UtilitasMathematica
ISSN 0315-3681 Volume 122, 2025

Totally e-continuous functions in $\hat{S}$ostak's fuzzy topological spaces

S. Rajakumar* [1] & M. Palanisamy† [2]

Abstract

The goal of this manuscript, we applied the concepts of e-open & e-closed sets within the context of $\hat{S}$ostak's fuzzy topological spaces. We also developed the notions of r-fuzzy totally e-open sets and r-fuzzy totally e-continuous functions in these spaces, which were used to interpret and explore a new idea of functions is r-f T e-continuous functions. Furthermore, we explored the connections this new cohort & other established classification of functions in the framework of $\widehat{S}$ ostak's fuzzy topology.

Keywords & phrases: Fuzzy topological spaces (FTS), fuzzy e-open set, fuzzy e-closed set, fuzzy totally e-continuity.

AMS (2000) subject classification: 54A40, 54C05, 54C08, 54D10, 54D65.

1 Introduction.

The conceptualization of fuzzy sets in his classical paper introduced by Zadeh[19] in the 1965, which laid the foundation for the field. Consequently, a substantial amount of research has been devoted to this area and its related fields, generating numerous applications and cross-disciplinary insights. Since then, numerous researchers have expanded upon this idea, applying it to various domains. One such development is initially developed by Chang [5], & later extended by other scholars. The fuzzy topological structure was introduced by another group of researchers [10], extending some result of ordinary fuzzy topology. [19] Chattopadhyay et al. [3] reanalyze this concept under the term *gradation openness.* Additionally, *Ramadan* [11] developed a similar type definition, namely it a smooth topological space for lattices $L = [0, 1]$.

* srkumar277@gmail.com
[1] Professor, Sri Manakula Vinayagar Engineering College, (Autonomous), Puducherry, India.
† palaniva26@gmail.com
[2] Assistant Professor, S. T. Hindu College, Nagercoil, Tamil Nadu-629 002.

The field evolved in several directions, with further work done by authors such as [4, 3, 6, 1, 14, 15, 2].

A substantial contribution emerged from Vadivel and Palanisamy [18], who introduced the new idea of *fuzzy totally e-continuous functions* (fTe-continuous function). Later, in 2018, Vadivel [17] defined r-fe-open sets and r-fe-closed sets in FTS. The new idea of relation between this new class and existing function classes were thoroughly explored in their work.

In this article, we use the following notations, let $\boldsymbol{\mathcal{X}}$ be a non-empty set, $\mathfrak{T}$ = [0, 1], $\mathfrak{T}_0$ = (0, 1]. A fuzzy set ψ of $\boldsymbol{\mathcal{X}}$ is a function (mapping) ψ: $\boldsymbol{\mathcal{X}} \to \mathfrak{T}$, and $\mathfrak{T}^{\boldsymbol{\mathcal{X}}}$ be the group of all fuzzy sets on $\boldsymbol{\mathcal{X}}$. The fuzzy subset ψ of a fuzzy topological space (FTS, in short) $(\boldsymbol{\mathcal{X}}, \tau)$, $\mathcal{C}_\zeta(\psi, r)$, $\mathfrak{T}_\zeta(\psi, r)$ & $\overline{1} - \psi$ denote the closure of ψ, the interior pf ψ and the complement of a fuzzy ψ, respectively.

Also, throughout of this article we use and follow fuzzy open set is f open set, fuzzy closed set is f closed set, fuzzy point if f point, fuzzy closure point is f closure point smooth fuzzy topology (SFT).

2 Preliminaries

Definition 2.1. [10] A mapping ζ: $\mathfrak{T}^{\mathbb{A}} \to \mathfrak{T}$ SFT on $\mathbb{A}$ if it holds for below statement:

i. $\zeta(\overline{0}) = 1$ & $\zeta(\overline{1}) = 1$,
ii. $\zeta(\bigvee_{n \in N} \omega_i) \geq \bigwedge_{n \in N} \zeta(\omega_i)$, for any $\{\omega\}_{n \in N} \subset I^{\mathbb{A}}$,
iii. $\zeta(\omega_1 \wedge \omega_2) \geq \zeta(\omega_1) \wedge \zeta(\omega_2)$, for all $\omega_1, \omega_2 \in I^{\mathbb{A}}$.

The set of $(\boldsymbol{\mathcal{X}}, \tau)$ is said to be a smooth topological space. A fuzzy set ψ is called an r-fuzzy open (In short r-f op.) if $\zeta(\psi) \geq r$. A fuzzy set ψ is called an r-fuzzy closed (In short r-f cl.) if $\zeta(\overline{1} - \psi) \geq r$.

Theorem 2.1. [4] Let a SFT's $(\boldsymbol{\mathcal{X}}, \zeta)$. That implies for every $\psi \in I^{\mathbb{A}}$ & r $\in \mathfrak{T}_0$, Define $\mathcal{C}_\zeta$: $I^{\mathbb{A}} \times \mathfrak{T}_0 \to \mathfrak{T}^{\boldsymbol{\mathcal{X}}}$ act in accordance with $\mathcal{C}_\zeta(\psi, r) = \wedge\{\omega \in \mathfrak{T}^{\boldsymbol{\mathcal{X}}}: \psi \leq \omega, \zeta(\overline{1} - \psi) \geq r\}$. $\psi, \omega \in \mathfrak{T}^{\mathbb{A}}$ & r, t $\in \mathfrak{T}_0$, the operator $\mathcal{C}_\tau$ holds the below conditions:

$(\mathcal{C}_i)$ $\mathcal{C}_\zeta(\overline{0}, r) = \overline{0}$,

$(\mathcal{C}_{ii})$ $\psi \leq \mathcal{C}_\zeta(\psi, r)$,

$(\mathcal{C}_{iii})$ $\mathcal{C}_\zeta(\psi, r) \vee \mathcal{C}_\zeta(\omega, r) = \mathcal{C}_\zeta(\psi \vee \omega, r)$,

$(\mathcal{C}_{iv})$ $\mathcal{C}_\zeta(\psi, r) \leq \mathcal{C}_\zeta(\psi, p)$ if r ≤ p,

$(\mathcal{C}_{v})$ $\mathcal{C}_\zeta(\mathcal{C}_\zeta(\psi, r)$, r) = $\mathcal{C}_\zeta(\psi, r)$.

Theorem 2.2. [4] A SFT is $(\mathbb{A}, \tau)$. That implies for each $\psi \in \mathfrak{T}^{\mathcal{X}}$ and r ∈ $\mathfrak{T}_0$, Define $\mathfrak{T}_\zeta: \mathfrak{T}^{\mathcal{X}} \times \mathfrak{T}_0 \to \mathfrak{T}^{\mathbb{A}}$ follows; $\mathfrak{T}_\zeta(\psi, r) = \wedge\{\omega \in \mathfrak{T}^{\mathcal{X}}: \omega \leq \psi, \zeta(\psi) \geq r\}$. $\psi, \omega \in I^{\mathbb{A}}$ & r, t ∈ $\mathfrak{T}_0$, the operator $\mathfrak{T}_\tau$ holds the following conditions:

$(\mathfrak{T}_i)$ $\mathfrak{T}_\zeta(\overline{1}, r) = \overline{1}$,

$(\mathfrak{T}_{ii})$ $\psi \geq \mathfrak{T}_\zeta(\psi, r)$,

$(\mathfrak{T}_{iii})$ $\mathfrak{T}_\zeta(\psi, r) \wedge \mathfrak{T}_\zeta(\omega, r) = \mathfrak{T}_\zeta(\psi \wedge \omega, r)$,

$(\mathfrak{T}_{iv})$ $\mathfrak{T}_\zeta(\psi, r) \leq \mathfrak{T}_\zeta(\psi, s)$ if r ≤ s,

$(\mathfrak{T}_{v})$ $\mathfrak{T}_\zeta(\mathfrak{T}_\zeta(\psi, r)$, r) = $\mathfrak{T}_\zeta(\psi, r)$,

$(\mathfrak{T}_{vi})$ $\mathfrak{T}_\zeta(\overline{1} - \psi, r) = \overline{1} - \mathcal{C}_\zeta(\psi, r)$ & $\mathcal{C}_\zeta(\overline{1} - \psi, r) = \overline{1} - \mathfrak{T}_\zeta(\psi, r)$.

Definition 2.2. Let a f point a of $\mathbb{A}$ is called a r-fuzzy δ-cluster [8] f point of ψ if $\mathfrak{T}_\zeta(\mathcal{C}_\zeta(v, r)$, r) $\wedge \psi = \phi$, for every r-f open set v of $\mathbb{A}$ containing $\mathbb{A}$. The collection of all r-fuzzy δ-luster points of ψ is said to be a r-fuzzy δ-closure of ψ is written by $\delta\mathcal{C}_\zeta(\psi)$.

Definition 2.3. A fuzzy set ψ is r-fuzzy δ-closed (In short, r-f δ-losed) ⇔ $\psi = \delta\mathcal{C}_\zeta(\psi, r)$. The complement of a r-fδ-losed set is called r-fuzzy δ-open (In short, r-fδ-open) [8]

Definition 2.4. *[16] A fuzzy set* ψ of a FTS $\mathbb{A}$ is a r-fuzzy e-open set (In short, r-f e-op) if $\psi \leq \mathcal{C}_\zeta(((\delta\mathfrak{T}_\zeta, r)$, r) $\psi)$ $\vee$ $\mathfrak{T}_\zeta(\mathcal{C}_\zeta\psi), where\ \mathcal{C}_\zeta\psi = \wedge\{\omega: \omega \geq \psi, \omega\ is\ f\ closed \in \chi\}$ & $\mathfrak{T}_\zeta(\psi) = \vee\{\omega: \omega \leq \psi, \omega\ is\ f\ open \in \mathbb{A}\}$. If ψ is f e-op., that implies 1- ψ is f e-clo.

The complement of a r-fuzzy e-open sets is a r-fuzzy e-closed. [17]

The ∩ of all r-fuzzy e-open sets is written by r-f e-open ($\boldsymbol{\mathcal{X}}$).

Lemma 2.1. [16] A fuzzy subset ψ of a FTS ($\mathbb{A}, \zeta$). Then:

i. $f\delta p\mathfrak{T}_\zeta(\psi, r) = \psi \wedge \mathfrak{T}_\zeta(\delta\mathcal{C}_\zeta(\psi, r)$, r) and $f\delta p\mathcal{C}_\zeta(\psi, r) = \psi \vee \mathcal{C}_\zeta(\delta\mathfrak{T}_\zeta(\psi, r)$, r)
ii. $f\beta\mathfrak{T}_\zeta(\psi, r) = \psi \wedge \mathcal{C}_\zeta(\mathfrak{T}_\zeta(\mathcal{C}_\zeta(\psi, r)$, r), r) and $f\beta\mathcal{C}_\zeta(\psi, r) = \psi \vee \mathfrak{T}_\zeta(\mathcal{C}_\zeta(\mathfrak{T}_\zeta(\psi, r)$, r), r).

Remark 2.1. $If\ a\ fuzzy\ set\ \psi\ of\ \boldsymbol{\mathcal{X}}, 1 - e\mathfrak{T}_\zeta(\psi, r)$ = e$\mathcal{C}_\zeta(1 - \psi, r)$.

Theorem 2.3. [16] $In\ a\ FTS\ \boldsymbol{\mathcal{X}}, \psi\ is\ a\ r - fuzzy\ e - closed\ (respectively\ r - f\ e - op)$ iff $\psi = e\mathcal{C}_\zeta(\psi)$ (respectively $\psi = e\mathfrak{T}_\zeta(\psi)$).

Definition 2.5. [16] A FTS $(\boldsymbol{\mathcal{X}}, \zeta)$ is called a r-fuzzy e-T_1 space (In short, r-f e-T_1) if for every set of two different points a and b of $\mathcal{A}\ there\ exists\ r - fuzzy\ e - open\ sets\ \mathfrak{A}_1\ and\ \mathfrak{A}_2 \ni a \in \mathfrak{A}_1\ and\ b \in \mathfrak{A}_2, a \notin \mathfrak{A}_2\ and\ b \notin \mathfrak{A}_1$.

Definition 2.6. [16] A FTS $(\boldsymbol{\mathcal{X}}, \zeta)$ is called a r-fuzzy e-T_2 space (In short, r-fe-T_2, $also\ say\ that\ r - fuzzy\ e - hausdorff$), if for every set of two different points a and b of $\mathcal{A}\ there\ exists\ different\ r - fuzzy\ e - open\ sets\ \mathfrak{A}\ and\ \mathfrak{B} \ni a \in \mathfrak{A}\ and\ b \in \mathfrak{B}$.

Definition 2.7. [16] A FTS $(\mathcal{A}, \zeta)$ is called a r-fuzzy e-regular (In short, r-f e-reg.) if every closed set $\mathbb{k}$ of $\mathcal{A}$ & each $\mathcal{A} \in \mathcal{A}$-$\mathbb{k}$, there exists different r-fuzzy e-open sets $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathcal{A} \in \mathfrak{B}$ and $\mathbb{k} \leq \mathfrak{B}$.

Definition 2.8. [16] A FTS $(\mathcal{A}, \zeta)$ is called a r-fuzzy e-normal (In short, r-f e-nor.) if every pair of different fuzzy closed sets $\mathfrak{R}\ and\ \mathfrak{I}$ of $\mathcal{A}$, there exist two different r-fuzzy e-open sets $\mathfrak{R}\ and\ \mathfrak{I}$ such that $\mathfrak{R} \leq \mu\ and\ \mathfrak{I} \leq \lambda$ & $\mu \wedge \lambda = 0$.

3 r-Fuzzy Totally e-continuous Function

Definition 3.1. Let $(\mathcal{A}, \zeta)$ and $(\mathcal{B}, \psi)$ be two FTS's. A map $h: (\mathcal{A}, \zeta) \to (\mathcal{B}, \psi)$ & $r \in \mathfrak{T}_0$. Then h is called *r-fuzzy totally e-continuous (*briefly r-fuzzy T e-contin.) if the preimage of all r-fuzzy open set in $(\mathcal{B}, \psi)$ is r-fuzzy e-closed open (In short r-fuzzy e-closed open) (i.e., r-fuzzy e-open and r-fuzzy e-closed) set in $(\mathcal{A}, \zeta)$.

Definition 3.2. A map $h: (\mathcal{A}, \zeta) \to (\mathcal{B}, \psi)$ & $r \in \mathfrak{T}_0$ is called a r-fuzzy strongly e-continuous (briefly fuzzy strongly e-continuous) iff $h^{-1}(\omega)$ is fuzzy e-closed open, whenever $\omega \in \mathfrak{T}^y$.

Now every r-fuzzy totally continuous (In short r-f T-contin.) function is r-fuzzy T e-continuous and r-fuzzy T e-continuous function is r-fuzzy e-continuous (In short r-f e-contin.)

The both above statement are not reversible, it is be in view the below examples.

Example 3.1. Let $\boldsymbol{\mathcal{X}} = \{a_0, b_0\}$ and $\omega_1, \omega_2, \omega_3, \omega_4 \in I^{\mathcal{X}}$ defined as follows:

$\omega_1(a_0) = 0.3$, $\omega_1(b_0) = 0.4$,
$\omega_2(a_0) = 0.6$, $\omega_2(b_0) = 0.7$,
$\omega_3(a_0) = 0.2$, $\omega_3(b_0) = 0.4$,
$\omega_4(a_0) = 0.6$, $\omega_4(b_0) = 0.5$,
Define a fuzzy topologies $\zeta, \psi: \mathfrak{T}^{\mathcal{X}} \to \mathfrak{T}$ as follows:

$$\zeta(\phi) = \begin{cases} 1 & if\ \phi \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} & if\ \phi \in \{\omega_1\}, \\ \frac{2}{3} & if\ \phi = \{\omega_2\}, \\ 0 & if\ otherwise. \end{cases}$$

$$\psi(\phi) = \begin{cases} 1 & if\ \phi \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} & if\ \phi \in \{\omega_3\}, \\ \frac{2}{3} & if\ \phi = \{\omega_4\}, \\ 0 & if\ otherwise. \end{cases}$$

⇒ identity function $I_x: (\mathcal{A}, \zeta) \to (\mathcal{B}, \psi)$ is $\frac{1}{2}$ -f T e-continuous but not $\frac{1}{2}$ -f T-continuous.
As, ω_3 of $(\mathfrak{T}, \zeta_2)$, ${I_x}^{-1}(\omega_3) = \omega_3$, is fuzzy open in $(\mathfrak{T}, \zeta_1)$ but not fuzzy closed in $(\mathfrak{T}, \zeta_1)$.

Example 3.2 Let $\boldsymbol{\mathcal{X}} = \{a_0, b_0, c_0\}$ and $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathfrak{T}^{\mathcal{X}}$ defined as follows:
$\omega_1(a_0) = 0.6$, $\omega_1(b_0) = 0.5$, $\omega_1(c_0) = 0.6$,
$\omega_2(a_0) = 0.4$, $\omega_2(b_0) = 0.5$, $\omega_2(c_0) = 0.6$,
$\omega_3(a_0) = 0.4$, $\omega_3(b_0) = 0.5$, $\omega_3(c_0) = 0.6$,
$\omega_4(a_0) = 0.4$, $\omega_4(b_0) = 0.5$, $\omega_4(c_0) = 0.4$,
Define a fuzzy topologies $\zeta, \psi: \mathfrak{T}^{\mathcal{X}} \to \mathfrak{T}$ as follows:

$$\zeta(\phi) = \begin{cases} 1 & if\ \phi \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} & if\ \phi \in \{\omega_1\}, \\ \frac{1}{2} & if\ \phi = \{\omega_2\}, \\ \frac{1}{2} & if\ \phi = \{\omega_1 \vee \omega_2\}, \\ \frac{1}{2} & if\ \phi = \{\omega_1 \wedge \omega_2\} \\ 0 & if\ otherwise. \end{cases}$$

$$\psi(\phi) = \begin{cases} 1 & if\ \phi \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} & if\ \phi \in \{\omega_3\}, \\ \frac{1}{2} & if\ \phi = \{\omega_4\}, \\ 0 & if\ otherwise. \end{cases}$$

⇒ identity function $I_x : (\mathcal{A}, \zeta) \to (\mathcal{A}, \psi)$ is $\frac{1}{2}$ -f e-continuous but not $\frac{1}{2}$ -f T e-continuous.

As, for the f open set ω_3 of $(\mathfrak{I}, \zeta_2)$, $f^{-1}(\omega_3) = \omega_3$, is fuzzy open in $(\mathfrak{I}, \zeta_1)$ but not fuzzy closed in $(\mathfrak{I}, \zeta_1)$.

Example 3.3. Let $\mathcal{A} = \{a_0, b_0, c_0\}\ also\ \mathcal{B} = \{a_0, b_0, c_0\}$ and $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathfrak{I}^{\mathcal{X}}$ defined as follows:
$\omega_1(a_0)$ = 0.4, $\omega_1(b_0)$ = 0.5, $\omega_1(c_0)$ = 0.6,
$\omega_2(a_0)$ = 0.4, $\omega_2(b_0)$ = 0.5, $\omega_2(c_0)$ = 0.4,
$\omega_3(a_0)$ = 0.6, $\omega_3(b_0)$ = 0.5, $\omega_3(c_0)$ = 0.4,
$\omega_4(a_0)$ = 0.4, $\omega_4(b_0)$ = 0.5, $\omega_4(c_0)$ = 0.6,
Describe fuzzy topologies ζ, ψ: $\mathfrak{I}^{\mathcal{X}} \to \mathfrak{I}$ as listed below:

$$\zeta(\phi) = \begin{cases} 1 & if\ \phi \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} & if\ \phi \in \{\omega_1\}, \\ \frac{1}{2} & if\ \phi = \{\omega_2\}, \\ 0 & if\ otherwise. \end{cases}$$

$$\psi(\phi) = \begin{cases} 1 & if\ \phi \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} & if\ \phi \in \{\omega_3\}, \\ \frac{1}{2} & if\ \phi = \{\omega_4\}, \\ \frac{1}{2} & if\ \phi = \{\omega_3 \vee \omega_4\}, \\ \frac{1}{2} & if\ \phi = \{\omega_3 \wedge \omega_4\} \\ 0 & if\ otherwise. \end{cases}$$

⇒ The identity mapping $f_{id} : (\mathcal{A}, \zeta) \to (\mathcal{B}, \psi)$ then f is $\frac{1}{2}$ -f T e-continuous but not $\frac{1}{2}$ -f Sto. e-continuous.

Theorem 3.1 Every r-fuzzy totally e-continuous function onto a discrete FTS is r-fuzzy Sto. e-continuous.

Proof. Straight forward.

Definition 3.3. Let $(\boldsymbol{\mathcal{X}}, \zeta)$ be a FTS. A r-fuzzy e-Separation of $\mathcal{X}$ is a pair ψ, ω of disjoint non-empty r-f y o subsets of $\mathcal{X}$ whose union is $\mathcal{X}$. The space $\boldsymbol{\mathcal{X}}$ is said to be r-fuzzy e-connected (briefly r-f e-continuous) if there does not exist r-f yeo sets $\psi, \omega \ \ni \psi + \omega, \psi \neq 0 \ and \ \omega \neq 0$.

Theorem 3.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two sets and $h: (\mathcal{X}, \zeta_{\mathcal{X}}) \rightarrow (\mathcal{Y}, \zeta_{\mathcal{Y}})$ is r-f T e-contin. Function from r-f e-contin. Space $(\mathcal{X}, \zeta_{\mathcal{X}})$ into any functions $(\mathcal{Y}, \zeta_{\mathcal{Y}})$, then $(\mathcal{Y}, \zeta_{\mathcal{Y}})$, is indiscrete FTS.

Proof. Suppose we take $(\mathcal{Y}, \zeta_{\mathcal{Y}})$ is not indiscrete
$\Rightarrow (\mathcal{Y}, \zeta_{\mathcal{Y}})$ has a proper fuzzy open set ψ (say) (i.e., $(\neq 0 \ and \ \neq 1)$).
$\Rightarrow$ from hypothesis h, $h^{-1}(\psi)$ is a proper r-fe-closed and r-f e-open subset of $(\mathcal{X}, \zeta_{\mathcal{X}})$, which is a contradiction to our given assumption $(\mathcal{X}, \zeta_{\mathcal{X}})$ is r-f e-continuous.

Definition 3.4. Let $(\mathcal{X}, \zeta_{\mathcal{X}})$ is FTS and any two different fuzzy points $\mathcal{X}_t \ and \ \mathcal{X}_s$, there exists r-f e-open sets ψ and ω such that $\mathcal{X}_t \in \psi, \mathcal{X}_s \in \omega \ \ e\mathcal{C}_\zeta(\psi, r) \leq 1 - e\mathcal{C}_\zeta(\omega, r)$ is called r-fuzzy e-T_2. (briefly r-f T_2) space.

Lemma 3.1. Let $h: (\mathcal{A}, \zeta) \rightarrow (\mathcal{B}, \psi)$ is one-one r-f T e-continuous function. If $\mathcal{Y}$ is r f T_0, then $\boldsymbol{\mathcal{X}}$ is r-f e-T_2 space.

Proof. We $\mu_{\boldsymbol{\mathcal{X}}}$ & $\gamma_{\boldsymbol{\mathcal{Y}}}$ are different fuzzy points of $\mathcal{A}$.
$\Rightarrow$ h $(\mu_{\boldsymbol{\mathcal{X}}}) \neq$ h $(\gamma_{\boldsymbol{\mathcal{Y}}})$.

Since $\boldsymbol{\mathcal{Y}}$ is r-f T_0 space, there exist a fuzzy open set namely
ψ in $\mathcal{Y} \ni \Rightarrow$ h $(\mu_{\boldsymbol{\mathcal{X}}}) \in \psi and$ h $(\gamma_{\boldsymbol{\mathcal{Y}}}) \notin \psi$.
$\Rightarrow \mu_{\boldsymbol{\mathcal{X}}} \in h^{-1}(\psi)$ and $\gamma_{\boldsymbol{\mathcal{Y}}} \notin h^{-1}(\psi)$.

Since h is r-f T e-continuous, $h^{-1}(\psi)$ is r-f e-closed open set of $\mathcal{X}$. Also $\mu_{\boldsymbol{\mathcal{X}}} \in h^{-1}(\psi)$ and $\gamma_{\boldsymbol{\mathcal{Y}}} \in 1 - h^{-1}(\psi)$. Now put $\omega = 1 - h^{-1}(\psi)$. Then $h^{-1}(\psi) = e\mathcal{C}_\zeta(h^{-1}(\psi, r))$ and $e\mathcal{C}_\zeta(1 - h^{-1}(\psi, r)) = e\mathcal{C}_\zeta(\omega, r) = 1 - f^{-1}(\psi)$ (since $h^{-1}(\psi)$ is r-f e-closed) & $e\mathcal{C}_\zeta(h^{-1}(\psi, r)) = h^{-1}(\psi) = 1 - e\mathcal{C}_\zeta(1 - h^{-1}(\psi, r)) = 1 - e\mathcal{C}_\zeta(\omega, r) \leq 1 - e\mathcal{C}_\zeta(\omega, r)$.

Theorem 3.3. Let $(\mathcal{A}, \zeta)$, $(\mathcal{B}, \psi)$ and $(\mathcal{C}, \Lambda)$ are FTS's. A map $h: (\mathcal{A}, \zeta) \rightarrow (\mathcal{B}, \psi)$ is r-f T e-contin. & k: $(\mathcal{B}, \psi) \rightarrow (\mathcal{C}, \Lambda)$ is r-fuzzy contin., $k \circ$h: $(\mathcal{A}, \zeta) \rightarrow (\mathcal{C}, \Lambda)$ is a r-f Te-contin. Map.

Proof. Easy to prove.

Theorem 3.4. Let $\prod_i: \boldsymbol{\mathcal{X}}_1 \times \boldsymbol{\mathcal{X}}_2 \rightarrow \boldsymbol{\mathcal{X}}_i (i = 1,2)$ be the projection of $\boldsymbol{\mathcal{X}}_1 \times \boldsymbol{\mathcal{X}}_2 \ on \ \boldsymbol{\mathcal{X}}_i$. If h: $\boldsymbol{\mathcal{X}} \rightarrow \boldsymbol{\mathcal{X}}_1 \times \boldsymbol{\mathcal{X}}_2$ is r-f T e-contin., then $\prod_i \circ$ h is also r-f Te-contin.

Proof. In the theorem proof use from Theorem 3.3

Theorem 3.5. Let h: $\mathcal{X}_1 \rightarrow \mathcal{X}_2$ be a function. If the graph k: $\mathcal{X}_1 \rightarrow \mathcal{X}_1 \times \mathcal{X}_2$ of h is r-f T e-contin., implies that h is also r-f T e-contin.

Proof. The theorem proof use by Theorem 3.4.

Theorem 3.6. Let $(\mathcal{A}, \zeta)$, $(\mathcal{B}, \psi)$ and $(\mathcal{C}, \Lambda)$ are FTS's. A map h: $(\mathcal{A}, \zeta) \rightarrow (\mathcal{B}, \psi)$ is r-f e-irresolute & k: $(\mathcal{B}, \psi) \rightarrow (\mathcal{C}, \Lambda)$ is r-fuzzy T e-contin., that implies $k \circ$h: $(\mathcal{A}, \zeta) \rightarrow (\mathcal{C}, \Lambda)$ is a r-f e-contin. Mapping.

Proof. Direct proof.

Definition 3.5. A mapping $h: \mathcal{A} \rightarrow \mathcal{B}$ is called a r-fuzzy totally e-open map(function) (r-fTeO map) there exist r-fuzzy e-Closed-open of $\mathcal{Y}$, if every domain of r-fO subset of $\mathcal{X}$.

Theorem 3.7. If $h: \mathcal{A} \rightarrow \mathcal{B}$ is r-fO & k: $\mathcal{B} \rightarrow \mathcal{C}$ is r-f Te O-function, then $k \circ$ f is r-fT eO-function.

Proof. Straight forward.

Definition 3.6. A mapping $h: \mathcal{A} \rightarrow \mathcal{B}$ is called a r-fuzzy almost e-open mapping (briefly r-f AeO-map), there exists a r-fO subset of $\mathcal{Y}$, if every domain r-f e-closed-open subset of $\mathcal{X}$.

Theorem 3.8. Let $h: \mathcal{A} \rightarrow \mathcal{B}$ is a r-f AeO-function also 1-1 and r-fTe-continuous function and k: $\mathcal{B} \rightarrow \mathcal{C}$ is a function such that $k \circ$ h is r-fTe-continuous, then k is a r-fuzzy continuous.

Proof. We consider ψ be a r-fO subset of $\mathcal{C}$.
$\Rightarrow (k \circ \text{h})^{-1}(\psi)$ is a r-fe-closed-open subset of $\mathcal{A}$. So, h$(k \circ \text{h})^{-1}(\psi) = (k)^{-1}(\psi)$ is a r-fO subset of $\mathcal{B}$. Therefore, k is r-fuzzy continuous.

Theorem 3.9. Let $\kappa: \mathcal{P} \rightarrow \mathcal{Q}\ is\ r - fTe - continuous\ mapping$ & $\mathfrak{X}\ is\ f - open\ ordinary\ subset\ of\ \mathcal{P}$, implies that $f_{\mathfrak{X}}: \mathcal{P} \rightarrow \mathcal{Q}$ is also a r-fTe-continuous.

Proof. Consider ψ is r-fO $\subset \mathcal{Q}$, that implies $(\kappa)^{-1}(\psi)$ is a r-fe-closed-open in $\mathcal{P}$. Here, $(\kappa)^{-1}(\psi)$ is r-fe-closed-open in $\mathcal{P}$ & $\mathcal{P}$ has a r-fO ordinary subset of a FTS $\mathcal{P}$,
$\Rightarrow (\kappa)^{-1}(\psi) \cap \mathcal{P}$ occur r-fe-closed-open set inside $\mathcal{P}$. So, $(\kappa)^{-1}(\psi) \cap \mathcal{P} = \kappa_{\mathcal{A}}{}^{-1}(\psi)$ implies r-fe-closed-open set in $\mathcal{A}$. Finally $h_{\mathcal{A}}$ has a r-fTe-continuous.

Definition 3.7. A mapping $h: \mathcal{P} \rightarrow \mathcal{Q}$ is r-fuzzy slightly e-continuous (briefly r-fSe-continuous) if $h^{-1}(\psi)$ occur r-fe-closed in $\mathcal{P}$ for all r-fuzzy closed-open set ψ of $\mathcal{Y}$.

Theorem 3.10. Let $(\mathcal{P}, \zeta)$, $(\mathcal{Q}, \psi)$ and $(\mathcal{R}, \Lambda)$ are FTS's. A map h: $(\mathcal{P}, \zeta) \rightarrow (\mathcal{Q}, \psi)$ is a r-fe-irresolute & k: $(\mathcal{Q}, \psi) \rightarrow (\mathcal{R}, \Lambda)$ is r-fSe-continuous, then $k \circ$ h: $(\mathcal{P}, \zeta) \rightarrow (\mathcal{R}, \Lambda)$ is r-f Se-continuous.

Proof. Consider ψ is any r-fuzzy closed-open set of $\mathcal{R}$. Take from given statement k, $k^{-1}(\psi)$ is a r-fe-closed in $\mathcal{Q}$. Here $(k \circ \text{h})^{-1}(\psi) = h^{-1}((k)^{-1}(\psi))$ & also, use the given h, $(k \circ \text{h})^{-1}(\psi)$ is r-fe-closed. Therefore, $k \circ$ h is r-fSe-continuous.

Definition 3.8. If $\mathcal{X}$ is a FTS:

i. *For all r-fe-closed-open cover of $\boldsymbol{\mathcal{X}}$ has a finite subcover is called r-fuzzy mildly e-compact (briefly r-f Me-Cpt.);*
ii. *For all r-fe-closed-open countably cover of $\mathcal{X}$ has a finite subcover is called r-fuzzy mildly countably e-compact (briefly r-fM countable e-Cpt.);*
iii. *For all over of $\mathcal{X}$ by r-fe-closed-open sets has a countable subcover is called r-fuzzy mildly e-Lindelöf (briefly r-fMe.Lidf.);*
iv. *If every f-fuzzy open cover of $\mathcal{X}$ has a finite subcover is called r-fuzzy compact (briefly r-f cpt.);*
v. *For all countable r-fuzzy open covering of $\mathcal{X}$ contains a finite subcollection that covers $\mathcal{X}$ is said to be a fuzzy countably compact (briefly r-f counb. Cpt.).*

Theorem 3.11. If $h\colon \mathcal{P} \to \mathcal{Q}$ be a r-fTe-contin. Onto mapping. Implies the ensuing statement satisfies:

i. $if\ \mathcal{P}\ has\ r - fMept., \Rightarrow\ \ \mathcal{Q}\ is\ r - f\ cpt.$
ii. $if\ \mathcal{P}\ has\ r - r - fMe - Lidf., \Rightarrow\ \ \mathcal{Q}\ is\ r - f\ Lid.$
iii. $if\ \mathcal{X}\ has\ r - fM\ ounb.e.Cpt., \Rightarrow\ \ \mathcal{Y}\ is\ r - f\ ounb.Cpt.$.

Proof. (i) Consider $\{\psi_\alpha : \alpha \in I\}$. Take any r-fuzzy open cover of $\boldsymbol{\mathcal{Y}}$. Then h is r-fTe-contin., $\Rightarrow$ $\{h^{-1}(\psi_\alpha) : \alpha \in I\}$ is rfe-closed-open of $\mathcal{X}$. Since § is r-f Me-cpt., there exists a finite subset $\mathfrak{I}_1\ of\ \mathfrak{I} \ni \vee\{h^{-1}(\psi_\alpha) : \alpha \in \mathfrak{I}_1\} = 1$. Thus, we have $\vee\{\psi_\alpha : \alpha \in \mathfrak{I}_1\} = 1$ and $\mathcal{Y}$ is r-fcpt..
The remaining follows above proofs.

Definition 3.9. A FTS $\mathcal{P}$ is said to be a r-fuzzy e-$\widehat{co} - T_1$ (briefly r-fe-$\widehat{co} - T_1$) if all the set of two of different fuzzy points μ_x & μ_y of $\mathcal{X}$ there exists r-fe-closed-open sets ψ & ω containing μ_x & μ_y, respectively such that $\mu_y \notin \psi$ & $\mu_x \notin \omega$.

Theorem 3.12. Let $\kappa: \mathcal{P} \to \mathcal{Q}$ is a r-fTe-contin. 1-1 function & $is\, r - f - T_1 \Rightarrow \mathcal{P}\, is\, r - fe - co - T_1$.

Proof. Assume, $\mathcal{Q}$ is r-f-T_1. All the pair of different f points μ_x & $\mu_y \in \mathcal{X}$, there exists r-fuzzy open sets $\psi, \omega \in \mathcal{Y} \ni \kappa(\mu_x) \in \psi,\ \kappa(\mu_y) \notin \psi,\ \kappa(\mu_x) \notin \omega$ & $h(\mu_y) \in \omega, so\ that\ \mu_x \in \kappa^{-1}(\psi), \mu_y \notin \kappa^{-1}(\omega), \mu_x \notin \kappa^{-1}(\omega)$ & $\mu_y \in \kappa^{-1}(\omega)$. That implies $\mathcal{P}$is r-fe-co-T_1.

Definition 3.10. A FTS $\mathcal{P}$ is r-fuzzy e-co-Hausdorff (briefly r-fe-co-Haus.) if each set of two of different fuzzy points μ_x & $\mu_y \ni \mu_x \neq \mu_y\ in\ \mathcal{X}$, there exists different r-fe-closed-open sets ψ & $\omega \in \mathcal{X} \in \mu_x \in \psi\ and\ \mu_y \in \omega$.

Theorem 3.13. Let $\kappa: \mathcal{P} \to \mathcal{Q}$ is a r-fTe-contin. 1-1 function & $is\ r - f - T_2$, that implies § is r-fe-co-T_2.

Proof. Assume $\mathcal{Q}$ is r-f T_2 space. For anyone set of two different fuzzy points μ_x & $\mu_y \in \mathcal{P}$, $\ni$ different r-fuzzy open sets ψ & $\omega \in \mathcal{Y}, \ni \kappa(\mu_x) \in \psi$ & $\kappa(\mu_y) \in \omega$, so that κ is r-fTe-contin. Function, we have $\kappa^{-1}(\psi)\ and\ \kappa^{-1}(\omega)$ are r-fe-closed-open sets in $\mathcal{P}$ containing μ_x & μ_y, respectively, use the definition $\kappa^{-1}(\psi) \wedge \kappa^{-1}(\omega) = \kappa^{-1}(\psi \wedge \omega) = \kappa^{-1}(0) = 0$, and Therefore $\mathcal{X}$ is r-fe-co-T_2.

Definition 3.11. A FTS $\mathcal{P}$ is said to be a r-r-fuzzy e-co-regular (briefly r-fe-co-reg.) if every r-fuzzy e-closed-open set ψ & every r-fuzzy point $\mu_x \notin \psi, \ni$ different r-fuzzy open sets ω & $\rho, \ni \psi \leq \omega$ & $\mu_x \in \rho$.

Definition 3.12. A FTS $\mathcal{P}$ is said to be r-fuzzy e-co-normal (briefly r-fe-co-nor.) if each set f two different r-fuzzy e-closed-open set ψ_1 & $\psi_2 \in \mathcal{X}, \ni different\ r - fuzzy\ open\ sets\ \omega$ & $\eta, \ni \psi_1 \leq \omega$ & $\psi_2 \leq \eta$.

Theorem 3.14. Let a function $h: \mathcal{P} \to \mathcal{Q}$ is a r-fTe-contin. 1-1 r-f-open function & $\mathcal{P}$ is a r-fe-co-reg. space, implies that $\mathcal{Q}$ is r-fuzzy regular.

Proof. We take ψ is a r-f open set of $\mathcal{Q}$ & a fuzzy point $\mu_y \notin \psi$. Here we take $\mu_y = h(\mu_x)$. As h is r-fTe-contin., $h^{-1}(\psi)$ is a r-fe-closed-open set of $\mathcal{X}$. Again, we take $\omega = h^{-1}(\psi)$. That implies $\mu_x \notin \omega$. As $\mathcal{X}$ is r-fe-co-reg., there exists different r-f-op. sets $\eta\ and\ \rho\ in\ \mathcal{X} \ni \omega \leq \eta$ & $\mu_x \in \rho$. Here find that $\psi = h(\omega) \leq h(\eta)$ & $\mu_y = h(\mu_x) \in h(\rho) \ni h(\eta)$ & $h(\rho)$ are different r-fuzzy open sets of $\mathcal{Q}$. Hence $\mathcal{Q}$ is r-fuzzy regular.

Theorem 3.15. Let a map f: $\mathcal{P} \rightarrow \mathcal{Q}$ is a r-fTe-contin. 1-1 r-f-open function & $\mathcal{P}$ is a r-fe-co-nor. Space, $\Rightarrow \mathcal{Q}$ is a r-fuzzy normal.

Proof. We consider ψ_1 and ψ_2 are different r-fuzzy open sets in $\mathcal{Y}$. As h is r-fTe-contin., $h^{-1}(\psi_1)$ & $h^{-1}(\psi_2)$ are r-fe-closed-open sets in $\mathcal{X}$. Here we take $\beta = h^{-1}(\psi_1)$ & $\omega = h^{-1}(\psi_2)$. Then we have $\beta \wedge \omega = 0$. As $\mathcal{X}$ is r-fe-co-nor., $\ni$ different r-fuzzy open sets ψ & $\rho, \ni \beta \leq \psi$ & $\omega \leq \rho$. Here find that $\psi_1 = h(\beta) \leq h(\psi)$ & $\psi_2 = h(\omega) \leq h(\rho) \ni$ $h(\psi)$& $h(\rho)$ are different r-fuzzy open sets. Hence, $\mathcal{Y}$ is r-fuzzy normal.

Definition 3.13. A graph $\kappa(h)$ of a function h: $\mathcal{X} \rightarrow \mathcal{Y}$ is called a r-fuzzy-co-e-closed (briefly r-f-co-e-closed) if every $(\mu_x, \mu_y) \in (\mathcal{X} \times \mathcal{Y})/\kappa(h), \ni r - fe - closed - open\ set\ \psi \in$ $\mathcal{X}$ containing μ_x & a r-fuzzy open set $\omega \in \mathcal{Y}$ containing $\mu_y \ni h(\psi) \wedge \omega = 0$.

Theorem 3.16. Let the mapping h: $\mathcal{X} \rightarrow \mathcal{Y}$ is r-fTe-contin. & $\mathcal{Y}$ is r-fuzzy Hausdorff, that implies $\kappa(h)$ is r-f-co-e-closed. $\in \mathcal{X} \times \mathcal{Y}$.

Proof. Consider $(\mu_x, \mu_y) \in (\mathcal{P} \times \mathcal{Q})/\kappa(h)$, implies h $(\mu_x) \neq \mu_y$. As $\mathcal{Y}$ is r-fuzzy Hausdorff, $\ni$ r-fuzzy open sets ψ & $\omega \in \mathcal{Y}\ with\ h(\mu_x) \in \psi$ & $\mu_y \in \omega, \ni \psi \wedge \omega = 0$. As h is r-fTe-contin., $\ni$ a r-fe-closed-open set $\eta \in \mathcal{X}$ containing $\mu_x \ni h(\eta) \leq \psi$. Hence, we find $\mu_y \in \omega\ and\ h(\eta) \wedge \omega = 0$. Therefore, $\kappa(h)$ is r-f-co-e-clo..

Theorem 3.17. A mapping h: $\mathcal{P} \rightarrow \mathcal{Q}$ has a r-f-co-e-clo. Graph $\kappa(h)$. If h is 1-1, implies that $\mathcal{X}$ is r-fe-T_1.

Proof. We consider $\mu_x\ and\ \mu_x$ be any pair of different points of $\mathcal{X}$, implies that, $(\mu_x, h(\mu_y)) \in (\mathcal{X} \times \mathcal{Y})/\kappa(h)$, we use the known definition of r-f-co-e-clo. Graph, $\ni$ a r-fe-closed-open set $\psi \in \mathcal{X}$ & $a\ r - fuzzy\ open\ set\ \omega\ in\ \mathcal{Y} \ni \mu_x \in \psi, h(\mu_y) \in \omega$ & $h(\psi) \wedge \omega = 0$; Therefore, $\psi \wedge h^{-1}(\omega) = 0$. Hence, $\mu_y \notin \psi.\ Then\ \mathcal{X}$ is r-fe-T_1.

Theorem 3.18. Let a function h: $\mathcal{P} \rightarrow \mathcal{Q}$ has a r-f-co-e-clo. Graph $\kappa(h)$. If h is 1-1 r-fe-contin., $\Rightarrow \mathcal{P}$ is r-fe-T_2.

Proof. Consider $\mu_x\ and\ \mu_x$ be any set of different points of $\mathcal{P}$, implies that, $(\mu_x, h(\mu_y)) \in (\mathcal{P} \times \mathcal{Q})/\kappa(h)$, we use the known definition of r-f-co-e-clo. Graph, $\ni$ a r-fe-closed-open set $\psi \in \mathcal{P}$ & $a\ r - fuzzy\ open\ set\ \omega\ in\ \mathcal{Q} \ni \mu_x \in \psi,\ f(y_\beta) \in \omega$ & $h(\psi) \wedge \omega = 0; As\ h\ is\ r -$

$fe - contin., then\ h^{-1}(\omega)\ is\ r - fe - open\ set\ in\ \mathcal{P} \ni\ h^{-1}(\omega)(\mu_y) = \omega\left(h(\mu_y)\right)\&\ \psi \wedge h^{-1}(\omega) = 0.\ \ There\ \mathcal{P}$ is r-fe-T_2.

References:

[1] R. k. Al-Hamido, "Neutrosophic Generalized Topological Space," *Plithogenic Logi and Computation,* vol. 2, pp. 1-9, 2024.

[2] V. Amarendra Bapu and J. Aswini, "Fuzzy supra Neutrosophic Topological Spaces," *Advances and Applications in Mathematical Sciences,* vol. 20, no. 8, pp. 1339-1347, June 2021.

[3] K. C. Chattopadhyay, R. Hazra and S. Samanta, "Gradation of Openness: Fuzzy Topology," *Fuzzy Sets and Systems,* vol. 49, no. 2, pp. 237-242, 1992.

[4] K. C. Chattopadhyay and S. Samanta, "Fuzzy Topology: Fuzzy Closure Operator," *Fuzzzy Compatness and Fuzzy Connectedness,* vol. 54, no. 2, pp. 207-212, 1993.

[5] C. L. Chang, "Fuzzy Topological Spaces," *Journal of Mathematical Analysis and Application,* vol. 24, pp. 182-190, 1968.

[6] R. Hazra, S. Samanta and K. C. Chattopadhyay, "Fuzzy Topology redefined," *Fuzy sets and Systems,* vol. 45, pp. 79-82, 1992.

[7] Y. C. Kim and J. W. Park, "r-Fuzzy δ-closure and r-Fuzzy θ-closure sets," *Journal of Korean Fuzzy Logic and Intelligent Systems,* vol. 1, no. 6, pp. 557-563, 2000.

[8] Y. C. Kim , A. A. Ramadan and S. E. Abbas, "r-Fuzzy strongly pre-open sets in fuzzy topological spaces," *Matematicki Vesnik (Mat. Vesnik),* vol. 55, pp. 1-13, 2003.

[9] A. Prabhu, A. Vadivel and A. Vijayalakshmi, "Fuzzy e*-Continuous and fuzzy e*-open sets in S ˆostaks Fuzzy Topological Spaces," *International Journal of pure and Applied Mathematics,* vol. 118, no. 10, pp. 21-29, 2018.

[10] A. Prabhu, "S ˆostak, On a Fuzzy Topological Structure," *Rendiconi del Circolo Matematico di Palemo (2) Supplemento,* vol. 11, pp. 89-103, 1985.

[11] A. A. Ramadan, "Smooth Topological Spaces," *Fuzzy Sets and Systems,* vol. 48, no. 3, pp. 371-275, 1992.

[12] A. A. Ramadan, S. E. Abbas and D. Coker, "Fuzzy γ-Continuous in Sˆostaks fuzzy Topology," *Journal of Fuzzy Mathematics,* vol. 10, no. 1, pp. 69-80, 2002.

[13] A. A. Ramadan, S. E. Abbas and Y. Chan Kim, "Weaker Forms of Continuity in S^ ostaks Fuzzy Topological Spaces," *Indian Journal of Pure and Applied Mathematics,* vol. 34, no. 2, pp. 311-333, 2003.

[14] Runu Dhar, "Separation Axioms in Neutrosophic Fuzzy supra Topological Spaces," *Advanced and Applications in Mathematical Sciences,* vol. 22, no. 12, pp. 2255-2267, 2023.

[15] Samriddhi Ghosh, Sonam, Ramakant Bhardwaj and Satyendra Narayan, "On Neutrosophic Fuzzy Metric Space and its Topological Properties," *Mathematical Intelligence,* vol. 16, no. 5, pp. 613-624, 2024.

[16] D. Sobana, V. Chandrasekar and A. Vadivel, "Fuzzy e-Continuous in S^ ostaks Fuzzy Topological Spaces," *AIP Conference Proceedings,* 2019.

[17] D. Sobana, V. Chandrasekar and A. Vadivel, "Fuzzy e-Open Sets in S^ ostaks Fuzzy Topological Spaces," *International Journal of Pure and Applied Mathematics,* vol. 119, no. 9, pp. 21-29, 2018.

[18] A. Vadivel and M. Palanisamy, "Fuzzy totally e-continuous function," *Annals of Fuzzy Mathematics and Information,* vol. 10, no. 2, pp. 239-247, 2015.

[19] L.A. Zadeh, "Fuzzy sets," *Information and Control,* vol. 8, pp. 338-353, 1965.