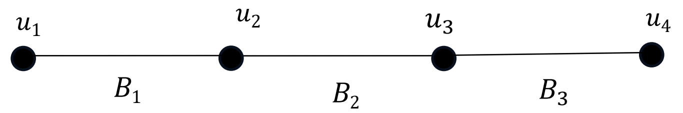

Figure 3: Path Graph P_4

Let us consider the edges $v_1, w_1$and v_2 w_2 for the duplication process.

Then the possible cases for duplications are:

Case (i): Duplication of the blocks $B_1$and B_2.

N(B_1)={ B_2} N (B_2)={ B_1, B_3}

Duplication of block B_1can be done as in Proposition 3.1 and duplication of block B_2 can be done by joining v_2 and w_2 with u_2, which is common to B_2 and its neighbor $B_1$and with u_3 , which is common to B_2 and its neighbor B_3.

Thus, the duplicated graph is obtained as follows,

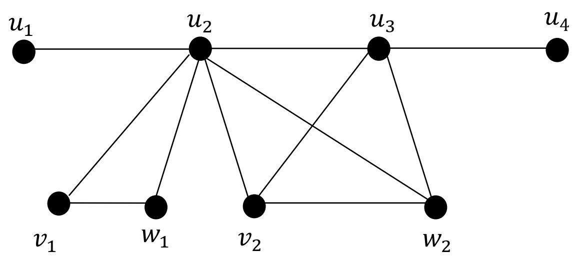

Figure 4: Duplicated graph $D^2_{P_4}$ (B_1B_2)

Case (ii): Duplication of the blocks $B_1$and B_3.

N (B_1)={B_2} N (B_3)={ B_2}.

Duplication of block B_1 can be done as in Proposition 3.1. and duplication of B_3 can be done by joining v_2 and w_2 with u_3, which is common to B_3 and its neighbor B_2.

Thus, the duplicated graph is obtained as follows,

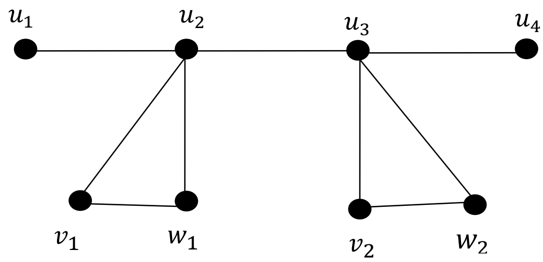

Figure 5: Duplicated graph $D^2_{P_4}$ (B_1B_3)

Case(iii): Duplication of the blocks B_2 and B_3.

N (B_2)={B_1,B_3} N (B_3)={ B_2}.

The block B_2 can be duplicated as in case (i) and the block B_3 can be duplicated as in case (ii)

Thus, the duplicated graph is obtained as follows:

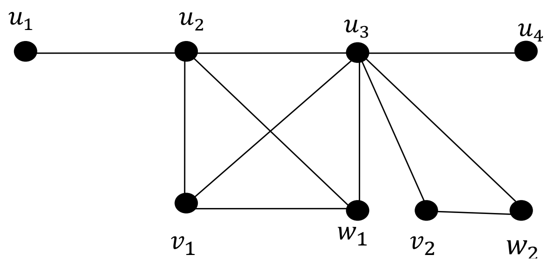

Figure 6: Duplicated graph $D^2_{P_4}$ (B_2B_3)

PROPOSITION 3.3: 2 - block duplication of P_5 by distinct edges results in Duplicated graphs $D^2_{P_5}$ (B_1B_2), $D^2_{P_5}$ (B_1B_3), $D^2_{P_5}$ (B_1B_4), $D^2_{P_5}$ (B_2B_3), $D^2_{P_5}$ (B_2B_4), $D^2_{P_5}$ (B_3B_4) as in figure 8,9,10,11,12 and 13 respectively.

Proof:

Let u_1, u_2, u_3, u_4 , u_5 be the vertices and B_1 , B_2 , B_3 , B_4 be the block of the path graph P_5.

Let us consider the edges v_1 w_1 and for the v_2 w_2 for the duplication process.

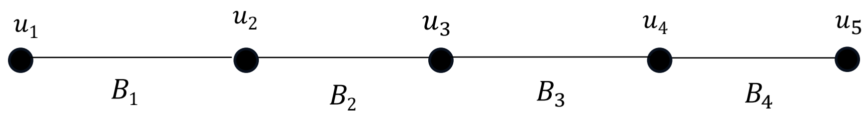

Figure 7: Path Graph P_5

Then the possible cases for duplications are

Case (i): Duplication of B_1 and B_2

This can be done as in case (i) of Proposition 3.2

Thus, the duplicated graph is obtained as follows:

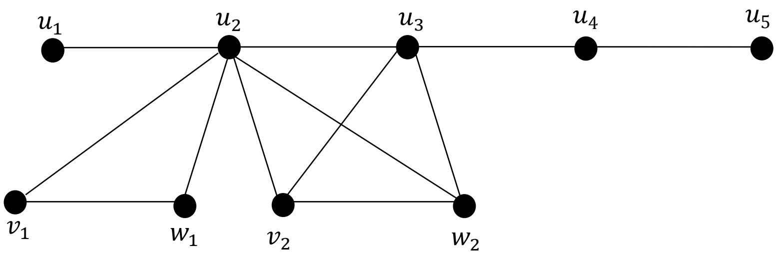

Figure 8: Duplicated graph $D^2_{P_5}$ (B_1B_2)

Case (ii): Duplication of B_1 and B_3

N (B_1)={B_2} N (B_3)={B_1 , B_2}

Duplication of B_1 is done by joining v_1 and w_1 with u_2 , which is common to B_1 and its neighbor B_2 and duplication of B_3 can be done by joining v_2 and u_2 with u_3, which is common to B_3 and its neighbor B_2 and with u_4 , which is common to B_3 and its neighbor B_4.

Thus, the duplicated graph is obtained as follows:

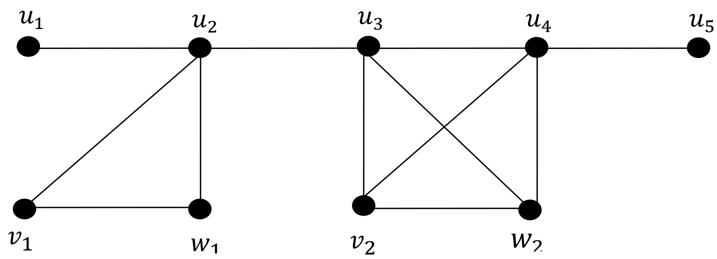

Figure 9: Duplicated graph $D^2_{P_5}$ (B_1B_3)

Case (iii) : Duplication of B_1 and B_4

N (B_1)={B_2} N (B_4)={B_3}

Duplication of B_1 can be done by joining v_1 and w_1 with u_2 and duplication of B_4 is done by joining v_2 and w_2 with u_4. Thus, the duplicated graph is obtained as follows,

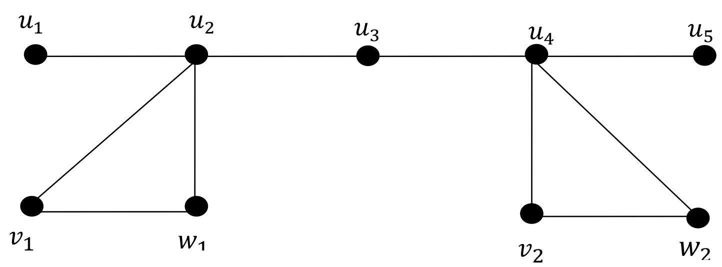

Figure 10: Duplicated graph $D^2_{P_5}$ (B_1B_4)

Case (iv): Duplication of B_2 and B_3

N (B_2)={B_1, B_3} N (B_3)={B_2, B_4}

Duplication of B_2 can be done by joining v_1 and w_1 with u_2 and u_3 which are common to B_2 and its neighbors B_1 and B_3 respectively.

Duplication of B_3 can be done by joining u_2 and w_2 with u_4 which are common to B_2 and its neighbors B_2 and B_4 respectively. Thus, the duplicated graph is obtained as follows

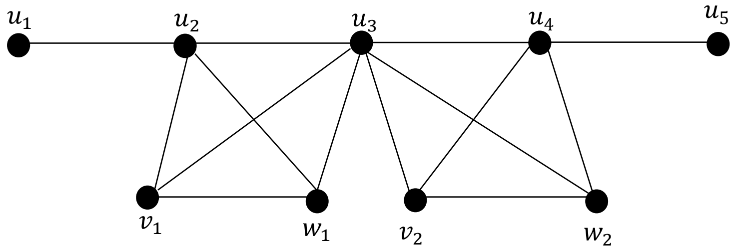

Figure 11: Duplicated graph $D_{P_5}^2$ (B_2B_3)

Case (v): Duplication of B_2 and B_4

N (B_2)={B_1, B_3} N (B_4)={B_3}

Duplication of B_2 can be done as in case (iv) and duplication of B_4 can be done by joining v_2 and w_2 with u_4 which is common to B_4 and its neighbor B_3. Thus, the duplicated graph is obtained as follows,

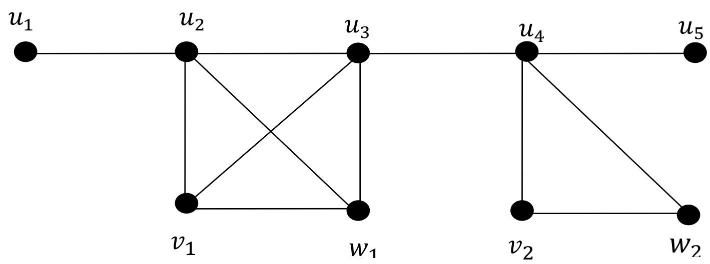

Figure 12: Duplicated graph $D_{P_5}^2$ (B_2B_4)

Case (vi): Duplication of B_3 and B_4

N (B_3)={B_2, B_4} N (B_4)={B_3}

Duplication of B_3 can be done by joining a both v_1 and w_1 with u_3 and u_4 which are common to B_3 and its neighbors B_2 and B_4 respectively. Duplication of B_4 is done by joining v_2 and w_2 with u_4 which is common to B_4 and its neighbor B_3.

Thus, the duplicated graph is obtained as follows:

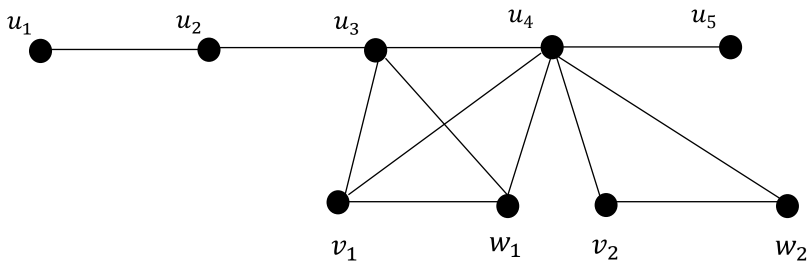

Figure 13: Duplicated graph $D^2_{P_5}$ (B_3B_4)

PROPOSITION 3.4: 2 - block duplication of P_6 by distinct edges results in Duplicated graphs $D^2_{P_6}$ (B_1B_2), $D^2_{P_6}$ (B_1B_3), $D^2_{P_6}$ (B_1B_4), $D^2_{P_6}$ (B_1B_5), $D^2_{P_6}$ (B_2B_3), $D^2_{P_6}$ (B_2B_4), $D^2_{P_6}$ (B_2B_5), $D^2_{P_6}$ (B_3B_4), $D^2_{P_6}$ (B_3B_5) and $D^2_{P_6}$ (B_4B_5) as in figure 15,16,17,18,19,20,21,22,23 and 24 respectively.

Proof:

Let u_1, u_2, u_3, u_4 , u_5, u_6 be the vertices and B_1 , B_2 , B_3 , B_4, B_5 be the block of the path graph P_6.

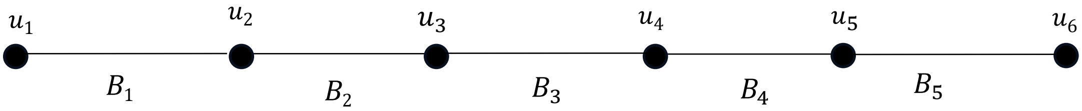

Figure 14: Path Graph P_6

∴ The neighbor sets are N (B_1)={B_2}, N (B_2)={B_1, B_3}, N (B_3)={B_2, B_4},

N (B_4)={B_3, B_5} and N (B_5)={ B_4}

Then the possible cases for duplications are

Case (i): Duplication of B_1 and B_2.

Duplication of B_1 can be done by joining v_1 and w_1 with u_2, which is common to B_1 and its neighbor B_2.

Duplication of B_2 can be done by adding both v_2 and w_2 with u_2 and u_3, which are common to B_2 and its neighbors B_1 and B_3 respectively.

Thus, the duplicated graph is obtained as follows

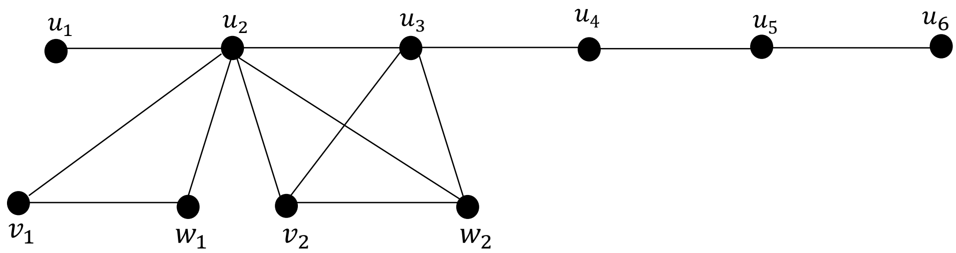

Figure 15: Duplicated graph $D_{P_6}^2$ (B_1B_2)

Case (ii): Duplication of $B_1$and B_3

This can be done as in case (ii) of Proposition 3.3.

Thus, the duplicated graph is obtained as follows:

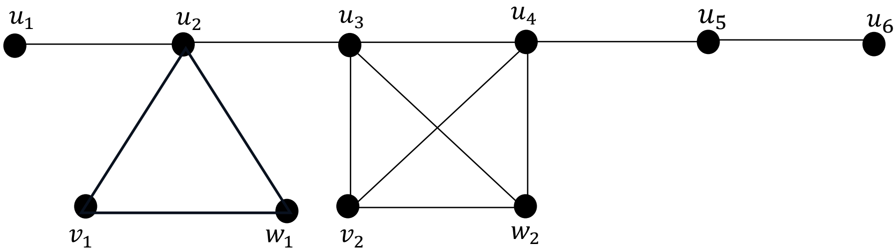

Figure 16: Duplicated graph $D_{P_6}^2$ (B_1B_3)

Case (iii): Duplication of $B_1$and B_4

Duplication of $B_1$is done by joining $v_1$and $w_1$with u_2, which is common to B_1 and its neighbor B_2. Duplication of B_4 is done by joining both v_2 and w_2 with u_4 and u_5 which are common to B_4 and its neighbors B_3 and B_5 respectively.

Thus, the duplicated graph is obtained as follows:

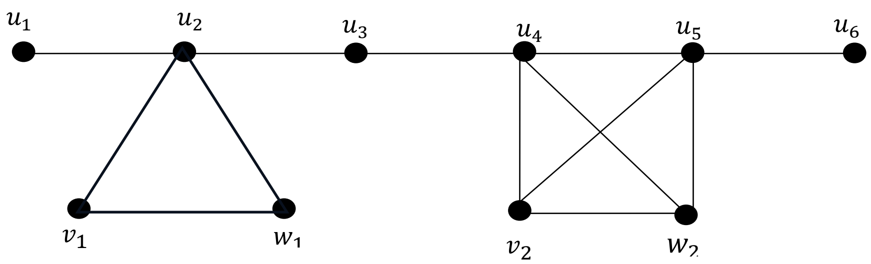

Figure 17: Duplicated graph $D_{P_6}^2$ (B_1B_4)

Case (iv): Duplication of B_1 and B_5

Duplication of B_1 is done by adding v_1 and w_1 with u_2 and duplication of B_5 is done by adding v_2 and w_2 with u_5.

Thus, the duplicated graph is obtained as follows:

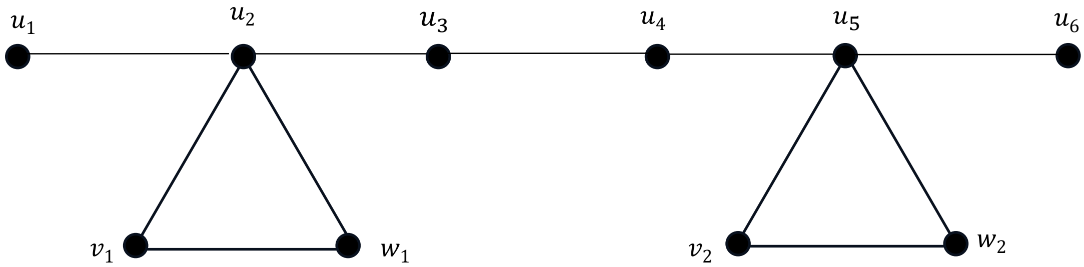

Figure 18: Duplicated graph $D_{P_6}^2$ (B_1B_5)

Case (v): Duplication of B_2 and B_3

This can be done as in case (iv) of Proposition 3.3.

Thus, the duplicated graph is obtained as follows:

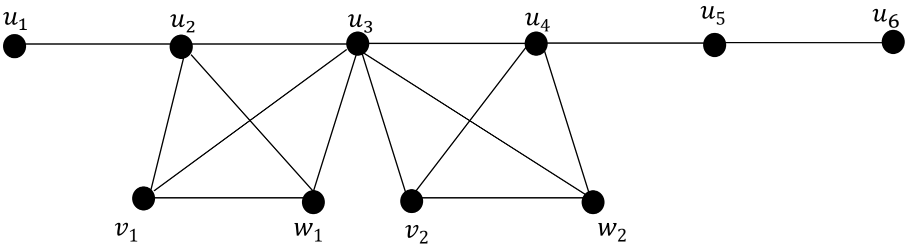

Figure 19: Duplicated graph $D_{P_6}^2$ (B_2B_3)

Case (vi): Duplication of B_2 and B_4

Duplication of the block B_2 can be done by joining both v_1 and w_1 with the vertices u_2 and u_3 which are common to block B_2 and its neighbors B_1 and B_3 respectively.

Duplication of the block B_4 can be done by joining both v_2 and w_2 with the vertices u_4 and u_5 which are common to B_4 and its neighbors B_3 and B_5 respectively.

Thus, the duplicated graph is obtained as follows:

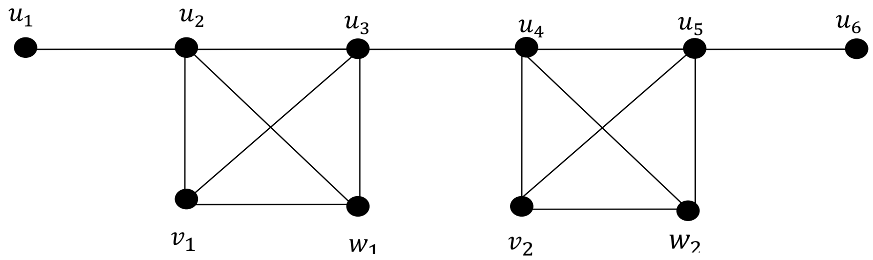

Figure 20: Duplicated graph $D_{P_6}^2$ (B_2B_4)

Case (vii): Duplication of B_2 and B_5

Duplication of B_2 can be done as in case (vi) duplication of B_5 done by joining v_2 and w_2 with u_5, which is common to B_5 and its neighbor B_4.

Thus, the duplicated graph is obtained as follows:

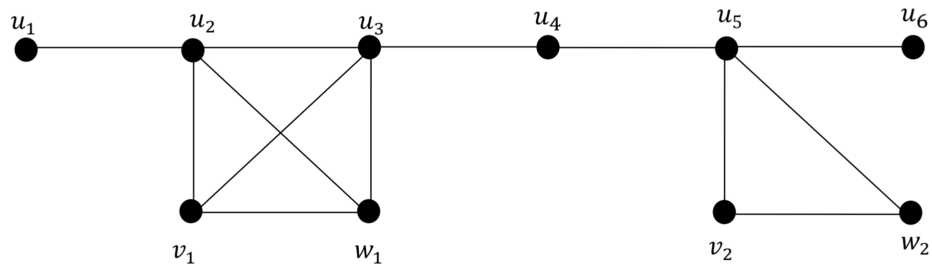

Figure 21: Duplicated graph $D_{P_6}^2$ (B_2B_5)

Case (viii): Duplication of B_3 and B_4

Duplication of B_3 can be done by joining both v_1 and w_1 with the vertices u_3 and u_4 which are common to the block B_3 and its neighbors B_2 and B_4 respectively.

Duplication of B_4 is done as in case (vi) thus the duplicated graph is obtained as follows

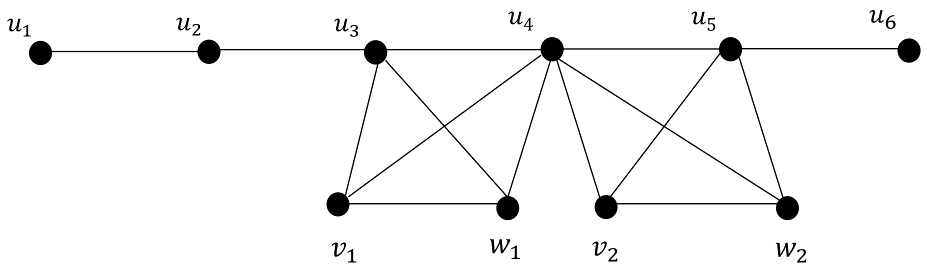

Figure 22: Duplicated graph $D_{P_6}^2$ (B_3B_4)

Case (ix): Duplication of B_3 and B_5

Duplication of B_3 can be done as in case (viii). Duplication of B_5 is done as in case (vii)

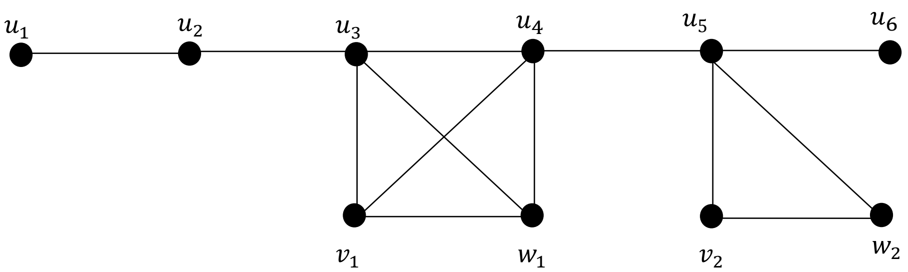

Figure 23: Duplicated graph $D_{P_6}^2$ (B_3B_5)

Case (x): Duplication of B_4 and B_5

Duplication of B_4 can be done as in case (vi) and duplication of B_5 can be done as in case (vi). Thus, the duplicated graph is obtained as follows:

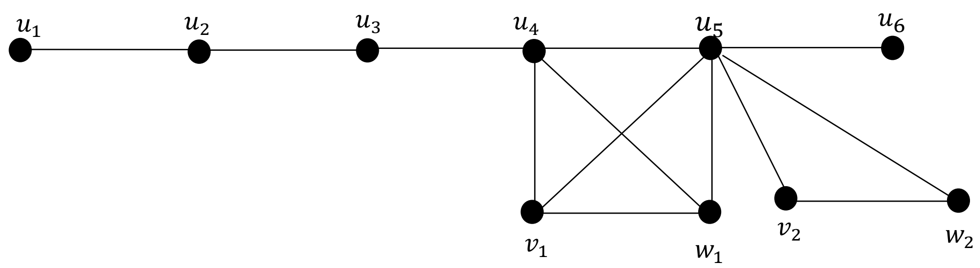

Figure 24: Duplicated graph $D^2_{P_6}$ (B_4B_5)

PROPOSITION 3.5: 2 - block duplication of P_n by distinct edges results in Duplicated graphs $D^2_{P_n}$ (B_1B_i) $(2 \le i \le n-2)$, $D^2_{P_n}$ (B_1B_{n-1}), $D^2_{P_n}$ (B_iB_j), $(2 \le i \le n-2)$ $(2 \le j \le n-2)$ and $D^2_{P_n}$ (B_iB_n) $(2 \le i \le n-2)$ as in figure 26,27,28 and 29 respectively.

Proof:

Let u_1, u_2, u_3, u_4 , u_5, u_6,…,u_n be the vertices and B_1 , B_2 , B_3 , B_4, B_5,…,B_{n-1} be the blocks of the path graph P_n.

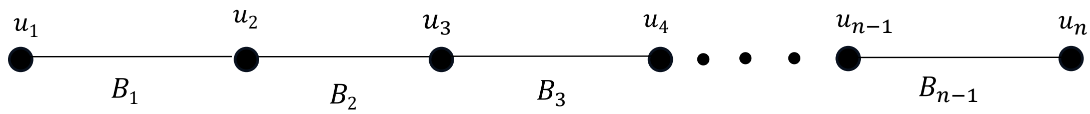

Figure 25: Path Graph P_n

∴ The neighbor sets are N (B_1)={B_2}, N (B_i)={B_{i-1}, B_{i+1}} for $2 \le i \le n-2$

and N (B_{n-1})={ B_{n-2}}

Then the possible cases for duplications are

Case (i): Duplication of B_1 and B_i(for $2 \le i \le n-2$).

Duplication of B_1 can be done by joining v_1 and w_1 with u_2, which is common to B_1 and its neighbor B_2.

Duplication of B_i can be done by adding both v_2 and w_2 with u_i and u_{i+1}, which are common to B_i and its neighbors B_{i-1} and B_{i+1} respectively.

Thus, the duplicated graph is obtained as follows:

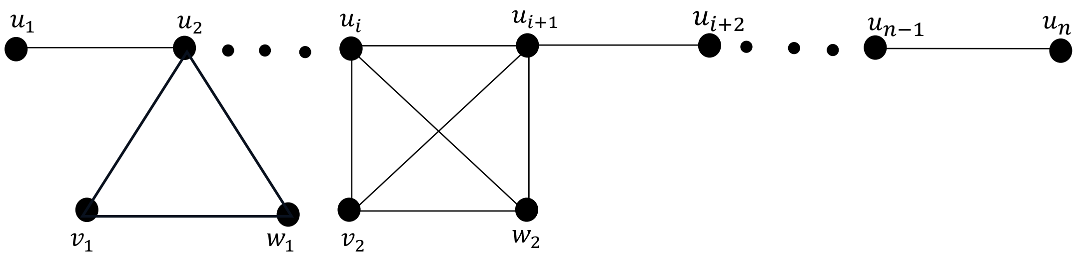

Figure 26: Duplicated graph $D^2_{P_n}$ (B_1B_i)

Case (ii): Duplication of B_1 and B_{n-1}

Duplication of B_1 is done by adding v_1 and w_1 with u_2 and duplication of B_{n-1} is done by adding v_2 and w_2 with u_{n-1}.

Thus, the duplicated graph is obtained as follows

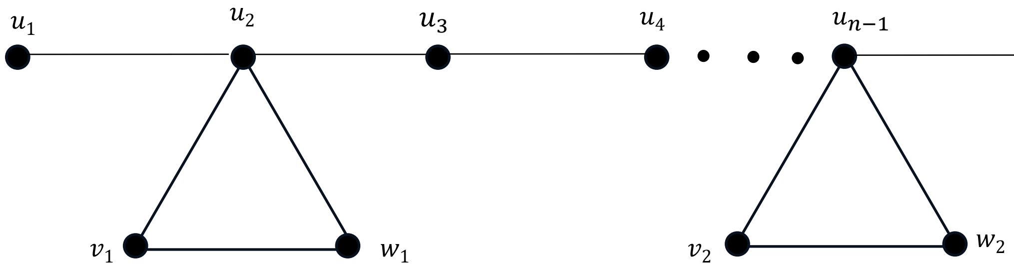

Figure 27: Duplicated graph $D^2_{P_n}$ (B_1B_{n-1})

Case (iii): Duplication of B_i(for $2 \leq i \leq n-2$) and B_j (for $2 \leq j \leq n-2$)

Duplication of B_i can be done as in case (i).

Duplication of B_j can be done by adding both v_2 and w_2 with u_j and u_{j+1}, which are common to B_j and its neighbors B_{j-1} and B_{j+1} respectively.

Thus, the duplicated graph is obtained as follows:

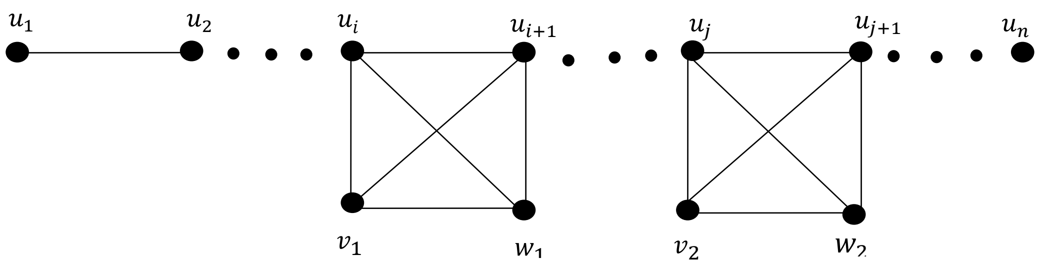

Figure 28: Duplicated graph $D^2_{P_n}$ (B_iB_j)

Case (iv): Duplication of B_i (for $2 \leq i \leq n-2$) and B_{n-1}

Duplication of B_i can be done as in case (i). duplication of B_{n-1} is done by adding v_2 and w_2 with u_{n-1}. Thus, the duplicated graph is obtained as follows:

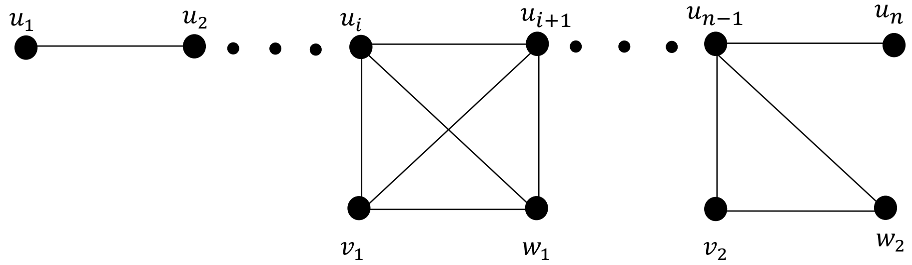

Figure 29: Duplicated graph $D^2_{P_n}$ $(B_i B_{n-1})$

Conclusion:

In this paper, we have investigated 2 – block duplication in path graphs, providing suitable examples, wherever necessary. There is a scope for extending the results to other families of graphs with applications to various fields like transportation management, network problems, decision making etc.,

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