

Extended Results on Minimum Pendant Dominating Distance Graph Energy

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Abstract

Recently, we introduced the concept of minimum pendant dominating energy [19]. Building on that work, we now propose the concept of minimum pendant dominating distance energy, denoted by $E_{PD_d}(G)$ for a graph G . We compute the minimum pendant dominating distance energies for several standard graphs, including the complete bipartite graph, star graph, bi-star graph, barbell graph, and cocktail party graph. Additionally, we establish both upper and lower bounds for $E_{PD_d}(G)$.

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1 Introduction

The notion of graph energy was first introduced by I. Gutman in 1978 [9]. Graph energy is a concept derived from spectral graph theory, which is a field that studies the relationships between a graph's structure and the properties of its associated matrices. This concept is inspired by the idea that the energy of a system in physics can be related to the interactions between particles, and similarly, the graph energy reflects the interactions between its vertices and edges.

The distance matrix $D(G)$ of a graph G is defined such that each entry d_{ij} represents the distance between vertices i and j . The distance between two vertices is the length of the shortest path connecting them, measured by the number of edges. In a graph G , the distance $d(u, v)$ between two vertices u and v is defined as the length of the shortest path that connects them. The eigenvalues of the distance matrix are computed just as for any square matrix.

Let G be a graph with n vertices and m edges, and let $A = (a_{ij})$ be the adjacency matrix of the graph. The symbols $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of graph G , respectively. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ are arranged in the non-increasing order. Since the adjacency matrix $A(G)$ is real and symmetric, its eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ are real, and their sum is equal to the domination number. The energy of a graph G is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. i.e.

$$E(G) = \sum_{i=1}^n |\lambda_i|$$

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the distinct eigenvalues of G with multiplicities $\alpha_1, \alpha_2, \dots, \alpha_n$ respectively. The multi-set

$$\text{Spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \end{pmatrix}$$

of eigenvalues of $A(G)$ is called adjacency $\text{Spec}(G)$. For a comprehensive discussion of the mathematical aspects of graph energy theory, refer to the reviews [10], the paper [11], and the references cited within them. The fundamental properties, including various bounds for the energy of a graph, were established in [16]. The distance matrix of G is the square matrix of order n , where the (i, j) – entry represents the distance, or the length of the shortest path, between the vertices v_i and v_j . Let $\phi_1, \phi_2, \dots, \phi_n$ be the eigenvalues of the distance matrix of G . The distance energy DE is defined as

$$DE(G) = \sum_{i=1}^n |\phi_i|$$

In 2012, Chandrashekar Adiga et al. defined the minimum covering energy $E_C(G)$, of a graph which depends on its specific minimum cover C . Motivated on this, Rajesh Kanna et al. introduced the concept of the minimum dominating energy of a graph [19]. In 2009, Chandrashekar Adiga and Smitha introduced the concept of maximum degree energy $E_{DM}(G)$ for a graph G . Recently, we introduced the concept of minimum pendant dominating energy. Building on that work, we now propose the concept of minimum pendant dominating distance energy, denoted by $E_{PD_d}(G)$ for a graph G . We compute the minimum pendant dominating distance energies for several standard graphs, including the complete bipartite graph, star graph, bi-star graph, barbell graph, and cocktail party graph. Additionally, we establish both upper and lower bounds for $E_{PD_d}(G)$.

2. Results related to bounds on energy of a graph

Theorem 2.1. Let a_i and b_i where $1 \leq i \leq n$ are non-negative real numbers. Then

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) - \left(\sum_{i=1}^n a_i b_i\right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2$$

where $M_1 = \max_{1 \leq i \leq n} (a_i)$, $M_2 = \max_{1 \leq i \leq n} (b_i)$, $m_1 = \min_{1 \leq i \leq n} (a_i)$ and $m_2 = \min_{1 \leq i \leq n} (b_i)$

Theorem 2.2. Let a_i and b_i where $1 \leq i \leq n$ be positive real numbers then

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \leq \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left(\sum_{i=1}^n a_i b_i\right)^2$$

where M_i and m_i are similar to previous theorem.

Theorem 2.3. Let a_i and b_i where $1 \leq i \leq n$ be positive real numbers then

$$\left| n \sum_{i=1}^n a_i b_i - \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right) \right| \leq \alpha(n)(A - a)(B - b)$$

where a, b, A and B are real constants and $1 \leq i \leq n$ for all i . Also, $a \leq a_i \leq A$, $b \leq b_i \leq B$ and

$$\alpha(n) = n \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right)$$

Theorem 2.4. Let a_1, a_2, \dots, a_n are non-negative numbers then

$$n \left(\frac{1}{n} \sum_{i=1}^n a_i - \left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}} \right) \leq n \sum_{i=1}^n a_i - \left(\sum_{i=1}^n \sqrt{a_i}\right)^2 \leq n(n-1) \left(\frac{1}{n} \sum_{i=1}^n a_i - \left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}} \right)$$

Theorem 2.5. Let a_i and b_i where $1 \leq i \leq n$ be positive real numbers then

$$\sum_{i=1}^n b_i^2 + rR \sum_{i=1}^n a_i^2 \leq (r + R) \left(\sum_{i=1}^n a_i b_i\right)$$

where r and R are real constants for all i and $1 \leq i \leq n$ holds for $ra_i \leq b_i \leq Ra_i$

3. Minimum Pendant Dominating Distance Energy of a Graph

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and the edge

set $E(G)$. A subset S of V is called a pendant dominating set of G such that the induced subgraph $\langle S \rangle$ contains atleast one pendant vertex. A pendant dominating set with minimum cardinality is called a minimum pendant dominating set. Let S be a minimum pendant dominating set of a graph G . The minimum pendant dominating degree matrix of G is the square matrix of order $n \times n$, defined by $A_{PD_d}(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} d(v_i, v_j), & \text{if } v_i v_j \in E(G) \\ 1, & \text{if } i = j \text{ and } v_i \in S \\ 0, & \text{otherwise} \end{cases}$$

The characteristic polynomial of $A_{PD_d}(G)$ is given by $f_n(G, \phi) = \det(\phi I - A_{PD_d}(G))$. The minimum pendant eigenvalues of a graph G are the eigenvalues of $A_{PD_d}(G)$. Since $A_{PD_d}(G)$ is a real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\phi_1 \geq \phi_2 \geq \dots \geq \phi_n$. The minimum pendant dominating energy of G is defined as

$$E_{PD_d}(G) = \sum_{i=1}^n |\phi_i|$$

Note that the trace of $A_{PD_d}(G)$ is equal to the pendant domination number $\gamma_{pe}(G)$

4. Example

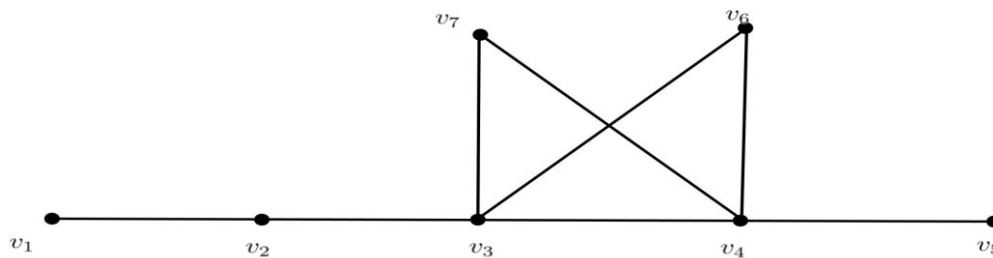


FIGURE-01

Let G be a graph with the vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ as illustrated in FIGURE-01. The minimum pendant dominating sets are as follows:

- (i) $D_1 = \{v_1, v_3, v_4\}$ (ii) $D_2 = \{v_1, v_2, v_4\}$ (iii) $D_3 = \{v_2, v_3, v_4\}$ (iv) $D_4 = \{v_2, v_3, v_5\}$
- (v) $D_5 = \{v_2, v_4, v_5\}$ (vi) $D_6 = \{v_2, v_4, v_6\}$ (vii) $D_7 = \{v_2, v_4, v_7\}$

(i) Let $D_1 = \{v_1, v_3, v_4\}$ be the minimum pendant dominating set. Then, the minimum pendant dominating distance energy matrix of G is as follows:

$$A_{PD_d}[D_1(G)] = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 3 & 3 \\ 1 & 0 & 1 & 2 & 3 & 2 & 2 \\ 2 & 1 & 1 & 1 & 2 & 1 & 1 \\ 3 & 2 & 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 & 2 & 2 \\ 3 & 2 & 1 & 1 & 2 & 0 & 2 \\ 3 & 2 & 1 & 1 & 2 & 2 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{PD_d}[D_1(G)]$ is $f_n(G, \phi) = (\phi + 2)(\phi^6 - 5\phi^5 - 79\phi^4 - 140\phi^3 - 2\phi^2 + 29\phi - 4) = 0$ and the minimum pendant dominating distance energy eigenvalues are $\phi_1 = -2$, $\phi_2 = 0.1623$, $\phi_3 = 0.3046$, $\phi_4 = -0.6291$, $\phi_5 = -2.0326$, $\phi_6 = -5.1339$ and $\phi_7 = 12.3287$. Hence the minimum pendant dominating maximum distance energy of a graph G is approximately 22.5912. i.e., $E_{PD_d}[D_1(G)] = 22.5912$.

(ii) Let $D_5 = \{v_2, v_4, v_5\}$ be the minimum pendant dominating set. Then, the minimum pendant dominating maximum distance energy matrix of G is as follows:

$$A_{PD_d}[D_5(G)] = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 & 1 & 1 \\ 3 & 2 & 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 & 2 & 0 & 2 \\ 3 & 2 & 1 & 1 & 2 & 2 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{PD_d}[D_5(G)]$ is $f_n(G, \phi) = \phi(\phi + 2)(\phi^5 - 5\phi^4 - 79\phi^3 - 141\phi^2 - 22\phi + 24) = 0$ and the minimum pendant dominating distance energy eigenvalues are $\phi_1 = 0, \phi_2 = -2, \phi_3 = 0.3192, \phi_4 = -0.6413, \phi_5 = -1.8332, \phi_6 = -5.1838$ and $\phi_7 = 12.3391$. Hence the minimum pendant dominating maximum distance energy of G is approximately 22.3166. i. e., $E_{PD_d}[D_5(G)] = 22.3166$.

The above example illustrates that the minimum pendant dominating distance energy of a graph G depends on the choice of the minimum pendant dominating distance energy set. In other words, the minimum pendant dominating distance energy is influenced by the pendant dominating set.

5 . Properties of the Minimum Pendant Dominating Eigenvalues of Graph

Theorem 5.1. Let $G = (n, m)$ be a simple graph of order n , degree m and $|D|$ be the minimum pendant dominating set. Let $f_n(G, \phi) = a_0\phi^n + a_1\phi^{n-1} + a_2\phi^{n-2} + \dots + a_n$ be the characteristic equation of minimum pendant dominating eigenvalues. Then

- (i) $a_0 = 1$
- (ii) $a_1 = -|D|$
- (iii) $a_2 = \binom{|D|}{2} + 2m$

Proof: (i) Proof is followed by the definition of $f_n(G, \phi)$.

(ii) We know that the sum of the principal diagonal elements of $A_{PD_d}(G)$ is equal to $|D| = \gamma_{pe}(G)$. That is, the trace of $A_{PD_d}(G)$ is 1×1 principal submatrices, is equal to $|D|$. Thus, we conclude $(-1)^1 a_1 = \gamma_{pe}(G)$ and hence $a_1 = -\gamma_{pe}(G) = -|D|$

(iii) $(-1)^2 a_2$ is equal to the sum of the determinants of all 2×2 principal submatrices of $A_{PD_d}(G)$. We have,

$$\begin{aligned} a_2 &= \sum_{1 \leq i < j \leq n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \sum_{1 \leq i < j \leq n} (a_{ii}a_{jj} + a_{ij}a_{ji}) \\ &= \sum_{1 \leq i < j \leq n} a_{ii}a_{jj} + \sum_{1 \leq i < j \leq n} (a_{ij})^2 \\ &\therefore a_2 = \binom{|D|}{2} + 2m \end{aligned}$$

Theorem 5.2. Let $\phi_1, \phi_2, \dots, \phi_n$ be the eigenvalues of $A_{PD_d}(G)$, then

- (i) $\sum_{i=1}^n \phi_i = \gamma_{pe}(G) = |D|$
- (ii) $\sum_{i=1}^n \phi_i^2 = |D| + 2m + 2M$ where $\sum_{i < j, d(v_i, v_j) \neq 1} (d(v_i, v_j))^2$ and $m = |E|$

Proof: (i) We know that the sum of eigenvalues of $A_{PD_d}(G)$ is equal to the trace of $A_{PD_d}(G)$. Hence, we have

$$\sum_{i=1}^n \phi_i = \sum_{i=1}^n a_{ii} = |D| = \gamma_{pe}(G)$$

(ii) Similarly, the sum of the squares of the eigenvalues of $A_{PD_d}(G)$ is equal to the trace of $(A_{PD_d}(G))^2$. Therefore,

$$\begin{aligned} \sum_{i=1}^n \phi_i^2 &= \sum_{i=1}^n \phi_i \sum_{j=1}^n \phi_j = \sum_{i=1}^n \sum_{j=1}^n \phi_i \phi_j = \sum_{i=1}^n \sum_{j=1}^n a_{ij} a_{ji} \\ &= \sum_{i=1}^n \sum_{j=1}^n (a_{ij} a_{ji} + a_{ji} a_{ij} + a_{ii} a_{jj}) = \sum_{i=1}^n \sum_{j=1}^n (2a_{ij} a_{ji} + a_{ii}^2) \\ &= \sum_{j=1}^n a_{ii}^2 + 2 \sum_{i < j} (a_{ij})^2 = \sum_{j=1}^n a_{ii}^2 + 2 \sum_{i < j} (d(v_i, v_j))^2 \\ &= |D| + 2 \sum_{i=1}^n (d(v_i, v_j))^2 = |D| + 2m + 2M \\ \therefore \sum_{i=1}^n \phi_i^2 &= |D| + 2m + 2M \text{ where } M = \sum_{i \leq j, d(v_i, v_j) \neq 1} (d(v_i, v_j))^2 \end{aligned}$$

Theorem 5.3. Let G be a (n, m) simple graph with diameter 2 and $D = \{u_1, u_2, \dots, u_k\}$. be a minimum pendant dominating set. If $\phi_1, \phi_2, \dots, \phi_n$ are the eigen values of minimum pendant dominating distance matrix $A_{PD_d}(G)$ then

$$\sum_{i=1}^n \phi_i^2 = k + 2(2n^2 - 2n - 3m)$$

Proof: We know that in $A_{PD_d}(G)$ there are $2m$ elements with 1 and $n(n - 1) - 2m$ elements with 2 and hence corollary follows from the above theorem.

Theorem 5.4. Let $G = (n, m)$ be a graph, and let $\phi_1(G)$ denote the largest minimum pendant dominating distance energy eigenvalue of $A_{PD_d}(G)$ then,

$$\phi_1(G) \geq \frac{2W(G) + |D|}{n}$$

where $W(G)$ is the Wiener index of G .

Proof: Let G be a graph and let ϕ_1 denote the minimum pendant dominating distance energy eigenvalue of $A_{PD_d}(G)$. Then, we have $\phi_1(A_{PD_d}(G)) = \max_{X \neq 0} \left(\frac{X^T A_{PD_d} X}{X^T X} \right)$ where X is any non-zero vector, X^T is its transpose and A is a matrix. If we set $X = J = (1 \ 1 \ \dots \ 1)^T$ where J is a unit matrix. Then, we get

$$\phi_1(A_{PD_d}(G)) \geq \frac{J^T A_{PD_d}(G) J}{J^T J} = \frac{2 \sum_{i < j} (d(v_i, v_j)) + \gamma_{pe}(G)}{n} = \frac{2W(G) + \gamma_{pe}(G)}{n}$$

Theorem 5.5. Let $G = (n, m)$ be a graph with diameter 2 and let $\phi_1(G)$ denote the largest minimum pendant dominating distance energy eigenvalue of $A_{PD_d}(G)$, then

$$\phi_1(G) \geq \frac{2n^2 - 2m - 2n + |D|}{n}$$

where $|D|$ is a pendant domination set.

Proof: Let G be a connected graph with diameter 2 and d_i denotes the degree of vertex v_i . Clearly i^{th} of $A_{PD_d}(G)$ consists of d_i one's and $n - d_i - 1$ twos. By using Rayleigh's principle for $J = [1, 1, \dots, 1]$ we have

$$\begin{aligned} \phi_1(A_{PD_d}(G)) &\geq \frac{J^T A_{PD_d}(G) J}{J^T J} = \frac{2 \sum_{i=1}^n [d_i * 1 + (n - d_i - 1) * 2] + |D|}{n} \\ \phi_1(G) &\geq \frac{2n^2 - 2m - 2n + |D|}{n} \end{aligned}$$

Similar to Koolen and Moulton’s [15] upper bound for energy of a graph, upper bound for $E_{PD_d}(G)$ is given in the following bounds.

6. Bounds for the Minimum Pendant Dominating Distance Energy

Theorem 6.1. Let $G = (n, m)$ be a connected graph. If $\Delta = \det(A_{PD_d}(G))$ and $|D|$ is the minimum pendant dominating set, then

$$\sqrt{|D| + 2m + 2M + n(n - 1)\Delta^{\frac{2}{n}}} \leq E_{PD_d}(G) \leq \sqrt{n(|D| + 2m + 2M)}$$

Proof: The Cauchy’s-Schwarz inequality is

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right) \cdot \left(\sum_{i=1}^n b_i^2\right)$$

Put $a_i = 1$ and $b_i = |\phi_i|$ in the above inequality

$$\begin{aligned} \left(\sum_{i=1}^n |\phi_i|\right)^2 &\leq \left(\sum_{i=1}^n 1\right) \left(\sum_{i=1}^n |\phi_i|^2\right) \\ (E_{PD_d}(G))^2 &\leq n \left(\sum_{i=1}^n |\phi_i|^2\right) \end{aligned}$$

From Theorem 5.2, we have

$$\begin{aligned} (E_{PD_d}(G))^2 &\leq n(2m + 2M + |D|) \\ \Rightarrow (E_{PD_d}(G))^2 &\leq \sqrt{n(2m + 2M + |D|)} \end{aligned}$$

Since arithmetic mean is not smaller than geometric mean, we have

$$\begin{aligned} \frac{1}{n(n-1)} \sum_{i \neq j} |\phi_i| |\phi_j| &\geq \left(\prod_{i \neq j} |\phi_i| |\phi_j|\right)^{\frac{1}{n(n-1)}} \\ \frac{1}{n(n-1)} \sum_{i \neq j} |\phi_i| |\phi_j| &= \left(\prod_{i \neq j} |\phi_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= \left(\prod_{i \neq j} |\phi_i|\right)^{\frac{2}{n}} = \left(\prod_{i \neq j} \phi_i\right)^{\frac{2}{n}} \\ &= |\det(A_{PD_d}(G))|^{\frac{2}{n}} = \Delta^{\frac{2}{n}} \\ \therefore \sum_{i \neq j} |\phi_i| |\phi_j| &\geq n(n-1)\Delta^{\frac{2}{n}} \end{aligned}$$

Now, consider

$$(E_{PD_d}(G))^2 = \sum_{i=1}^n |\phi_i|^2 = \sum_{i=1}^n |\phi_i|^2 + \sum_{i \neq j} |\phi_i| |\phi_j|$$

From Theorem 5.2, we have

$$\begin{aligned} (E_{PD_d}(G))^2 &\geq |D| + 2m + 2M + n(n-1)\Delta_n^{\frac{2}{n}} \\ \Rightarrow (E_{PD_d}(G))^2 &\geq \sqrt{|D| + 2m + 2M + n(n-1)\Delta_n^{\frac{2}{n}}} \\ \therefore \sqrt{|D| + 2m + 2M + n(n-1)\Delta_n^{\frac{2}{n}}} &\leq E_{PD_d}(G) \leq \sqrt{n(|D| + 2m + 2M)} \end{aligned}$$

Theorem 6.2. Let $G = (n, m)$ be a connected graph. Then,

$$\sqrt{|D| + 2m + 2M} \leq E_{PD_d}(G) \leq \sqrt{n(|D| + 2m + 2M)}$$

Proof: The Cauchy's-Schwarz inequality is

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right) \cdot \left(\sum_{i=1}^n b_i^2\right)$$

Put $a_i = 1$ and $b_i = |\phi_i|$ in the above expression

$$\begin{aligned} \left(\sum_{i=1}^n |\phi_i|\right)^2 &\leq \left(\sum_{i=1}^n 1\right) \cdot \left(\sum_{i=1}^n |\phi_i|^2\right) \\ (E_{PD_d}(G))^2 &\leq n \cdot \left(\sum_{i=1}^n |\phi_i|^2\right) \end{aligned}$$

From Theorem 5.2, we have

$$\begin{aligned} (E_{PD_d}(G))^2 &\leq n(|D| + 2m + 2M) \\ \Rightarrow E_{PD_d}(G) &\leq \sqrt{n(|D| + 2m + 2M)} \end{aligned}$$

Now, consider

$$(E_{PD_d}(G))^2 = \sum_{i=1}^n |\phi_i|^2$$

From Theorem 5.2, we have

$$\begin{aligned} (E_{PD_d}(G))^2 &\geq |D| + 2m + 2M \\ \Rightarrow E_{PD_d}(G) &\geq \sqrt{|D| + 2m + 2M} \\ \therefore \sqrt{|D| + 2m + 2M} &\leq E_{PD_d}(G) \leq \sqrt{n(|D| + 2m + 2M)} \end{aligned}$$

Theorem 6.3. Let $G = (n, m)$ be a graph with diameter 2 and $\frac{|D|+2n^2-2m-2n}{n} \geq 1$ then

$$E_{PD_d}(G) \leq \left(\frac{|D| + 2n^2 - 2m - 2n}{n}\right) + \sqrt{(n-1) \left[|D| + 4n^2 - 6m - 4n - \left(\frac{|D| + 2n^2 - 2m - 2n}{n}\right)^2 \right]}$$

Proof: The Cauchy's-Schwarz inequality

$$\left(\sum_{i=2}^n a_i b_i\right)^2 \leq \left(\sum_{i=2}^n a_i^2\right) \cdot \left(\sum_{i=2}^n b_i^2\right)$$

Put $a_i = 1$ and $b_i = |\phi_i|$ in the above inequality

$$\begin{aligned} \left(\sum_{i=2}^n |\phi_i|\right)^2 &\leq \left(\sum_{i=2}^n 1\right) \cdot \left(\sum_{i=2}^n |\phi_i|^2\right) \\ (E_{PD_d}(G) - \phi_1)^2 &\leq (n-1)(|D| + 4n^2 - 6m - 4n - \phi_1^2) \end{aligned}$$

$$E_{PD_d}(G) \leq \phi_1 + \sqrt{(n-1)(|D| + 4n^2 - 6m - 4n - \phi_1^2)}$$

Let $f(x) = x + \sqrt{(n-1)(|D| + 4n^2 - 6m - 4n - x^2)}$

For decreasing function $f'(x) \leq 0$ we have,

$$\Rightarrow 1 + \frac{(n-1)(-2x)}{2\sqrt{(n-1)(|D| + 4n^2 - 6m - 4n - x^2)}} \leq 0$$

$$\sqrt{\frac{|D| + 4n^2 - 6m - 4n - x^2}{n-1}} \leq x$$

$$x \geq \sqrt{\frac{|D| + 4n^2 - 6m - 4n}{n}}$$

Therefore $f(x)$ is decreasing in

$$\left[\sqrt{\frac{|D| + 4n^2 - 6m - 4n}{n}}, |D| + 4n^2 - 6m - 4n \right]$$

Clearly $\sqrt{\frac{|D| + 4n^2 - 6m - 4n}{n}} \in \left[\sqrt{\frac{|D| + 4n^2 - 6m - 4n}{n}}, |D| + 4n^2 - 6m - 4n \right]$

Since $\frac{|D| + 2n^2 - 2m - 2n}{n} \geq 1$ we have $\sqrt{\frac{|D| + 2n^2 - 2m - 2n}{n}} \leq \frac{|D| + 2n^2 - 2m - 2n}{n} \leq \phi_1(G)$ (By Theorem 5.5.)

Therefore,

$$\Rightarrow f(\phi_1) \leq f\left(\frac{|D| + 2n^2 - 2m - 2n}{n}\right)$$

$$E_{PD_d}(G) \leq f\left(\frac{|D| + 2n^2 - 2m - 2n}{n}\right)$$

$$E_{PD_d}(G) \leq \left(\frac{|D| + 2n^2 - 2m - 2n}{n}\right) + \sqrt{(n-1) \left[|D| + 4n^2 - 6m - 4n - \left(\frac{|D| + 2n^2 - 2m - 2n}{n}\right)^2 \right]}$$

Bapat and S. Pati [2] proved that if the graph energy is a rational number, then it is an even integer. A similar result for the minimum dominating energy is presented in the following theorem.

Theorem 6.4. Let G be a (n, m) graph where n and m are the number of vertices and edges. Let $|\phi_1| \geq |\phi_2| \geq \dots \geq |\phi_n|$ be the minimum pendant dominating eigenvalues of $A_{PD_d}(G)$ which are arranged in descending order, then

$$E_{PD_d}(G) \geq \sqrt{n(|D| + 2m + 2M) - \alpha(n)(|\phi_n| + |\phi_1|)^2}$$

Here $\alpha(n) = n \lfloor \frac{n}{2} \rfloor \left(1 - \frac{1}{n} \lfloor \frac{n}{2} \rfloor\right)$ and $\lfloor n \rfloor$ indicates the integral part of a real number.

Proof. Let $A \geq a_i \geq a$ and $B \geq b_i \geq b \forall i = 1, 2, 3, \dots, n$ where $a, a_1, a_2, \dots, a_n, A$ and $b, b_1, b_2, \dots, b_n, B$ are the real numbers.

$$\alpha(n)(-a + A)(-b + B) \geq \left| n \sum_{i=1}^n a_i b_i - \left(\sum_{i=1}^n a_i\right) \left(\sum_{i=1}^n b_i\right) \right|$$

where $\alpha(n) = n \lfloor \frac{n}{2} \rfloor \left(1 - \frac{1}{n} \lfloor \frac{n}{2} \rfloor\right)$ and equality holds if and only if $b_1 = b_2 = \dots = b_n$ and $a_1 = a_2 = \dots = a_n$.

If $b_i = |\phi_i|, a_i = |\phi_i|, B = A = |\phi_1|$ and $b = a = |\phi_n|$, then

$$\left| n \sum_{i=1}^n |\phi_i|^2 - \left(\sum_{i=1}^n |\phi_i|\right)^2 \right| \leq \alpha(n)(|\phi_n| + |\phi_1|)^2$$

But

$$\sum_{i \geq 1}^n |\phi_i|^2 = |D| + 2m + 2M$$

and $E_{PD_d}(G) \leq \sqrt{n(|D| + 2m + 2M)}$ then the above inequality will be

$$n(|D| + 2m + 2M) - (E_{PD_d}(G))^2 \leq \alpha(n)(|-\phi_n| + |\phi_1|)^2$$

i.e.,

$$\sqrt{n(|D| + 2m + 2M) - \alpha(n)(|-\phi_n| + |\phi_1|)^2} \leq E_{PD_d}(G)$$

Theorem 6.5. Let G be a (n, m) graph. Let $|\phi_1| \geq |\phi_2| \geq \dots \geq |\phi_n| > 0$ be a non-incremental order of minimum covering eigenvalues of $A_{PD_d}(G)$ then

$$E_{PD_d}(G) \geq \frac{|D| + 2m + 2M + n|\phi_1||\phi_n|}{(|\phi_n| + |\phi_1|)}$$

Proof. The following inequality holds [Theorem 2, [12]] if $Ra_i \geq b_i \geq ra_iR$ where $r, R, a_i \neq 0$ and b_i are real numbers.

$$(R + r) \left(\sum_{i \geq 1}^n a_i b_i \right) \geq \sum_{i \geq 1}^n b_i^2 + rR \sum_{i \geq 1}^n a_i$$

Put $r = |\phi_n|, a_i = 1, R = |\phi_1|$ and $b_i = |\phi_i|$ then

$$(|\phi_1| + |\phi_n|) \left(\sum_{i \geq 1}^n |\phi_i| \right) \geq \sum_{i \geq 1}^n |\phi_i|^2 + |\phi_n||\phi_1| \left(\sum_{i \geq 1}^n 1 \right)$$

$$\text{i.e., } |C| + 2 \sum_{j > i} (d_i + d_j)^2 d_{ij}^2 + n|\phi_n||\phi_1| \leq (|\phi_n| + |\phi_1|) \epsilon_S^C(G)$$

$$\therefore E_{PD_d}(G) \geq \frac{|D| + 2m + 2M + n|\phi_1||\phi_n|}{(|\phi_n| + |\phi_1|)}$$

Theorem 6.6. Let G be a graph with the minimum pendant dominating maximum degree set $D(G)$ where $D(G)$ is the minimum pendant dominating set. If the minimum pendant dominating distance energy $E_{PD_d}(G)$ is a rational number, then

$$E_{PD_d}(G) \equiv |D| \pmod{2}$$

Proof: Let $\phi_1, \phi_2, \dots, \phi_n$ be the minimum pendant dominating distance energy eigenvalues of graph G of which $\phi_1, \phi_2, \dots, \phi_p$ are positive and the remaining are negative. Then

$$\sum_{i=1}^n |\phi_i|^2 = (\phi_1 + \phi_2 + \dots + \phi_p) - (\phi_{p+1} + \phi_{p+2} + \dots + \phi_n)$$

$$\sum_{i=1}^n |\phi_i|^2 = 2(\phi_1 + \phi_2 + \dots + \phi_p) - (\phi_1 + \phi_2 + \dots + \phi_n)$$

$$E_{PD_d}(G) = 2(\phi_1 + \phi_2 + \dots + \phi_p) - \sum_{i=1}^n |\phi_i|$$

$$E_{PD_d}(G) = 2(\phi_1 + \phi_2 + \dots + \phi_p) - |D|$$

$$\therefore E_{PD_d}(G) \equiv |D| \pmod{2}$$

7. The Minimum Pendant Dominating Distance Energy of Some Standard Graphs

Theorem 7.1. The minimum pendant dominating distance energy of a complete bipartite graph $K_{n,n}$ for $n \geq 2$ is $(4n - 8) + \sqrt{n^2 - 2n + 5} + \sqrt{9n^2 - 6n + 13}$

Proof: Let $K_{n,n}$ be a complete bipartite graph with the vertex set $V(G) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$. The minimum pendant dominating set is $D = \{u_1, v_1\}$. Then, the minimum pendant dominating distance energy matrix of the complete bipartite graph is

$$A_{PD_d}(K_{n,n}) = \begin{pmatrix} 1 & 2 & \dots & 2 & 1 & 1 & \dots & 1 \\ 2 & 0 & \dots & 2 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \dots & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 1 & 2 & \dots & 2 \\ 1 & 1 & \dots & 1 & 2 & 0 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 2 & 2 & \dots & 0 \end{pmatrix}_{2n \times 2n}$$

The characteristic polynomial of $A_{PD_d}(K_{n,n})$ is

$$(\phi + 2)^{2n-4} (\phi^2 - (n - 3)\phi + (n - 1))(\phi^2 - (3n - 3)\phi - (3n + 1)) = 0$$

The minimum pendant dominating maximum distance energy eigenvalues of $K_{n,n}$ are

$\phi = -2$ with multiplicity $(2n - 4)$, $\phi = \frac{(n-3) \pm \sqrt{n^2 - 2n + 5}}{2}$ and $\phi = \frac{(3n-3) \pm \sqrt{9n^2 - 6n + 13}}{2}$ with multiplicity one each.

Spectrum of $K_{n,n}$ is

$$\text{Spec}(K_{n,n}) = \left(\begin{array}{ccc} \frac{(n - 3) \pm \sqrt{n^2 - 2n + 5}}{2} & \frac{(3n - 3) \pm \sqrt{9n^2 - 6n + 13}}{2} & -2 \\ 1 & 1 & 2n - 4 \end{array} \right)$$

The minimum pendant dominating maximum degree energy of $K_{n,n}$ is

$$E_{PD_d}(K_{n,n}) = \left| \frac{(n - 3) + \sqrt{n^2 - 2n + 5}}{2} \right| + \left| \frac{(3n - 3) + \sqrt{9n^2 - 6n + 13}}{2} \right| + \left| \frac{(n - 3) - \sqrt{n^2 - 2n + 5}}{2} \right| + \left| \frac{(3n - 3) - \sqrt{9n^2 - 6n + 13}}{2} \right| + |-2|(2n - 4)$$

$$E_{PD_d}(K_{n,n}) = 2(2n - 4) + \sqrt{n^2 - 2n + 5} + \sqrt{9n^2 - 6n + 13}$$

$$\therefore E_{PD_d}(K_{n,n}) = 2(2n - 4) + \sqrt{n^2 - 2n + 5} + \sqrt{9n^2 - 6n + 13}$$

Theorem 7.2. The minimum pendant dominating distance energy of a star graph $K_{1,n-1}$ for $n \geq 3$ is almost $(6n - 10) + 2\sqrt{n + 2}$.

Proof: Let $K_{1,n-1}$ be a star graph with the vertex set $V(G) = \{v_0, v_1, v_2, \dots, v_{n-1}\}$ where $\{v_0\}$ is the center vertex. The minimum pendant dominating set is $D = \{v_0, v_1\}$. Then, the minimum pendant dominating distance matrix is

$$A_{PD_d}(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 2 & \dots & 2 \\ 1 & 2 & 0 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 2 & \dots & 0 \end{pmatrix}_{n \times n}$$

The characteristic polynomial of $A_{PD_d}(K_{1,n-1})$ is

$$f_n(K_{1,n-1}, \phi) = (-1)^n (\phi + 2)^{n-3} (\phi^3 - (2n - 4)\phi^2 - (n + 2)\phi + (n - 2)) \leq (-1)^n (\phi + 2)^{n-3} (\phi^3 - (2n - 4)\phi^2 - (n + 2)\phi + (n - 2) + (2n^2 - n - 6))$$

$$\begin{aligned} &= (-1)^n(\phi + 2)^{n-3}(\phi^3 - (2n - 4)\phi^2 - (n + 2)\phi + 2(n + 2)(n - 2)) \\ &= (-1)^n(\phi + 2)^{n-3}(\phi^2(\phi - (2n - 4)) - (n + 2)(\phi - (2n - 4))) \\ &\leq (-1)^n(\phi + 2)^{n-3}(\phi^2 - (n + 2))(\phi - (2n - 4)) \end{aligned}$$

The minimum pendant dominating distance energy eigenvalues of $K_{1,n-1}$ are

$\phi = -2$ with multiplicity $(n - 3)$, $\phi = (2n - 4)$ and $\phi = \pm\sqrt{n + 2}$ with multiplicity one each.

Spectrum of a star graph $K_{1,n-1}$ is

$$\text{Spec}(K_{1,n-1}) = \begin{pmatrix} \sqrt{n+2} & 2n-4 & -2 & -\sqrt{n+2} \\ 1 & 1 & (n-3) & 1 \end{pmatrix}$$

The minimum pendant dominating distance energy of $K_{1,n-1}$ is

$$E_{PD_d}(K_{1,n-1}) = |\sqrt{n + 2}|(1) + |2n - 4|(1) + |-2|(2n - 4) + |-\sqrt{n + 2}|(1)$$

$$E_{PD_d}(K_{1,n-1}) = 2(2n - 6) + 2\sqrt{n + 2} + (2n - 4)$$

$$\therefore E_{PD_d}(K_{1,n-1}) = (6n - 10) + 2\sqrt{n + 2}$$

Definition 7.1. The double star graph $S_{n,m}$ is the graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0 . A vertex set $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1}) = \{v_0, v_1, \dots, v_{n-1}, u_0, u_1, \dots, u_{m-1}\}$ and the edge set $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j \text{ where } 1 \leq i \leq n - 1, 1 \leq j \leq m - 1\}$

Theorem 7.3. The minimum pendant dominating distance energy of a double star graph $S_{n,n}$ for $n \geq 3$ is $(4n - 4) + \sqrt{25n^2 - 4n + 16} + \sqrt{n^2 + 8n + 4}$

Proof: Let $S_{n,n}$ be a star graph with the vertex set $V(G) = \{v_0, v_1, v_2, \dots, v_{n-1}, u_0, u_1, \dots, u_{n-1}\}$. The minimum pendant dominating set is $D = \{v_0, u_0\}$. Then, the minimum pendant dominating distance energy matrix is

$$A_{PD_d}(S_{n,n}) = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 & 2 & \dots & 2 \\ 1 & 0 & \dots & 2 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \dots & 0 & 2 & 3 & \dots & 3 \\ 1 & 2 & \dots & 2 & 1 & 1 & \dots & 1 \\ 2 & 3 & \dots & 3 & 1 & 0 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 3 & \dots & 3 & 1 & 2 & \dots & 0 \end{pmatrix}_{2n \times 2n}$$

The characteristic polynomial of $A_{PD_d}(S_{n,n})$ is

$$(\phi + 2)^{2n-2} (\phi^2 - 5n\phi + (n - 4))(\phi^2 + (n + 2)\phi - n) = 0$$

The minimum pendant dominating distance energy eigenvalues of $S_{n,n}$ are

$\phi = -2$ with multiplicity $(2n - 2)$, $\phi = \frac{5n \pm \sqrt{25n^2 - 4n + 16}}{2}$ and $\phi = \frac{-(n+2) \pm \sqrt{n^2 + 8n + 4}}{2}$ with multiplicity one each.

Spectrum of $S_{n,n}$ is

$$\text{Spec}(S_{n,n}) = \begin{pmatrix} \frac{5n \pm \sqrt{25n^2 - 4n + 16}}{2} & \frac{-(n+2) \pm \sqrt{n^2 + 8n + 4}}{2} & -2 \\ 1 & 1 & 2n-2 \end{pmatrix}$$

The minimum pendant dominating distance energy of $S_{n,n}$ is

$$E_{PD_d}(S_{n,n}) = \left| \frac{5n + \sqrt{25n^2 - 4n + 16}}{2} \right| + \left| \frac{-(n+2) + \sqrt{n^2 + 8n + 4}}{2} \right| + \left| \frac{5n - \sqrt{25n^2 - 4n + 16}}{2} \right| + |-2|(2n-2) + \left| \frac{-(n+2) - \sqrt{n^2 + 8n + 4}}{2} \right|$$

$$E_{PD_d}(S_{n,n}) = (4n - 4) + \sqrt{25n^2 - 4n + 16} + \sqrt{n^2 + 8n + 4}$$

$$\therefore E_{PD_d}(S_{n,n}) = (4n - 4) + \sqrt{25n^2 - 4n + 16} + \sqrt{n^2 + 8n + 4}$$

Theorem 7.4. The minimum pendant dominating distance energy of a barbell graph $B_{n,n}$ for $n \geq 3$ is $(2n - 4) + \sqrt{16n^2 - 20n + 13} + \sqrt{4n^2 - 3}$

Proof: Let $B_{n,n}$ be a barbell graph with the vertex set $V(G) = \{v_0, v_1, v_2, \dots, v_n, u_0, u_1, \dots, u_n\}$. The minimum pendant dominating set is $D = \{u_1, u_2\}$. Then, the minimum pendant dominating distance energy matrix is

$$A_{PD_d}(B_{n,n}) = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 2 & \dots & 2 \\ 1 & 0 & 1 & \dots & 1 & 2 & 3 & \dots & 3 \\ 1 & 1 & 0 & \dots & 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 & 2 & 3 & \dots & 3 \\ 1 & 2 & 2 & \dots & 2 & 1 & 1 & \dots & 1 \\ 2 & 3 & 3 & \dots & 3 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 3 & 3 & \dots & 3 & 1 & 1 & \dots & 0 \end{pmatrix}_{2n \times 2n}$$

The characteristic polynomial of $A_{PD_d}(B_{n,n})$ is

$$(\phi + 1)^{2n-4} (\phi^2 - (4n - 3)\phi - (n + 1))(\phi^2 + (2n - 1)\phi - (n - 1)) = 0$$

The minimum pendant dominating distance energy eigenvalues of $B_{n,n}$ are

$\phi = -1$ with multiplicity $(2n - 4)$, $\phi = \frac{(4n-3) \pm \sqrt{16n^2 - 20n + 13}}{2}$ and $\phi = \frac{-(2n-1) \pm \sqrt{4n^2 - 3}}{2}$ with multiplicity one each.

Spectrum of $B_{n,n}$ is

$$\text{Spec}(B_{n,n}) = \left(\begin{array}{ccc} \frac{(4n - 3) \pm \sqrt{16n^2 - 20n + 13}}{2} & \frac{-(2n - 1) \pm \sqrt{4n^2 - 3}}{2} & -1 \\ 1 & 1 & 2n - 4 \end{array} \right)$$

The minimum pendant dominating distance energy of $B_{n,n}$ is

$$E_{PD_d}(B_{n,n}) = \left| \frac{(4n - 3) + \sqrt{16n^2 - 20n + 13}}{2} \right| + \left| \frac{-(2n - 1) + \sqrt{4n^2 - 3}}{2} \right| + \left| \frac{(4n - 3) - \sqrt{16n^2 - 20n + 13}}{2} \right| + |-1|(2n - 4) + \left| \frac{-(2n - 1) - \sqrt{4n^2 - 3}}{2} \right|$$

$$E_{PD_d}(B_{n,n}) = (2n - 4) + \sqrt{16n^2 - 20n + 13} + \sqrt{4n^2 - 3}$$

$$\therefore E_{PD_d}(B_{n,n}) = (2n - 4) + \sqrt{16n^2 - 20n + 13} + \sqrt{4n^2 - 3}$$

Definition 7.2. The cocktail party graph denoted by K_{2*n} is the graph having the vertex set $V(G) = \cup_{i=1}^n \{u_i, v_i\}$ and the edge set $E(G) = \{u_i u_j, v_i v_j, u_i v_j, v_i u_j : 1 \leq i < j \leq n\}$ i.e., $|V(G)| = 2n$ and $E(G) = \frac{n^2 - 3n}{2}$

Theorem 7.5. For the cocktail party graph K_{2*n} where $n \geq 3$, the minimum pendant dominating distance energy is at most $(4n - 5) + \sqrt{5} + 2\sqrt{2n + 3}$

Proof: Let K_{2*n} be the cocktail party graph having the vertex set $V(K_{2*n}) = \cup_{i=1}^n \{u_i, v_i\}$. The minimum

pendant dominating set is $D = \{u_1, u_2\}$. Then, the minimum pendant dominating distance energy matrix is

$$A_{PD_d}(K_{2*n}) = \begin{pmatrix} 1 & 1 & \dots & 1 & 2 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 1 & 2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 & 1 & 1 & \dots & 2 \\ 2 & 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2 & 1 & 1 & \dots & 0 \end{pmatrix}_{2n \times 2n}$$

The characteristic polynomial of $A_{PD_d}(K_{2*n})$ is

$$\begin{aligned} f_n(K_{2*n}, \phi) &= \phi^{n-3}(\phi + 2)^{n-2}(\phi^2 + \phi - 1)(\phi^3 - (2n - 1)\phi^2 - (2n + 3)\phi + (2n - 4)) \\ &\leq \phi^{n-3}(\phi + 2)^{n-2}(\phi^2 + \phi - 1)(\phi^3 - (2n - 1)\phi^2 - (2n + 3)\phi + (2n - 4) + (4n^2 + 2n + 1)) \\ &= \phi^{n-3}(\phi + 2)^{n-2}(\phi^2 + \phi - 1)(\phi^3 - (2n - 1)\phi^2 - (2n + 3)\phi + (4n^2 + 4n + 3)) \\ &= \phi^{n-3}(\phi + 2)^{n-2}(\phi^2 + \phi - 1)(\phi^3 - (2n - 1)\phi^2 - (2n + 3)\phi + (2n - 1)(2n + 3)) \\ &= \phi^{n-3}(\phi + 2)^{n-2}(\phi^2 + \phi - 1)(\phi^2(\phi - (2n - 1)) - (2n + 3)(\phi - (2n - 1))) \\ &\leq \phi^{n-3}(\phi + 2)^{n-2}(\phi^2 + \phi - 1)(\phi^2 - (2n + 3))(\phi - (2n - 1)) \end{aligned}$$

The minimum pendant dominating distance energy eigenvalues of K_{2*n} are

$\phi = 0$ with multiplicity $(n - 3)$, $\phi = -2$ with multiplicity $(n - 2)$, $\phi = \frac{-1 \pm \sqrt{5}}{2}$, $\phi = \pm\sqrt{2n + 3}$ and $\phi = 2n - 1$ with multiplicity one each.

Spectrum of K_{2*n} is

$$\text{Spec}(K_{2*n}) = \begin{pmatrix} \sqrt{2n + 3} & 2n - 1 & \frac{-1 + \sqrt{5}}{2} & 0 & \frac{-1 - \sqrt{5}}{2} & -\sqrt{2n + 3} & -2 \\ 1 & 1 & 1 & n - 3 & 1 & 1 & n - 2 \end{pmatrix}$$

The minimum pendant dominating distance energy of K_{2*n} is

$$E_{PD_d}(K_{2*n}) = |\sqrt{2n + 3}| + |2n - 1| + \left| \frac{-1 + \sqrt{5}}{2} \right| + 0 + \left| \frac{-1 - \sqrt{5}}{2} \right| + |-\sqrt{2n + 3}| + |-2|(n - 2)$$

$$E_{PD_d}(K_{2*n}) = \sqrt{2n + 3} + (2n - 1) - \frac{1}{2} + \frac{\sqrt{5}}{2} + \frac{1}{2} + \frac{\sqrt{5}}{2} + \sqrt{2n + 3} + 2(n - 2)$$

$$\therefore E_{PD_d}(K_{2*n}) = (4n - 5) + \sqrt{5} + 2\sqrt{2n + 3}$$

Theorem 7.6. For any integer $n \geq 3$ the minimum pendant dominating distance energy of complete graph K_n is $(n - 3) + \sqrt{n^2 - 2n + 9}$

Proof: For complete graph K_n the minimum pendant dominating distance matrix is same as minimum pendant dominating matrix [19]. Therefore, the minimum pendant dominating distance energy is equal to minimum pendant dominating energy.

8. Conclusion

The theory of domination and graph energy are both central topics in modern graph theory, with numerous applications in fields such as network theory, computer science, chemistry, and even physics. In recent years many scholars are working in this area and also, they are introducing new domination parameters. In this article we have initiated the pendant domination parameter for distance energy. We have made significant progress in our research by calculating the exact values of the minimum pendant dominating distance energy for well-known families of graphs and establishing bounds for this parameter in terms of other graph properties, such as the degree and order of the graph.

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