

Primitive Idempotents in R_{8p^n} and Corresponding Codes

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Abstract: Let F be a finite field of prime power order q , where q is of the form $8k + 3$. If q is primitive root modulo p^n , then the semi-simple group algebra FG of the cyclic group G of order $8p^n$ over F , where p is an odd prime and $n \geq 1$, has $8n + 5$ primitive idempotents. Explicit expressions for these primitive idempotents are obtained. Generating polynomials, minimum distances and dimensions of the corresponding minimal cyclic codes are also obtained.

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1 Introduction

Let F be a finite field of prime power order q and G be a cyclic group of order m such that $\gcd(m, q) = 1$. Then FG , the group algebra of the cyclic group G over F , is semi-simple and has only a finite number of primitive idempotents which equals the number of cyclotomic cosets modulo m . Let t be the multiplicative order of q modulo m , then $1 \leq t \leq \phi(m)$ [4]. If $t = \phi(m)$ and $m = 2, 4, p^n, 2p^n$, the complete sets of primitive idempotents were calculated by Pruthi and Arora [1, 6]. The minimal quadratic residue cyclic codes of length p^n were obtained by Batra and Arora [3]. The minimal cyclic codes of length $p^n q$ were discussed by Bakshi and Raka [2]. The minimal cyclic codes of length $8p^n$ over field of prime power order q , where q is of the form $8k + 5$, were obtained by Singh and Arora [7]. Explicit expression of primitive idempotents in FC_{4p^n} and corresponding codes were obtained by Singh and Ahlawat [10, 11]. The minimal cyclic codes of length $8p^n$ were discussed by Singh and Arora [12, 13].

The equivalence relation $a \sim b$ whenever $a \equiv bq^i \pmod{8p^n}$ partitions the set $S = \{1, 2, \dots, 8p^n\}$ into

$$\Omega_{rp^n} = \{rp^n\} \text{ for } r = 0, 4$$

$$\Omega_{sp^n} = \{sp^n, sp^n q\} \text{ for } s = 1, 2, v = 1 + 4p^n$$

and for $0 \leq j \leq n - 1$,

$$\Omega_{tp^j} = \{tp^j, tp^jq, tp^jq^2, \dots, tp^jq^{\phi(p^{n-j})-1}\}$$

where $t = 1, 2, 4, 8, \lambda = 1 + 2p^n, \mu = 2(1 + 2p^n), \nu = 1 + 4p^n, \chi = 1 + 6p^n$.

In the following lemmas some results for cyclotomic cosets are obtained:

Lemma 1.1. *If $p \equiv 1(mod 4)$, then $(1 + 6p^n)q^{\frac{\phi(p^{2n})}{2}} \equiv -1(mod 8p^n)$ and if $p \equiv 3(mod 4)$ then $(1 + 4p^n)q^{\frac{\phi(p^{2n})}{2}} \equiv -1(mod 8p^n)$.*

Proof. For an odd prime p , so $\frac{\phi(p^n)}{2}$ is odd iff $p \equiv 3(mod 4)$.

Now, $q^{\frac{\phi(p^{2n})}{2}} \equiv -1(mod p^n)$ and $(1 + 6p^n) \equiv 1(mod p^n)$, so $(1 + 6p^n)q^{\frac{\phi(p^{2n})}{2}} \equiv -1(mod p^n)$.

Further, $p^n = 4t + 1$, thus $(1 + 6p^n)q^{\frac{\phi(p^{2n})}{2}} \equiv -1(mod 8p^n)$.

When $p^n = 4t + 3$, then $(1 + 4p^n)q^{\frac{\phi(p^{2n})}{2}} \equiv -1(mod 8)$ and hence $(1 + 4p^n)q^{\frac{\phi(p^{2n})}{2}} \equiv -1(mod 8p^n)$. □

Remark 1.2. *By above lemma, we obtained the following "If $p \equiv 1(mod 4)$, then $-\Omega_1 = \Omega_\chi$ and if $p \equiv 3(mod 4)$ then $-\Omega_1 = \Omega_\nu$."*

Lemma 1.3. *For cyclotomic cosets $\Omega_{p^j}, 0 \leq j \leq n - 1$.*

(i) $\lambda^2\Omega_{p^j} = \Omega_{p^j} = \lambda\Omega_{\lambda p^j}$.

(ii) $\nu^2\Omega_{p^j} = \Omega_{p^j} = \nu\Omega_{\nu p^j}$.

(iii) $\chi^2\Omega_{p^j} = \Omega_{p^j} = \chi\Omega_{\chi p^j}$.

(iv) $\mu^2\Omega_{p^j} = 4\Omega_{p^j} = \mu\Omega_{\mu p^j} = 2\Omega_{2p^j} = \Omega_{4p^j}$.

Proof. Since $\lambda^2 = (1 + 2p^n)^2 = 1 + 4p^{2n} + 4p^n = 1 + 4p^n(p^n + 1)$.

As p is odd, so $p^n + 1 \equiv 0(mod 2)$ and hence $\lambda^2 \equiv 1(mod 8p^n)$.

Therefore, $\lambda^2\Omega_{p^j} = \Omega_{p^j}$. Also,

$$\lambda^2\Omega_{p^j} = \{\lambda^2 p^j, \lambda^2 p^jq, \dots, \lambda^2 p^jq^{\phi(p^{n-j})-1}\} = \lambda \{\lambda p^j, \lambda p^jq, \dots, \lambda p^jq^{\phi(p^{n-j})-1}\} = \lambda\Omega_{\lambda p^j}$$

Thus, $\lambda^2\Omega_{p^j} = \Omega_{p^j} = \lambda\Omega_{\lambda p^j}$. □

Proof of remaining parts will go on similar lines.

Lemma 1.4. *For cyclotomic cosets $\Omega_{p^n}, \Omega_{2p^n}, \Omega_{4p^n}$ and $\Omega_{\nu p^n}$:*

(i) $-\Omega_{p^n} = \Omega_{\nu p^n}$

(ii) $-\Omega_{2p^n} = \Omega_{2p^n}$

(iii) $-\Omega_{4p^n} = \Omega_{4p^n}$

Proof. Proof is trivial. □

Lemma 1.5. *If $r = 1, 2, 4, 8, \lambda, \mu, \nu$ or χ then for $0 \leq j \leq n - 1$.*

$$r\Omega_{8p^j} = \Omega_{8p^j} = 8\Omega_{rp^j} = 4\Omega_{\mu p^j}$$

Proof. Since $8rp^j \equiv lp^jq^k(mod 8p^n)$ for $l = 1, 2, 4, 8, \lambda, \mu, \nu, \chi$. So, $8rp^j \equiv 8p^jq^k(mod 8p^n)$. Hence the required result holds. □

Lemma 1.6. *For cyclotomic cosets $\Omega_{2p^j}, \Omega_{4p^j}, \Omega_{8p^j}, \Omega_{\lambda p^j}, \Omega_{\mu p^j}, \Omega_{\nu p^j}$ and $\Omega_{\chi p^j}$:*

(i) *If $p \equiv 1(mod 4)$, then $-\Omega_{2p^j} = \Omega_{\mu p^j}$.*

- (ii) If $p \equiv 3 \pmod{4}$, then $-\Omega_{2pj} = \Omega_{2pj}$ and $-\Omega_{\mu pj} = \Omega_{\mu pj}$,
- (iii) $-\Omega_{4pj} = \Omega_{4pj}$ and $-\Omega_{8pj} = \Omega_{8pj}$.
- (iv) If $p \equiv 1 \pmod{4}$, then $-\Omega_{\lambda pj} = \Omega_{\nu pj}$.
- (v) If $p \equiv 3 \pmod{4}$, then $-\Omega_{\chi pj} = \Omega_{\lambda pj}$

Proof of these can be obtained by using lemma 1.1.

The $8n + 5$ primitive idempotents are obtained in Theorem 2.8. In Section 3, we discuss generating polynomials and dimensions for the corresponding minimal cyclic codes of length $8p^n$. The minimum distances or the bounds for these codes are obtained in Section 4. Minimal cyclic codes of length 24 for $q = 11$ are calculated in Section 5.

Notation 1.7. Let α be a fixed primitive $8p^n$ th root of unity in some extension field of F . For $0 \leq j \leq n-1$, we define:

$$T_j = p^j \sum_{s \in \Omega_{pj}} \alpha^s, S_j = p^j \sum_{s \in \Omega_{\nu pj}} \alpha^s, Q_j = p^j \sum_{s \in \Omega_{\lambda pj}} \alpha^s, P_j = p^j \sum_{s \in \Omega_{\chi pj}} \alpha^s, \text{ and } R_j = p^j \sum_{s \in \Omega_{2pj}} \alpha^s.$$

Since $(T_j)^q = T_j$. Thus $T_j \in F$. Similarly, S_j, Q_j, P_j and $R_j \in F$.

2 Primitive Idempotents

Throughout this paper, α is assumed to be $8p^n$ th root of unity in some extension field of F , M_s the minimal ideal in $R_{8p^n} = \frac{F[x]}{\langle x^{8p^n} - 1 \rangle} \cong FG$; generated by $\frac{(x^{8p^n} - 1)}{m \cdot x}$ and $\vartheta_s(x)$, the primitive idempotent in R_{8p^n} , corresponding to the minimal ideal M_s , given by $\vartheta_s(x) = \frac{1}{8p^n} \sum_{i=0}^{8p^n-1} \epsilon^s x^i$ where $\epsilon^s = \sum_{j \in \Omega_s} \alpha^{-ij}$ [3].

Also, denote $\bar{C}_s = \sum_{s \in \Omega_s} x^s$.

Lemma 2.1. For any odd prime p and a positive integer k , if δ is a primitive p^k th root of unity, ζ is a primitive $2p^k$ th root of unity in some extension field of F and q a primitive root modulo p^k ,

$$\sum_{s=0}^{\phi(p^k)-1} \delta^{qs} = \begin{cases} -1, & k = 1 \\ 0, & k \geq 2 \end{cases} \quad \text{and} \quad \sum_{s=0}^{\phi(2p^k)-1} \delta^{qs} = \begin{cases} 1, & k = 1 \\ 0, & k \geq 2 \end{cases}$$

Proof. Since $\{1, q, \dots, q^{\phi(p^k)-1}\}$ and $\{1, q, \dots, q^{\phi(2p^k)-1}\}$ are reduced residue system modulo p^k and $2p^k$ respectively and q is a primitive root modulo p^k . Thus,

$$\sum_{s=0}^{\phi(p^k)-1} \delta^{qs} = \delta^s - \sum_{s=1}^{p^k-1} \delta^{ps}$$

$$\text{and } \sum_{s=0}^{\phi(2p^k)-1} \zeta^{qs} = \zeta^s - \sum_{s=1}^{2p^k-1} \zeta^{2s} - \sum_{s=1}^{p^k-1} \zeta^{ps} + \sum_{s=1}^{p^k-1} \zeta^{2ps}.$$

Now, the result can be obtained using the properties of δ and ζ . □

Lemma 2.2. For $0 \leq j \leq n - 1$, $T_j + S_j = 0$ and $P_j + Q_j = 0$.

Proof. Since $\alpha^{\nu pj} = \alpha^{(1+4p^n)p^j} = \alpha^{p^j} \alpha^{4p^n p^j} = -\alpha^{p^j}$ and $\alpha^{\chi pj} = \alpha^{(1+6p^n)p^j} = -\alpha^{\lambda pj}$. Using these the result follows. □

Lemma 2.3. For $0 \leq i, j \leq n-1$,

$$\sum_{s \in \Omega_{pj}} \alpha^{p^i s} = \sum_{s \in \Omega_{vpj}} \alpha^{vp^i s} = \sum_{s \in \Omega_{\lambda pj}} \alpha^{\lambda p^i s} = \sum_{s \in \Omega_{\chi pj}} \alpha^{\chi p^i s} = - \sum_{s \in \Omega_{\lambda pj}} \alpha^{\chi p^i s} = - \sum_{s \in \Omega_{pj}} \alpha^{vp^i s} = - \sum_{s \in \Omega_{vpj}} \alpha^{p^i s} =$$

$$\begin{cases} \frac{\phi(p^{n-j})}{2} (\alpha^{p^{i+j}} + \alpha^{3p^{i+j}}), & \text{if } i+j \geq n, \\ \frac{1}{p^j} T_{i+j}, & \text{if } i+j \leq n-1. \end{cases}$$

Proof. Let $\delta = \alpha^{p^{i+j}}$. Then $\sum_{s \in \Omega_{pj}} \alpha^{p^i s} = \sum_{t=0}^{\phi(p^{n-i})-1} \alpha^{p^{i+j} q^t} = \sum_{t=0}^{\phi(p^{n-i})-1} \delta^{q^t}$.

Consider the following cases:

Case:1. If $i+j \geq n$, then $\delta = \alpha^{p^n}$ is 8th root of unity and

$$\delta^{q^l} \equiv \delta^{q^r} \text{ iff } q^l \equiv q^r \pmod{8} \text{ iff } l \equiv r \pmod{\phi(8)}.$$

Hence $\sum_{s \in \Omega_{pj}} \alpha^{p^i s} = \sum_{t=0}^{\phi(p^{n-i})-1} \delta^{q^t} = \frac{\phi(p^{n-j})}{\phi(8)} \sum_{t=0}^{\phi(8)-1} \delta^{q^t} = \frac{\phi(p^{n-j})}{2} (\alpha^{p^{i+j}} + \alpha^{3p^{i+j}}).$

Case 2. If $i+j \leq n-1$, then δ is $8p^{n-i-j}$ th root of unity and

$$\delta^{q^l} \equiv \delta^{q^r} \text{ iff } q^l \equiv q^r \pmod{8p^{n-i-j}} \text{ iff } l \equiv r \pmod{\phi(p^{n-i-j})}$$

Therefore, $\sum_{s \in \Omega_{pj}} \alpha^{p^i s} = \frac{\phi(p^{n-j})}{\phi(p^{n-i-j})} \sum_{t=0}^{\phi(p^{n-i-j})-1} \delta^{q^t} = \frac{1}{p^j} p^{i+j} \sum_{t=0}^{\phi(p^{n-i-j})-1} \alpha^{p^{i+j} q^t} = \frac{1}{p^j} p^{i+j} \sum_{s \in \Omega_{pi+j}} \alpha^s$ Since $\sum_{s \in \Omega_{pi+j}} \alpha^s = T_{i+j}$, therefore, $\sum_{s \in \Omega_{pj}} \alpha^{p^i s} = \frac{1}{p^j} T_{i+j}$. □

The proof of the following lemmas will go on similar lines as of the above lemma.

Lemma 2.4. For $0 \leq i, j \leq n$,

$$\sum_{s \in \Omega_{\lambda pj}} \alpha^{p^i s} = \sum_{s \in \Omega_{pj}} \alpha^{\lambda p^i s} = - \sum_{s \in \Omega_{\lambda pj}} \alpha^{vp^i s} = \sum_{s \in \Omega_{vpj}} \alpha^{\chi p^i s} = - \sum_{s \in \Omega_{vpj}} \alpha^{\lambda p^i s} = \sum_{s \in \Omega_{\chi pj}} \alpha^{vp^i s} = - \sum_{s \in \Omega_{pj}} \alpha^{\chi p^i s} =$$

$$\begin{cases} \frac{\phi(p^{n-j})}{2} (\alpha^{p^{i+j}} - \alpha^{3p^{i+j}}) \theta^{p^{i+j}}, & \text{if } i+j \geq n, \\ -\frac{1}{p^j} T_{i+j}, & \text{if } i+j \leq n-1. \end{cases}$$

Lemma 2.5. For $0 \leq i \leq n; 0 \leq j \leq n-1$

$$\sum_{s \in \Omega_{pj}} \alpha^{4p^i s} = \sum_{s \in \Omega_{2pj}} \alpha^{2p^i s} = \sum_{s \in \Omega_{\lambda pj}} \alpha^{4p^i s} = \sum_{s \in \Omega_{4pj}} \alpha^{p^i s} = \sum_{s \in \Omega_{4pj}} \alpha^{\lambda p^i s} = \sum_{s \in \Omega_{vpj}} \alpha^{4p^i s} = \sum_{s \in \Omega_{4pj}} \alpha^{vp^i s} =$$

$$\sum_{s \in \Omega_{\chi pj}} \alpha^{4p^i s} = \sum_{s \in \Omega_{4pj}} \alpha^{\chi p^i s} = \sum_{s \in \Omega_{\mu pj}} \alpha^{2p^i s} = \sum_{s \in \Omega_{2pj}} \alpha^{\mu p^i s} = \sum_{s \in \Omega_{\mu pj}} \alpha^{\mu p^i s} = \begin{cases} -\phi(p^{n-j}), & \text{if } i+j \geq n \\ p^{n-j-1}, & \text{if } i+j = n-1 \\ 0, & \text{if } i+j < n-1. \end{cases}$$

Lemma 2.6. For $0 \leq i \leq n; 0 \leq j \leq n-1$

$$\sum_{s \in \Omega_{pj}} \alpha^{8p^i s} = \sum_{s \in \Omega_{4pj}} \alpha^{2p^i s} = \sum_{s \in \Omega_{2pj}} \alpha^{4p^i s} = \sum_{s \in \Omega_{\lambda pj}} \alpha^{8p^i s} = \sum_{s \in \Omega_{8pj}} \alpha^{\lambda p^i s} = \sum_{s \in \Omega_{vpj}} \alpha^{8p^i s} = \sum_{s \in \Omega_{8pj}} \alpha^{vp^i s} =$$

$$\sum_{s \in \Omega_{\chi pj}} \alpha^{8p^i s} = \sum_{s \in \Omega_{8pj}} \alpha^{\chi p^i s} = \sum_{s \in \Omega_{\mu pj}} \alpha^{4p^i s} = \sum_{s \in \Omega_{4pj}} \alpha^{\mu p^i s} = \sum_{s \in \Omega_{\mu pj}} \alpha^{8p^i s} = \sum_{s \in \Omega_{8pj}} \alpha^{\mu p^i s} = \sum_{s \in \Omega_{pj}} \alpha^{16p^i s} =$$

$$\sum_{s \in \Omega_{\mu pj}} \alpha^{16p^i s} = \sum_{s \in \Omega_{pj}} \alpha^{32p^i s} = \sum_{s \in \Omega_{pj}} \alpha^{64p^i s} = \begin{cases} \phi(p^{n-j}), & \text{if } i+j \geq n, j = n \\ -p^{n-j-1}, & \text{if } i+j = n-1 \\ 0, & \text{if } i+j < n-1. \end{cases}$$

Lemma 2.7. For $0 \leq i, j \leq n - 1$,

$$\sum_{s \in \Omega_{pj}} \alpha^{2p^i s} = \sum_{s \in \Omega_{2pj} / j=n} \alpha^{p^i s} = - \sum_{s \in \Omega_{\mu pj} / j=n} \alpha^{p^i s} = - \sum_{s \in \Omega_{\lambda pj} / j=n} \alpha^{2p^i s} = \sum_{s \in \Omega_{\nu pj} / j=n} \alpha^{2p^i s} =$$

$$- \sum_{s \in \Omega_{\nu pj} / j=n} \alpha^{\mu p^i s} = \sum_{s \in \Omega_{\chi pj} / j=n} \alpha^{\mu p^i s} = - \sum_{s \in \Omega_{\chi pj} / j=n} \alpha^{2p^i s} = \sum_{s \in \Omega_{\lambda pj} / j=n} \alpha^{\mu p^i s}$$

$$= \sum_{s \in \Omega_{\mu pj} / j=n} \alpha^{\lambda p^i s} = \begin{cases} 0, & \text{if } i+j \geq n \\ \frac{1}{p^j} R_{i+j}, & \text{if } i+j \leq n-1. \end{cases}$$

Theorem 2.8. The explicit expressions for the $8n+5$ primitive idempotents in R_{8p^n} are given by

$$\vartheta_0(x) = \frac{1}{8p^n} [\bar{C}_0 + \bar{C}_{p^n} + \bar{C}_{2p^n} + \bar{C}_{4p^n} + \bar{C}_{\nu p^n} + \sum_{i=0}^{n-1} \{ \bar{C}_{p^i} + \bar{C}_{2p^i} + \bar{C}_{4p^i} + \bar{C}_{8p^i} + \bar{C}_{\lambda p^i} + \bar{C}_{\mu p^i} + \bar{C}_{\nu p^i} + \bar{C}_{\chi p^i} \}]$$

$$\vartheta_{p^n}(x) = \frac{1}{8p^n} [2\bar{C}_0 - (\alpha^{p^{2n}} + \alpha^{3p^{2n}}) \bar{C}_{p^n} - 2\bar{C}_{4p^n} + (\alpha^{p^{2n}} + \alpha^{3p^{2n}}) \bar{C}_{\nu p^n} - \sum_{i=0}^{n-1} \{ (\alpha^{p^{n+i}} + \alpha^{3p^{n+i}}) \bar{C}_{p^i} + 2\bar{C}_{4p^i} - 2\bar{C}_{8p^i} + (\alpha^{p^{n+i}} - \alpha^{3p^{n+i}}) \beta^{p^{n+i}} \bar{C}_{\lambda p^i} - (\alpha^{p^{n+i}} + \alpha^{3p^{n+i}}) \bar{C}_{\nu p^i} - (\alpha^{p^{n+i}} - \alpha^{3p^{n+i}}) \beta^{p^{n+i}} \bar{C}_{\chi p^i} \}]$$

$$\vartheta_{2p^n}(x) = \frac{1}{4p^n} [\bar{C}_0 - \bar{C}_{2p^n} + \bar{C}_{4p^n} - \sum_{i=0}^{n-1} \{ 2\bar{C}_{2p^i} - \bar{C}_{4p^i} - \bar{C}_{8p^i} + \bar{C}_{\mu p^i} \}]$$

$$\vartheta_{4p^n}(x) = \frac{1}{8p^n} [\bar{C}_0 - \bar{C}_{p^n} + \bar{C}_{2p^n} + \bar{C}_{4p^n} - \bar{C}_{\nu p^n} - \sum_{i=0}^{n-1} \{ \bar{C}_{p^i} - \bar{C}_{2p^i} - \bar{C}_{4p^i} - \bar{C}_{8p^i} + \bar{C}_{\lambda p^i} - \bar{C}_{\mu p^i} + \bar{C}_{\nu p^i} + \bar{C}_{\chi p^i} \}]$$

$$\vartheta_{\nu p^n}(x) = \frac{1}{8p^n} [2\bar{C}_0 + (\alpha^{p^{2n}} + \alpha^{3p^{2n}}) \bar{C}_{p^n} - 2\bar{C}_{4p^n} - (\alpha^{p^{2n}} + \alpha^{3p^{2n}}) \bar{C}_{\nu p^n} + \sum_{i=0}^{n-1} \{ (\alpha^{p^{n+i}} + \alpha^{3p^{n+i}}) \bar{C}_{p^i} - 2\bar{C}_{4p^i} + 2\bar{C}_{8p^i} + (\alpha^{p^{n+i}} - \alpha^{3p^{n+i}}) \beta^{p^{n+i}} \bar{C}_{\lambda p^i} - (\alpha^{p^{n+i}} + \alpha^{3p^{n+i}}) \bar{C}_{\nu p^i} - (\alpha^{p^{n+i}} - \alpha^{3p^{n+i}}) \beta^{p^{n+i}} \bar{C}_{\chi p^i} \}]$$

and for $0 \leq j \leq n - 1$,

$$\vartheta_{4p^j}(x) = \frac{1}{8p^n} [\phi(p^{n-j}) \{ \bar{C}_0 - \bar{C}_{p^n} + \bar{C}_{2p^n} + \bar{C}_{4p^n} - \bar{C}_{\nu p^n} \} + p^{n-j-1} \{ \bar{C}_{p^{n-j-1}} - \bar{C}_{2p^{n-j-1}} - \bar{C}_{4p^{n-j-1}} - \bar{C}_{8p^{n-j-1}} + \bar{C}_{\lambda p^{n-j-1}} - \bar{C}_{\mu p^{n-j-1}} + \bar{C}_{\nu p^{n-j-1}} + \bar{C}_{\chi p^{n-j-1}} \} - \phi(p^{n-j}) \sum_{i=0}^{n-1} \{ \bar{C}_{p^i} - \bar{C}_{2p^i} - \bar{C}_{4p^i} - \bar{C}_{8p^i} + \bar{C}_{\lambda p^i} - \bar{C}_{\mu p^i} + \bar{C}_{\nu p^i} + \bar{C}_{\chi p^i} \}]$$

$$\vartheta_{8p^j}(x) = \frac{1}{8p^n} [\phi(p^{n-j}) \{ \bar{C}_0 + \bar{C}_{p^n} + \bar{C}_{2p^n} + \bar{C}_{4p^n} + \bar{C}_{\nu p^n} \} - p^{n-j-1} \{ \bar{C}_{p^{n-j-1}} + \bar{C}_{2p^{n-j-1}} - \bar{C}_{4p^{n-j-1}} + \bar{C}_{8p^{n-j-1}} + \bar{C}_{\lambda p^{n-j-1}} + \bar{C}_{\mu p^{n-j-1}} + \bar{C}_{\nu p^{n-j-1}} + \bar{C}_{\chi p^{n-j-1}} \} + \phi(p^{n-j}) \sum_{i=n-j} \{ \bar{C}_{p^i} + \bar{C}_{2p^i} + \bar{C}_{4p^i} + \bar{C}_{8p^i} + \bar{C}_{\lambda p^i} + \bar{C}_{\mu p^i} + \bar{C}_{\nu p^i} + \bar{C}_{\chi p^i} \}]$$

If $p^n \equiv 3 \pmod{4}$

$$\vartheta_{p^j}(x) = \frac{1}{8p^n} [\phi(p^{n-j}) \{ \bar{C}_0 - \frac{(\alpha^{p^{n+j}} + \alpha^{3p^{n+j}})}{n} \bar{C}_{p^n} - \bar{C}_{4p^n} + \frac{(\alpha^{p^{n+j}} + \alpha^{3p^{n+j}})}{2} \bar{C}_{\nu p^n} \} + p^{n-j-1} \{ \bar{C}_{4p^{n-j-1}} - \bar{C}_{8p^{n-j-1}} - \frac{1}{p^j} \sum_{i=0}^{j-1} \{ T_{i+j, p} \epsilon_{i-R} \epsilon_{i+P} \epsilon_{i+R} \epsilon_{i-T} \epsilon_{i+Q} \epsilon_{i+X} \} - \sum_{i=n-j}^{n-1} \{ \frac{(\alpha^{p^{i+j}} + \alpha^{3p^{i+j}})}{2} \bar{C}_{p^i} + \bar{C}_{4p^i} - \bar{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} - \alpha^{3p^{i+j}}) \beta^{p^{i+j}}}{2} \bar{C}_{\lambda p^i} - \frac{(\alpha^{p^{i+j}} + \alpha^{3p^{i+j}})}{2} \bar{C}_{\nu p^i} - \frac{(\alpha^{p^{i+j}} - \alpha^{3p^{i+j}}) \beta^{p^{i+j}}}{2} \bar{C}_{\chi p^i} \} \}]$$

$$\vartheta_{2p^j}(x) = \frac{1}{8p^n} [\phi(p^{n-j}) \{ \bar{C}_0 - \bar{C}_{2p^n} + \bar{C}_{4p^n} \} + p^{n-j-1} \{ \bar{C}_{2p^{n-j-1}} - \bar{C}_{4p^{n-j-1}} - \bar{C}_{8p^{n-j-1}} + \bar{C}_{\mu p^{n-j-1}} \} + \frac{1}{p^j} \sum_{i=0}^{j-1} \{ R_{i+j} \bar{C}_p - R_{i+j} \bar{C}_{\lambda p^i} + R_{i+j} \bar{C}_{\nu p^i} \}]$$

$$\begin{aligned}
 & -R_{i+j} \bar{C}_{\chi p^i} \} - \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \{ \bar{C}_{2p^i} - \bar{C}_{4p^i} - \bar{C}_{8p^i} + \bar{C}_{\mu p^i} \} \\
 \vartheta_{\lambda p^j}(x) &= \frac{1}{8p^n} [\phi(p^{n-j}) \{ \bar{C}_0 - \frac{(\alpha^{p^{n+j}} - \alpha^{3p^{n+j}})\beta^{p^{n+j}}}{n \sum_{i=1}^{n-1} \bar{C}_{p^i}} \bar{C}_{p^n} - \bar{C}_{4p^n} + \frac{(\alpha^{p^{n+j}} - \alpha^{3p^{n+j}})\beta^{p^{n+j}}}{2} \bar{C}_{vp^n} \} + \\
 p^{n-j-1} \{ & \bar{C}_{4p^{n-j-1}} - \bar{C}_{8p^{n-j-1}} \} + \frac{1}{p^i} \{ P_{i+j} \epsilon_{i-R} \epsilon_{i-T} \epsilon_{i+R} \epsilon_{i-P} \epsilon_{i+T} \epsilon_{i+} \} - \\
 \phi(p^{n-j}) & \sum_{i=n-j}^{n-1} \left\{ \frac{(\alpha^{p^{i+j}} - \alpha^{3p^{i+j}})\beta^{p^{i+j}}}{2} \bar{C}_{p^i} + \bar{C}_{4p^i} - \bar{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} + \alpha^{3p^{i+j}})}{2} \bar{C}_{\lambda p^i} - \right. \\
 & \left. \frac{(\alpha^{p^{i+j}} - \alpha^{3p^{i+j}})\beta^{p^{i+j}}}{2} \bar{C}_{vp^i} - \frac{(\alpha^{p^{i+j}} + \alpha^{3p^{i+j}})}{2} \bar{C}_{\chi p^i} \right\} \\
 \vartheta_{\mu p^j}(x) &= \frac{1}{8p^n} [\phi(p^{n-j}) \{ \bar{C}_0 - \bar{C}_{2p^n} + \bar{C}_{4p^n} \} + \\
 p^{n-j-1} \{ & \bar{C}_{2p^{n-j-1}} - \bar{C}_{4p^{n-j-1}} - \bar{C}_{8p^{n-j-1}} + \bar{C}_{\mu p^{n-j-1}} \} - \frac{1}{p^i} \sum_{i=0}^{n-1} \{ R_{i+j} \epsilon_{i-R} \epsilon_{i+R} \epsilon_{i+} \} - \\
 & \sum_{i=n-j}^{n-1} \{ \bar{C}_{2p^i} - \bar{C}_{4p^i} - \bar{C}_{8p^i} + \bar{C}_{\mu p^i} \} \\
 \vartheta_{vp^j}(x) &= \frac{1}{8p^n} [\phi(p^{n-j}) \{ \bar{C}_0 + \frac{(\alpha^{p^{n+j}} + \alpha^{3p^{n+j}})}{2} \bar{C}_{p^n} - \bar{C}_{4p^n} - \frac{(\alpha^{p^{n+j}} + \alpha^{3p^{n+j}})}{2} \bar{C}_{vp^n} \} + \\
 p^{n-j-1} \{ & \bar{C}_{4p^{n-j-1}} - \bar{C}_{8p^{n-j-1}} \} + \frac{1}{p^i} \sum_{i=0}^{n-1} \{ T_{i+j} \epsilon_{i+R} \epsilon_{i-P} \epsilon_{i-R} \epsilon_{i-T} \epsilon_{i+P} \epsilon_{i+} \} + \\
 \phi(p^{n-j}) & \sum_{i=n-j}^{n-1} \left\{ \frac{(\alpha^{p^{i+j}} + \alpha^{3p^{i+j}})}{2} \bar{C}_{p^i} - \bar{C}_{4p^i} + \bar{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} - \alpha^{3p^{i+j}})\beta^{p^{i+j}}}{2} \bar{C}_{\lambda p^i} - \frac{(\alpha^{p^{i+j}} + \alpha^{3p^{i+j}})}{2} \bar{C}_{vp^i} - \right. \\
 & \left. \frac{(\alpha^{p^{i+j}} - \alpha^{3p^{i+j}})\beta^{p^{i+j}}}{2} \bar{C}_{\chi p^i} \right\} \\
 \vartheta_{\chi p^j}(x) &= \frac{1}{8p^n} [\phi(p^{n-j}) \{ \bar{C}_0 + \frac{(\alpha^{p^{n+j}} - \alpha^{7p^{n+j}})\beta^{p^{n+j}}}{n \sum_{i=1}^{n-1} \bar{C}_{p^i}} \bar{C}_{p^n} - \bar{C}_{4p^n} - \frac{(\alpha^{p^{n+j}} - \alpha^{3p^{n+j}})\beta^{p^{n+j}}}{2} \bar{C}_{vp^n} \} + \\
 p^{n-j-1} \{ & \bar{C}_{4p^{n-j-1}} - \bar{C}_{8p^{n-j-1}} \} - \frac{1}{p^i} \sum_{i=0}^{n-1} \{ P_{i+j} \epsilon_{i+R} \epsilon_{i-T} \epsilon_{i-R} \epsilon_{i-P} \epsilon_{i+T} \epsilon_{i+} \} + \\
 \phi(p^{n-j}) & \sum_{i=n-j}^{n-1} \left\{ \frac{(\alpha^{p^{i+j}} - \alpha^{3p^{i+j}})\beta^{p^{i+j}}}{2} \bar{C}_{p^i} - \bar{C}_{4p^i} + \bar{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} + \alpha^{3p^{i+j}})}{2} \bar{C}_{\lambda p^i} - \frac{(\alpha^{p^{i+j}} - \alpha^{3p^{i+j}})\beta^{p^{i+j}}}{2} \bar{C}_{vp^i} - \right. \\
 & \left. \frac{(\alpha^{p^{i+j}} + \alpha^{3p^{i+j}})}{2} \bar{C}_{\chi p^i} \right\}
 \end{aligned}$$

Further, when $p^n \equiv 1 \pmod{4}$, then the expressions for ϑ_{p^i} , ϑ_{2p^i} , $\vartheta_{\lambda p^i}$, $\vartheta_{\mu p^i}$, ϑ_{vp^i} and $\vartheta_{\chi p^i}$ in case when $p^n \equiv 3 \pmod{4}$, are the expressions for $\vartheta_{\lambda p^i}$, $\vartheta_{\mu p^i}$, ϑ_{p^i} , ϑ_{2p^i} , $\vartheta_{\chi p^i}$ and ϑ_{vp^i} respectively, where β is a 4th root of unity in F .

The expressions for T_{i+j} , P_{i+j} and R_{i+j} are given by:

if $p^n \equiv 3 \pmod{4}$

$$\sum_{i=0}^{n-1} \frac{1}{P_{i+j}} T_{i+j}^2 + \sum_{t=0}^{n-1} \frac{1}{P_{i+j}} p^{2t} = -p^n - p^{n-1},$$

$$\sum_{t=0}^{n-1} \frac{2}{P_{i+j}} T_{i+j} P_{i+j} = (-1)^n (p^{n-1} - p^n),$$

$$R_{n-1} = \sqrt{p^{2n-1}}, \text{ and for all } j \leq n-2, R_j = 0$$

and if $p^n \equiv 1 \pmod{4}$

$$\sum_{t=0}^{n-1} \frac{2}{P_{i+j}} T_{i+j} P_{i+j} = p^n + p^{n-1},$$

$$\sum_{t=0}^{n-j-1} \frac{1}{P_{i+j}} \{T_{i+j}^2 + P_{i+j}^2\} = p^{n-1} - p^n,$$

$$R_{n-1} = \sqrt[p^{2n-1}]{-p^{2n-1}}, \text{ and for all } j \leq n-2, R_j = 0.$$

3 Dimension and Generating Polynomial

If α is primitive $8p^n$ th root of unity, then $m_s(x) = \prod_{s \in \Omega_s} (x - \alpha^s)$ denote the minimal polynomial for α^s .

Remark 3.1. [5] If $m_s(x)$ denote the minimal polynomial for $\alpha^s, s \in \Omega_s$, then the generating polynomial for cyclic code of length $8p^n$ corresponding to the cyclotomic coset Ω_s is $\frac{x^{8p^n} - 1}{m_s(x)}$ and the dimension of minimal cyclic code M_s is equal to the cardinality of the class Ω_s .

Theorem 3.2. (i) The generating polynomials for the codes M_0, M_{2p^n} and M_{4p^n} are $(1 + x + x^2 + \dots + x^{8p^n - 1}), (x^6 - x + x^2 + \dots + x^{8p^n - 1}), (x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1)(1 + x^8 + \dots + x^{8(p^n - 1)})$ respectively.

(ii) The generating polynomials for the codes M_{4p^j} and M_{8p^j} are $(x^{p^{n-j-1}} + 1)(x^{p^{n-j}} - 1)(x^{2p^{n-j}} + 1)(x^{4p^{n-j}} + 1)(1 + x^8 + \dots + x^{8p^{n-j}(p^j - 1)}), (x^{p^{n-j-1}} - 1)(x^{p^{n-j}} + 1)(x^{2p^{n-j}} + 1)(x^{4p^{n-j}} + 1)(1 + x^8 + \dots + x^{8p^{n-j}(p^j - 1)})$ respectively.

(iii) The generating polynomials for $M_{p^n} \oplus M_{vp^n}$ is $(x^4 - 1)(1 + x^8 + \dots + x^{8(p^{n-1})})$.

(iv) The generating polynomials for $M_{p^j} \oplus M_{2p^j} \oplus M_{\lambda p^j} \oplus M_{\mu p^j} \oplus M_{vp^j} \oplus M_{\chi p^j}$ is $(x^{2p^{n-j-1}} + 1)(x^{4p^{n-j-1}} + 1)(x^{2p^{n-j}} - 1)(1 + x^8 + \dots + x^{8p^{n-j}(p^j - 1)})$.

Proof. (i) The minimal polynomials for α^0, α^{2p^n} and α^{4p^n} are $(x - 1), (x^2 + 1)$ and $(x + 1)$. Then, the corresponding generating polynomials are

$$(1 + x + x^2 + \dots + x^{8p^n - 1}), (x^6 - x^4 + x^2 - 1)(1 + x^8 + \dots + x^{8(p^n - 1)})$$

and $(x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1)(1 + x^8 + \dots + x^{8(p^n - 1)})$.

(ii) The minimal polynomials for α^{4p^j} and α^{8p^j} are $\frac{(x^{p^{n-j}} - 1)}{(x^{p^{n-j-1}} + 1)}$ and $\frac{(x^{p^{n-j}} - 1)}{(x^{p^{n-j-1}} - 1)}$ respectively. Then, the corresponding generating polynomials are

$$(x^{p^{n-j-1}} + 1)(x^{p^{n-j}} - 1)(x^{2p^{n-j}} + 1)(x^{4p^{n-j}} + 1)(1 + x^8 + \dots + x^{8p^{n-j}(p^j - 1)})$$

and $(x^{p^{n-j-1}} - 1)(x^{p^{n-j}} + 1)(x^{2p^{n-j}} + 1)(x^{4p^{n-j}} + 1)(1 + x^8 + \dots + x^{8p^{n-j}(p^j - 1)})$.

(iii) The product of minimal polynomial satisfied by α^{p^n} and α^{vp^n} is $(x^4 + 1)$. Therefore, the generating polynomial for $M_{p^n} \oplus M_{vp^n}$ is $(x^4 - 1)(1 + x^8 + \dots + x^{8(p^{n-1})})$.

(iv) The product of minimal polynomial satisfied by $\alpha^{p^j}, \alpha^{2p^j}, \alpha^{\lambda p^j}, \alpha^{\mu p^j}, \alpha^{vp^j}$ and $\alpha^{\chi p^j}$ is $\frac{(x^{2p^{n-j-1}} + 1)(x^{4p^{n-j-1}} + 1)}{(x^{2p^{n-j-1}} + 1)(x^{4p^{n-j-1}} + 1)}$.

Therefore, the corresponding generating polynomial for $M_{p^j} \oplus M_{2p^j} \oplus M_{\lambda p^j} \oplus M_{\mu p^j} \oplus M_{vp^j} \oplus M_{\chi p^j}$, is $(x^{2p^{n-j-1}} + 1)(x^{4p^{n-j-1}} + 1)(x^{2p^{n-j}} - 1)(1 + x^8 + \dots + x^{8p^{n-j}(p^j - 1)})$. □

4 Minimum Distance

Lemma 4.1. If I is a cyclic code of length m generated by $g(x)$ and its minimum distance is d , then the code \bar{I} of length mk generated by $g(x)(1 + x^m + x^{2m} + \dots + x^{(k-1)m})$ is a repetition code of I repeated k times and its minimum distance is dk [3].

Theorem 4.2. *The minimum distance of the codes M_0, M_{2p^n}, M_{4p^n} are $8p^n, 4p^n, 8p^n$ respectively and for the codes M_{p^n} and M_{vp^n} are greater than or equal to $2p^n$.*

Proof. Since the generating polynomial for the code M is $(1 + x + x^2 + \dots + x^{8p^n-1})$, which is itself a polynomial of length $8p^n$, hence its minimum distance is $8p^n$.

The minimum distance of the cyclic code M_{2p^n} with generating polynomial $(x^6 - x^4 + x^2 - 1)(1 + x^8 + \dots + x^{8(p^n-1)})$ is $4p^n$ as it is repetition of the cyclic code of length 8 with generating polynomial $(x^6 - x^4 + x^2 - 1)$ whose minimum distance is 4, repeated p^n times.

In a similar way, the minimum distance of the cyclic code M_{4p^n} with generating polynomial $(x^7 - x^6 + x^5 - x^4 - x^3 + x^2 - x + 1)(1 + x^8 + \dots + x^{8(p^n-1)})$ is $8p^n$ as it is repetition of the cyclic code of length 8 with generating polynomial $(x^7 - x^6 + x^5 - x^4 - x^3 + x^2 - x + 1)$ whose minimum distance is 8, repeated p^n times.

Since the generating polynomial for the cyclic codes M_{p^n} and M_{vp^n} is $(x^4 - 1)(1 + x^8 + \dots + x^{8(p^n-1)})$, therefore, if we take a code l of length 8 generated by the polynomial $(x^4 - 1)$, then the minimum distance of this code is 2. Since the cyclic code $M(p^n, vp^n)$ generated by the polynomial $(x^4 - 1)(1 + x^8 + \dots + x^{8(p^n-1)})$ is a repetition code of the code l , repeated p^n times. Hence its minimum distance is $2p^n$. The codes corresponding to Ω_{p^n} and Ω_{vp^n} are subcodes of above code so their minimum distances are greater than or equal to $2p^n$. □

Proof for theorem 4.3 is similar to that of theorem 4.1.

Theorem 4.3. *For $0 \leq j \leq n - 1$, the minimum distance of the cyclic codes M_{4p^j} and M_{8p^j} are $16p$ and for the codes $M_{p^j}, M_{2p^j}, M_{\lambda p^j}, M_{\mu p^j}, M_{vp^j}$ and $M_{\chi p^j}$ are greater than equal to $8p$.*

5 Example

Example 5.1. *cyclic code of length 24.*

Take $p = 3, n = 1, q = 11$. Then, the q -cyclotomic cosets modulo 24 are

$$\begin{aligned} \Omega_0 &= \{0\}, \Omega_1 = \{1, 11\}, \Omega_2 = \{2, 22\}, \Omega_3 = \{3, 9\}, \Omega_4 = \{4, 20\}, \\ \Omega_6 &= \{6, 18\}, \Omega_7 = \{5, 7\}, \Omega_8 = \{8, 16\}, \Omega_{12} = \{12\}, \Omega_{13} = \{13, 23\}, \\ \Omega_{14} &= \{10, 14\}, \Omega_{15} = \{15, 21\}, \Omega_{19} = \{17, 19\} \end{aligned}$$

Also, $T_0 = 5, R_0 = 5, P_0 = 2$ and the corresponding primitive idempotents in $\frac{GF(11)[x]}{\langle x^{24} - 1 \rangle}$ are

$$\begin{aligned} \Omega_0(x) &= \frac{1}{24} [\overline{C_0} + \overline{C_1} + \overline{C_2} + \overline{C_3} + \overline{C_4} + \overline{C_6} + \overline{C_7} + \overline{C_8} + \overline{C_{12}} + \overline{C_{13}} + \overline{C_{14}} + \overline{C_{15}} + \overline{C_{19}}] \\ \Omega_1(x) &= \frac{1}{24} [2\overline{C_0} + 6\overline{C_1} + 5\overline{C_2} + 3\overline{C_3} + \overline{C_4} + 2\overline{C_7} + 10\overline{C_8} + 9\overline{C_{12}} + 5\overline{C_{13}} + 6\overline{C_{14}} + 8\overline{C_{15}} + 9\overline{C_{19}}] \\ \Omega_2(x) &= \frac{1}{24} [2\overline{C_0} + 5\overline{C_1} + \overline{C_2} + 10\overline{C_4} + 9\overline{C_6} + 6\overline{C_7} + 10\overline{C_8} + 2\overline{C_{12}} + 5\overline{C_{13}} + \overline{C_{14}} + 6\overline{C_{19}}] \\ \Omega_3(x) &= \frac{1}{24} [2\overline{C_0} + 3\overline{C_1} + 3\overline{C_3} + 9\overline{C_4} + 8\overline{C_7} + 2\overline{C_8} + 9\overline{C_{12}} + 8\overline{C_{13}} + 8\overline{C_{15}} + 3\overline{C_{19}}] \\ \Omega_4(x) &= \frac{1}{24} [2\overline{C_0} + \overline{C_1} + 10\overline{C_2} + 9\overline{C_3} + 10\overline{C_4} + 2\overline{C_6} + \overline{C_7} + 10\overline{C_8} + 2\overline{C_{12}} + \overline{C_{13}} + 10\overline{C_{14}} + 9\overline{C_{15}} + \overline{C_{19}}] \\ \Omega_6(x) &= \frac{1}{24} [\overline{C_0} + 10\overline{C_2} + \overline{C_4} + 10\overline{C_6} + \overline{C_8} + \overline{C_{12}} + 10\overline{C_{14}}] \\ \Omega_7(x) &= \frac{1}{24} [2\overline{C_0} + 2\overline{C_1} + 6\overline{C_2} + 8\overline{C_3} + \overline{C_4} + 6\overline{C_7} + 10\overline{C_8} + 9\overline{C_{12}} + 9\overline{C_{13}} + 7\overline{C_{14}} + 3\overline{C_{15}} + 5\overline{C_{19}}] \\ \Omega_8(x) &= \frac{1}{24} [2\overline{C_0} + 10\overline{C_1} + 10\overline{C_2} + 2\overline{C_3} + 10\overline{C_4} + 2\overline{C_6} + 10\overline{C_7} + 10\overline{C_8} + 2\overline{C_{12}} + 10\overline{C_{13}} \\ &+ 10\overline{C_{14}} + 2\overline{C_{15}} + 10\overline{C_{19}}] \end{aligned}$$

$$\Omega_{12}(x) = \frac{1}{24} [\overline{C_0} + 10\overline{C_1} + \overline{C_2} + 10\overline{C_3} + \overline{C_4} + \overline{C_6} + 10\overline{C_7} + \overline{C_8} + \overline{C_{12}} + 10\overline{C_{13}} + \overline{C_{14}} + 10\overline{C_{15}} + 10\overline{C_{19}}]$$

$$\Omega_{13}(x) = \frac{1}{24} [2\overline{C_0} + 5\overline{C_1} + 5\overline{C_2} + 8\overline{C_3} + \overline{C_4} + 9\overline{C_7} + 10\overline{C_8} + 10\overline{C_{12}} + 6\overline{C_{13}} + 6\overline{C_{14}} + 3\overline{C_{15}} + 2\overline{C_{19}}]$$

$$\Omega_{14}(x) = \frac{1}{24} [2\overline{C_0} + 6\overline{C_1} + \overline{C_2} + 10\overline{C_4} + 9\overline{C_6} + 5\overline{C_7} + 10\overline{C_8} + 2\overline{C_{12}} + 6\overline{C_{13}} + \overline{C_{14}} + 5\overline{C_{19}}]$$

$$\Omega_{15}(x) = \frac{1}{24} [2\overline{C_0} + 8\overline{C_1} + 8\overline{C_3} + 9\overline{C_4} + 3\overline{C_7} + 2\overline{C_8} + 9\overline{C_{12}} + 3\overline{C_{13}} + 3\overline{C_{15}} + 8\overline{C_{19}}]$$

$$\Omega_{19}(x) = \frac{1}{24} [2\overline{C_0} + 9\overline{C_1} + 5\overline{C_2} + 3\overline{C_3} + \overline{C_4} + 6\overline{C_7} + 10\overline{C_8} + 10\overline{C_{12}} + 9\overline{C_{13}} + 6\overline{C_{14}} + 8\overline{C_{15}} + 5\overline{C_{19}}]$$

Minimal polynomials of $\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^6, \alpha^7, \alpha^8, \alpha^{12}, \alpha^{13}, \alpha^{14}, \alpha^{15}$ and α^{19} are

$x - 1, x^2 - 5x - 1, x^2 - 5x + 1, x^2 - 8x - 1, x^2 - x + 1, x^2 + 1, x^2 - 2x - 1, x^2 + x + 1, x + 1, x^2 + 5x - 1,$
 $x^2 - 6x + 1, x^2 + 8x - 1$ and $x^2 + 2x - 1$ respectively.

The minimal codes $M_0, M_1, M_2, M_3, M_4, M_6, M_7, M_8, M_{12}, M_{13}, M_{14}, M_{15}$ and M_{19} of length 24 are as follows:

Code	Dimension	Minimum Distance	Generating Polynomial
M_0	1	24	$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19} + x^{20} + x^{21} + x^{22} + x^{23}$
M_1	2	$8 \leq d \leq 22$	$1 + 6x + 4x^2 + 8x^3 + 8x^4 + x^5 + 3x^6 + 8x^7 + 7x^8 + 6x^9 + 10x^{10} + 10x^{12} + 5x^{13} + 7x^{14} + 3x^{15} + 3x^{16} + 10x^{17} + 8x^{18} + 3x^{19} + 4x^{20} + 5x^{21} + x^{22}$
M_2	2	$8 \leq d \leq 20$	$10 + 6x + 9x^2 + 6x^3 + 10x^4 + 8x^5 + x^6 + x^8 + 3x^9 + 10x^{10} + 10x^{12} + 8x^{13} + x^{14} + x^{16} + 3x^{17} + 10x^{18} + 10x^{20} + 8x^{21} + x^{22}$
M_4	2	16	$-1 - x + x^3 + x^4 - x^6 - x^7 + x^9 + x^{10} - x^{12} - x^{13} + x^{15} + x^{16} - x^{18} - x^{19} + x^{21} + x^{22}$
M_6	2	12	$-1 + x^2 - x^4 + x^6 - x^8 + x^{10} - x^{12} + x^{14} - x^{16} + x^{18} - x^{20} + x^{22}$
M_7	2	$8 \leq d \leq 22$	$1 + 9x + 5x^2 + 10x^3 + 7x^4 + 7x^5 + 4x^6 + 10x^7 + 6x^8 + 9x^9 + 10x^{10} + 10x^{12} + 2x^{13} + 6x^{14} + x^{15} + 4x^{16} + 4x^{17} + 7x^{18} + x^{19} + 5x^{20} + 2x^{21} + x^{22}$
M_8	2	16	$-1 + x - x^3 + x^4 - x^6 + x^7 - x^9 + x^{10} - x^{12} + x^{13} - x^{15} + x^{16} - x^{18} + x^{19} - x^{21} + x^{22}$
M_{12}	1	24	$-1 + x - x^2 + x^3 - x^4 + x^5 - x^6 + x^7 - x^8 + x^9 - x^{10} + x^{11} - x^{12} + x^{13} - x^{14} + x^{15} - x^{16} + x^{17} - x^{18} + x^{19} - x^{20} + x^{21} - x^{22} + x^{23}$
M_{13}	2	$8 \leq d \leq 22$	$1 + 5x + 4x^2 + 3x^3 + 8x^4 + 10x^5 + 3x^6 + 3x^7 + 7x^8 + 5x^9 + 10x^{10} + 10x^{12} + 6x^{13} + 7x^{14} + 8x^{15} + 3x^{16} + x^{17} + 8x^{18} + 8x^{19} + 4x^{20} + 6x^{21} + x^{22}$
M_{14}	2	$8 \leq d \leq 20$	$10 + 5x + 9x^2 + 5x^3 + 10x^4 + x^6 + 6x^7 + 2x^8 + 6x^9 + x^{10} + 10x^{12} + 5x^{13} + 9x^{14} + 5x^{15} + 10x^{16} + x^{18} + 6x^{19} + 2x^{20} + 6x^{21} + x^{22}$
M_{15}	2	$6 \leq d \leq 18$	$1 + 8x + 10x^2 + 10x^4 + 3x^5 + x^6 + x^8 + 8x^9 + 10x^{10} + 10x^{12} + 3x^{13} + x^{14} + x^{16} + 8x^{17} + 10x^{18} + 10x^{20} + 3x^{21} + x^{22}$
M_{19}	2	$8 \leq d \leq 22$	$1 + 2x + 5x^2 + x^3 + 7x^4 + 4x^5 + 4x^6 + x^7 + 6x^8 + 2x^9 + 10x^{10} + 10x^{12} + 9x^{13} + 6x^{14} + 10x^{15} + 4x^{16} + 7x^{17} + 7x^{18} + 10x^{19} + 5x^{20} + 9x^{21} + x^{22}$

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