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THE MINIMUM NONSPLIT PENDANT DOMINATION ENERGY OF A GRAPH

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Abstract: For a graph$G$, a subset D of $V(G)$ is called a nonsplit pendant dominating set if the induced graph $< V - D >$ is connected. The nonsplit pendant domination number $\gamma_{nsp}(G)$ is the minimum cardinality of a nonsplit pendant domination set. In this paper we introduce the minimum nonsplit pendant dominating energy $E_{nsp}(G)$ of a graph G and computed minimum nonsplit pendant dominating energies of some standard graphs. Upper and lower bounds for $E_{nsp}(G)$ are established.

Keywords: Nonsplit domination, Nonsplit pendant domination number, Energy of graph.

1. Introduction

Let $G = (V, E)$ be a graph with n vertices and m edges. The degree of v_i written by $d(v_i)$ is the number of edges incident with v_i. The maximum vertex of degree is denoted by $\Delta(G)$ and minimum vertex of degree is denoted by $\delta(G)$. The adjacency matrix $A(G)$ of G is defined by its entries as $a_{ij} = 1$ if $v_i v_j \in E(G)$, or $v_i \in D$ if $(i = j)$ where D is a nonsplit pendant dominating set of G and 0 otherwise. The eigen values of graph G are the eigenvalues of its adjacency matrix$A(G)$, denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. A graph G is said to be singular if at least one of its eigenvalues is equal to zero. For singular graphs, evidently, $detA = 0$. A graph is nonsingular if all its eigenvalues are different from zero. A graph G is said to be k-regular if every vertex in G has degree$k$.

The energy of a graph G is defined as$\mathrm{E(G)} = \sum_{\mathrm{i}=1}^{\mathrm{n}} |\lambda_{\mathrm{i}}|$. This concept was introduced by I. Gutman in 1978 [4] . Initially, the graph energy concept did not attract any noteworthy attention of mathematicians, but later they did realize its value and worldwide mathematical research of graph energy started. Nowadays, in connection with graph energy, energy like quantities were considered also for other matrices.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ for $r \leq n$ be the distinct eigenvalues of G with multiplicity $m_1, m_2, \ldots, m_r$ respectively, the multiset of eigenvalues of $A(G)$ is called the spectrum of G and denoted by

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ m_1 & m_2 & \ldots & m_n \end{pmatrix}$$

A set of vertices S is said to dominate the graph G, if for every vertex$v \notin S$, there is a vertex $u \in s$ with v adjacent to$u$. The minimum cardinality of any dominating set is called the domination number of G and is denoted by $\gamma(G)$. The concept of nonsplit domination was introduced by V. R. Kulli and B. Janakiram[6]. A dominating set D of a graph $G = (V, E)$ is a

nonsplit dominating set if the induced graph $< V - D >$ is connected. The nonsplit domination number γ_{ns} is the minimum cardinality of a nonsplit domination set.
The concept of pendant domination was introduced by Nayaka S. R., Puttaswamy and Purushothama S. [10]. A dominating set S of a graph $G = (V, E)$ is a pendant dominating set if the induced sub graph $< S >$ contains at least one pendant vertex. The pendant domination number γ_{pe} is the minimum cardinality of a pendant domination set.

1.1 The Minimum Nonsplit Domination Energy of Graphs

Let G be simple graph of order n with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set E. Let D be a subset of $V(G)$ which is said to be nonsplit dominating set if the induced graph $< V - D >$ is connected. The nonsplit domination number
$\gamma_{ns}(G)$ of G is the minimum cardinality of a nonsplit dominating set. Any nonsplit dominating set with minimum cardinality is called a MNS set. Let D be a MNS set of a graph G. The MNS matrix of G is the $n \times n$ matrix defined by $A_{ns}(G) = a_{ij}$ [9] where

$$a_{i,j} = \begin{cases} 1, & \text{if } (v_i, v_j) \epsilon E(G) \\ 1, & \text{if } i = j \text{ and } v_i \in D \\ 0, & \text{otherwise} \end{cases}$$

The characteristic polynomial of $A_{ns}(G)$, denoted by $f_n(G, \lambda) = \det(\lambda I - A_{ns}(G))$. The minimum nonsplit dominating eigenvalues of the graph G are the eigenvalues of $A_{ns}(G)$. Since $A_{ns}(G)$ is real and symmetric, its eigenvalues are real numbers and are labeled in non-increasing order $\lambda_1 \geq \lambda_2 \geq, \dots\dots, \geq \lambda_n$. The minimum nonsplit dominating energy of G is defined as

$$E_{ns}(G) = \sum_{i=1}^{n} |\lambda_i|$$

1.2 The Minimum Nonsplit Pendant Domination Energy of Graph

Let G be simple graph of order n with vertex set $V = \{v_1, v_2, \dots, v_n\}$ edge set E. Let S be a subset of $V(G)$ which is said to be nonsplit pendant dominating set if the induced graph $< V - S >$ is connected. The nonsplit pendant domination number
$\gamma_{nsp}(G)$ of G is the minimum cardinality of a nonsplit pendant dominating set. Any nonsplit pendant dominating set with minimum cardinality is called a minimum nonsplit pendant dominating set. Let S be a minimum nonsplit pendant dominating set of a graph G. The minimum nonsplit pendant dominating matrix of G is the $n \times n$ matrix defined by $A_{nsp}(G) = a_{ij}$ where

$$a_{i,j} = \begin{cases} 1, & \text{if } (v_i v)_j \epsilon E(G), \\ 1, & \text{if } i = j \text{ and } v_i \in S, \\ 0, & \text{otherwise}. \end{cases}$$

The characteristic polynomial of $A_{nsp}(G)$, denoted by $f_n(G, \lambda) = \det(\lambda I - A_{nsp}(G))$. The minimum nonsplit pendant dominating eigenvalues of the graph G are the eigenvalues of $A_{nsp}(G)$. Since $A_{nsp}(G)$ is real and symmetric, its eigenvalues are real numbers and are labelled in non-increasing order $\lambda_1 \geq \lambda_2 \geq, \ldots \ldots, \geq \lambda_n$. The minimum nonsplit pendant dominating energy of G is defined as

$$E_{nsp}(G) = \sum_{i=1}^{n} |\lambda_i|$$

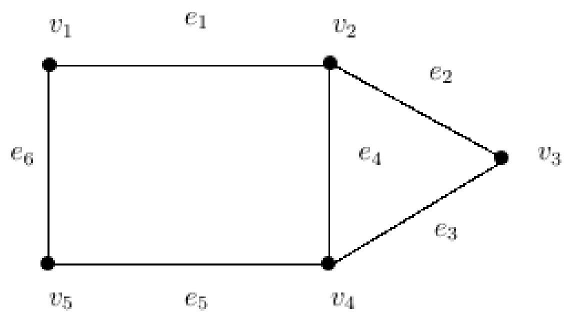

Figure 1. Example for the nonsplit pendant domination energy of a graph

Example 1. Let G be the graph in Figure 1 with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and its minimum nonsplit pendant dominating set be $D_1 = \{v_1, v_5, v_3\}$

$$A_{nsp}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The characteristic equation$f_n(G, \lambda) = \lambda^5 - 3\lambda^4 - 3\lambda^3 + 9\lambda^2 + \lambda - 3$. The spectrum of

$$A_{nsp}(G) = \begin{pmatrix} -1.6180 & -0.6180 & 0.6180 & 1.6180 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Therefore the minimum nonsplit pendant dominating energy of G is $E_{nsp} = 7.472$
Suppose if we take the minimum nonsplit pendant dominating set of G as$D_2 = \{v_1, v_2, v_3\}$. Then

$$A_{nsp}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic equation$f_n(G, \lambda) = \lambda^5 - 3\lambda^4 - 3\lambda^3 + 8\lambda^2 - 3$. The spectrum of

$$A_{nsp}(G) = \begin{pmatrix} -1.6447 & -0.3403 & 0.3950 & 1.4322 & 3.1578 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Therefore the minimum nonsplit pendant dominating energy of G is $E_{nsp} = 6.97$.
This illustrates the fact that the minimum nonsplit pendant dominating energy of the graph depends on the choice of the minimum nonsplit pendant dominating set.

2. Minimum Nonsplit Pendant Dominating Energy of Some Standard Graphs

Theorem 2.1 If K_n is the complete graph with n vertices, then$E_{nsp}(K_n) = (n-3) + \sqrt{n^2 - 2n + 9}$.

Proof: Let K_n be the complete graph with vertex set$V = \{v_1, v_2, \ldots, v_n\}$, then$\gamma_{nsp}(K_n) = 2$. Hence the minimum nonsplit pendant dominating set is $D = \{v_1, v_2\}$ and

$$A_{nsp}(K_n) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 0 & \ldots & 1 & 1 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 1 & 1 & 1 & \ldots & 0 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 0 \end{pmatrix}$$

Its characteristic polynomial is

$$\lambda\,[\lambda + 1]^{n-3}\,[\lambda^2 - (n-1)\lambda\, - 2].$$

The minimum nonsplit pendant dominating spectrum of K_n can be written as

$$Spec(K_n) = \begin{pmatrix} 0 & -1 & \frac{(n-1) + \sqrt{n^2 - 2n + 9}}{2} & \frac{(n-1) - \sqrt{n^2 - 2n + 9}}{2} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence the minimum nonsplit pendant dominating energy for complete graph is $E_{nsp}(K_n) = (n-3) + \sqrt{n^2 - 2n + 9}$.

Theorem 2.2 For$n \geq\ 2$, the minimum nonsplit pendant dominating energy of star graph $K_{1,n-1}$ is $(n-2) +\ 2\sqrt{n-1}$

Proof: Consider the star graph $K_{1,n-1}$with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ having the vertex v_1 at the center of $K_{1,n-1}$. The minimum nonsplit pendant dominating set is$D = \{v_1, v_2, \ldots, v_n\ \}$. Then the minimum nonsplit pendant dominating matrix is

$$A_{nsp}(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 1 & 0 & 0 & \dots & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Its characteristic polynomial is $\lambda^{n-2}\ [\lambda^2 - 2\lambda - (n-2)]$.
The minimum nonsplit pendant dominating spectrum of $K_{1,n-1}$ can be written as

$$Spec(K_{1,n-1}) = \begin{pmatrix} 1 & \frac{2+2\sqrt{n-1}}{2} & \frac{2-2\sqrt{n-1}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Hence the minimum nonsplit pendant dominating energy for star graph is
$E_{nsp}(K_{1,n-1}) = (n-2) + 2\sqrt{n-1}$.

Theorem 2.3. For$n \geq 3$, the minimum nonsplit pendant dominating energy of a complete bipartite graph $K_{n,n}$ is $(n+1) + \sqrt{n^2+2n-3}$

Proof: Consider the complete bipartite graph $K_{n,n}$ with vertex set $V = \{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n\}$. The minimum nonsplit pendant dominating set of $K_{m,n}$ is$D = \{v_1, u_1\}$. Then

$$A_{nsp}(K_{n,n}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 1 & 1 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 1 & 1 & 1 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 0 & 0 \end{pmatrix}$$

Its characteristic polynomial is

$$f_n(G,\lambda) = \lambda^{m+n-4}[\lambda^4 - 2\lambda^3 - (mn-1)\lambda^2 + (2mn-m-n)\lambda - (m-1)(n-1)]$$

In particular for $m = n$, we have

$$f_n(G,\lambda) = (\lambda^{n-2})\ (\lambda^{n-2})\ (\lambda^2 + (n-1)\lambda - (n-1))(\lambda^2 - (n+1)\lambda - (n-1))$$

The minimum nonsplit pendant dominating spectrum of $K_{n,n}$ can be written as

$$spec(K_{n,n}) = \begin{pmatrix} 0 & 0 & \frac{-(n-1)+\sqrt{n^2+2n-3}}{2} & \frac{-(n-1)-\sqrt{n^2+2n-3}}{2} & \frac{(n+1)+\sqrt{n^2-2n+5}}{2} & \frac{(n+1)+\sqrt{n^2-2n+5}}{2} \\ n-2 & n-2 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence the minimum nonsplit pendant dominating energy of a complete bipartite graph is $E_{nsp}(K_{n,n}) = (n+1) + \sqrt{n^2+2n-3}$.

3. Properties of Minimum Nonsplit Pendant Dominating Eigenvalues

Proposition 3.1. Let G be the graph of order$n$. Let $f_n(G,\lambda) = \lambda^n + c_1\,\lambda^{n-1} + \ldots + c_n$ be the characteristic polynomial of minimum nonsplit pendant dominating matrix of a graph G and D be the minimum nonsplit pendant dominating set of G. Then

(i) $c_1 = -|D|$

(ii) $c_2 = \binom{|D|}{2} - |E(G)|$

Theorem 3.1. Let $G = (V, E)$ be any simple (p, q) graph. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the minimum nonsplit pendant dominating eigenvalues of the matrix $A_{nsp}(G)$ and D be the minimum nonsplit pendant dominating set of G. Then the following conditions hold:

(i) $\sum_{i=1}^{n} \lambda_i = |D|$

(ii) $\sum_{i=1}^{n} {\lambda_i}^2 = |D| + 2\,|\,E(G)\,|$

Proof: (i) Since the sum of the eigenvalues of $A_{nsp}(G)$ is same as the trace of$A_{nsp}(G)$, it follows that

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |D|$$

(ii) Since the sum of squares of the eigen values of $A_{nsp}(G)$ is the trace of $\left(A_{nsp}(G)\right)^2$. Then

$$\sum_{i=1}^{n} {\lambda_i}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji}$$

$$= \sum_{i=1}^{n} (a_{ii})^2 + \sum_{i\neq j}^{n} (a_{ij}a_{ji})$$

$$= \sum_{i=1}^{n} {a_{ii}}^2 + 2\sum_{i<j}^{n} a_{ij}^2$$

$$= |\,D\,| + 2\,|\,E(G)\,|$$

$$\therefore \sum_{i=1}^{n} {\lambda_i}^2 = |D| + 2\,|\,E(G)\,|$$

Theorem 3.2. Let G be a graph of order n and let $\lambda_1(G)$ be the largest eigen value of$A_{nsp}(G)$. Then $\lambda_1(G) \geq \frac{2|E(G)|+D}{n}$ where D is the nonsplit pendant domination number.

Proof: Let G be a graph of order n and let $\lambda_1$be the largest nonsplit pendant dominating eigenvalue of$A_{nsp}(G)$. Then$\lambda_1 = \max_{X\neq 0}\frac{X^tA_{nsp}(G)X}{X^tX}$, where X is any nonzero vector and X^t is its transpose and $A_{nsp}(G)$ is a nonsplit pendant dominating matrix. If we take$X = J = \begin{pmatrix}1\\1\\1\\\vdots\\1\end{pmatrix}$.

Then $\lambda_1(G) \geq \frac{J^tA_{nsp}(G)J}{J^tJ} = \frac{2|E(G)|+D}{n}$, where D is the nonsplit pendant domination number.

Theorem 3.3. Let G_1 and G_2 be two graphs with n vertices. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigen values of $A_{nsp}(G_1)$ and $\lambda_1', \lambda_2', \dots, \lambda_n'$ are the eigen values of $A_{nsp}(G_2)$. Then

$$\sum_{i=1}^{n}\lambda_i\lambda_i' \leq \sqrt{(2\mid E(G_1)\mid + \mid D_1\mid)(2\mid E(G_2)\mid + \mid D_2\mid)}$$

Where $A_{nsp}(G_i)$ is the minimum nonsplit pendant dominating matrix of G_i ; $i = 1, 2$ and D_1, D_2 are the minimium nonsplit pendant dominating sets of G_1 and G_2 respectively.

Proof: Let $\lambda_1, \lambda_2, \dots, \lambda_n$are the eigen values of $A_{nsp}(G_1)$ and$\lambda_1', \lambda_2', \dots, \lambda_n'$ are the eigen values of $A_{nsp}(G_2)$. Then by Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n}{a_i}^2\right)\left(\sum_{i=1}^{n}{b_i}^2\right)$$

Put $\mathrm{a_i} = \lambda_\mathrm{i}$, $\mathrm{b_i} = \lambda_\mathrm{i}'$ then,

$$\left(\sum_{i=1}^{n}\lambda_i\lambda_i'\right)^2 \leq \left(\sum_{i=1}^{n}\lambda_\mathrm{i}^2\right)\left(\sum_{i=1}^{n}(\lambda_\mathrm{i}')^2\right)$$

$$\left(\sum_{i=1}^{n}\lambda_i\lambda_i'\right)^2 \leq (2\mid E(G_1)\mid + \mid D_1\mid)(2\mid E(G_2)\mid + \mid D_2\mid)$$

$$\left(\sum_{i=1}^{n}\lambda_i\lambda_i'\right) = \sqrt{(2\mid E(G_1)\mid + \mid D_1\mid)(2\mid E(G_2)\mid + \mid D_2\mid)}$$

4. Bounds for Minimum Nonsplit Pendant Dominating Energy of a Graph

Theorem: 4.1 Let G be a connected graph of order n. Then

$$\sqrt{2\mid E(G)\mid + \gamma_{nsp}(G)} \leq E_{nsp}(G) \leq \sqrt{n\big(2\mid E(G)\mid + \gamma_{nsp}(G)\big)}$$

Proof: By using Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n}{a_i}^2\right)\left(\sum_{i=1}^{n}{b_i}^2\right)$$

Put $a_i = 1, \; b_i = |\lambda_i|$ and by Theorem 3.1.

$$\left(E_{nsp}(G)\right)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

$$\leq n(2 \mid E(G) \mid + \mid D \mid)$$

$$\leq n\left(2 \mid E(G) \mid + \gamma_{nsp}(G)\right)$$

Therefore, the upper bound is hold. For the lower bound, since

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \geq \sum_{i=1}^{n} \lambda_i^2$$

it follows by Theorem 3.1 that

$$\left(E_{nsp}(G)\right)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = 2 \mid E(G) \mid + \mid D \mid = 2 \mid E(G) \mid + \gamma_{nsp}(G)$$

Therefore, the lower bound is hold.

Theorem 4.2 Let G be a graph with n vertices and let D be a minimum nonsplit pendant dominating set. Then

$$\sqrt{(2 \mid E(G) \mid + \mid D \mid) + (n-1)n \mid \det\left(A_{nsp}(G)\right) \mid^{\frac{2}{n}}} \leq E_{nsp}(G) \leq \sqrt{n(2 \mid E(G) \mid + \mid D \mid)}$$

Proof: For the upper bound, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of the minimum nonsplit pendant dominating matrix$A_{nsp}(G)$. By Theorem 3.1.

$$E_{nsp}(G) \leq \sqrt{n(2 \mid E(G) \mid + \mid D \mid)}$$

which is the upper bound. For the lower bound, by using arithmetic mean and geometric mean inequality, we have

$$\frac{\sum_{i \neq j} |\lambda_i| \; |\lambda_j|}{n(n-1)} \geq \left[\prod_{i \neq j} |\lambda_i| \, |\lambda_j|\right]^{\frac{1}{n(n-1)}}$$

$$\left[\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right]^{\frac{1}{n(n-1)}} = \left[\prod_{i=1}^{n} |\lambda_i|\right]^{\frac{2}{n}} = \mid \det\left(A_{nsp}(G)\right) \mid^{\frac{2}{n}}$$

$$i.e., \sum_{i \neq j} |\lambda_i| \; |\lambda_j| \geq n(n-1) \mid \det\left(A_{nsp}(G)\right) \mid^{\frac{2}{n}}$$

Consider

$$\left(E_{nsp}(G)\right)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i \neq j} |\lambda_i| \; |\lambda_j|.$$

$$\left(E_{nsp}(G)\right)^2 \geq \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 + n(n-1) \mid \det\left(A_{nsp}(G)\right) \mid^{\frac{2}{n}}$$

$$\geq 2 \mid E(G) \mid + \mid D \mid + n(n-1) \mid \det\left(A_{nsp}(G)\right) \mid^{\frac{2}{n}}$$

$$\left(E_{nsp}(G)\right) \geq \sqrt{(2 \mid E(G) \mid + \mid D \mid) + (n-1)n \mid \det\left(A_{nsp}(G)\right) \mid^{\frac{2}{n}}}$$

Which is the lower bound.

Theorem 4.3. Let G be a connected graph of order n and $2 \mid E(G) \mid + \gamma_{nsp}(G) \geq n$. Then

$$E_{nsp}(G) \leq \frac{2|E(G)|+\gamma_{nsp}(G)}{n} + \sqrt{(n-1)\left[2 \mid E(G) \mid + \gamma_{nsp}(G) - \left(\frac{2|E(G)|+\gamma_{nsp}(G)}{n}\right)^2\right]}$$

Proof: Consider the Cauchy-Schwartz inequality

$$\left(\sum_{i=2}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=2}^{n} {a_i}^2\right)\left(\sum_{i=2}^{n} {b_i}^2\right)$$

Put $\mathrm{a_i} = 1$, $\mathrm{b_i} = \mid \lambda_\mathrm{i} \mid$ and by Theorem 3.1

$$\left(\sum_{i=2}^{n} \mid \lambda_\mathrm{i} \mid\right)^2 \leq \left(\sum_{i=2}^{n} 1\right)\left(\sum_{i=2}^{n} \mid \lambda_\mathrm{i} \mid^2\right)$$

$$\left(E_{nsp}(G) - \lambda_1\right)^2 \leq (n-1)\left(2 \mid E(G) \mid + \gamma_{nsp}(G) - \lambda_1^2\right)$$

$$E_{nsp}(G) \leq \lambda_1 + \sqrt{(n-1)\left(2 \mid E(G) \mid + \gamma_{nsp}(G) - \lambda_1^2\right)}$$

From Theorem 3.1. We have $\lambda_1 \geq \frac{2|E(G)|+\gamma_{nsp}(G)}{n}$. Since $f(x) = x + \sqrt{(n-1)\left(2 \mid E(G) \mid + \gamma_{nsp}(G) - x^2\right)}$ is a decreasing function. It follows $x \geq \sqrt{\frac{2|E(G)|+\gamma_{nsp}(G)}{n}}$

Since$2 \mid E(G) \mid + \gamma_{nsp}(G) \geq n$, we have $\sqrt{\frac{2|E(G)|+\gamma_{nsp}(G)}{n}} \leq \frac{2|E(G)|+\gamma_{nsp}(G)}{n} \leq \lambda_1$

$$f(\lambda_1) \leq f\left(\frac{2 \mid E(G) \mid + \gamma_{nsp}(G)}{n}\right)$$

$$E_{nsp}(G) \leq f(\lambda_1) \leq f\left(\frac{2 \mid E(G) \mid + \gamma_{nsp}(G)}{n}\right)$$

$$E_{nsp}(G) \leq f\left(\frac{2 \mid E(G) \mid + \gamma_{nsp}(G)}{n}\right)$$

$$E_{nsp}(G) \leq \frac{2 \mid E(G) \mid + \gamma_{nsp}(G)}{n} \leq \sqrt{(n-1)\left[2 \mid E(G) \mid + \gamma_{nsp}(G) - \left(\frac{2 \mid E(G) \mid + \gamma_{nsp}(G)}{n}\right)^2\right]}$$

Conclusion:

Formula and bounds obtained in this paper are useful for theoretical chemists, for whom this value can take on physical significance. For mathematicians, the concept leads to many interesting problems which are not necessarily identical to determining the spectrum of a graph but can provide certain helpful information about the graph. The basic properties including various upper and lower bounds for minimum nonsplit pendant dominating energy of a graph have been established. The nonsplit pendant dominating energy of the graph can give the idea for the chemists to remove some carbon atoms that have the hydrogen bond with all the other carbon atoms and still they want the bonding has to be with all the other remaining carbon atoms.

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