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# Fuzzy Maximal $\mu$-open and Minimal $\mu$-closed Sets via Generalized Fuzzy Topology

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## Abstract

In this study, we propose the notions of fuzzy maximal $\mu$ -open and fuzzy minimal $\mu$ -closed sets within the framework of generalized topological spaces. We further investigate their fundamental properties and examine their roles in the underlying structure.

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## 1. Introduction

Zadeh[4] was the pioneer of fuzzy set theory which established in 1965. In 1968, Chang[1] brought forth the idea of fuzzy topology. In the realm of classical topology, Csa´sza´r, established the framework of generalized topology[3] . Let $1^X$ signifies power set of the non empty set X. A fuzzy subcollection $\mu$ of $1^X$ is defined as a generalized fuzzy topology [2] on X if $0_X \in \mu$ and $\vee\{\xi_k, k\epsilon\Delta\}$ whenever k $\epsilon$ $\Delta$ and (X, $\mu$) mean to generalized fuzzy topological space. A fuzzy set is $\xi \in \mu$ called as fuzzy $\mu$-open[2] of (X,$\mu$) . The complement of fuzzy $\mu$ -open set is known as fuzzy $\mu$ -closed of (X, $\mu$) . A fuzzy subset $\xi$ of X, the intersection of all fuzzy $\mu$ -closed sets containing $\xi$ is the generalized fuzzy closure [4] of $\xi$ and is denoted by $c_\mu(\xi)$. Also for a fuzzy subset $\xi$ of X, the union of all fuzzy $\mu$ -open sets contained in $\xi$ is the generalized fuzzy interior [4] of $\xi$ and is denoted by $i_\mu(\xi)$ . For our better understanding, a fuzzy subset $\xi$ of X is $\mu$ -open (resp. fuzzy $\mu$ -closed) iff
$\xi = i_\mu(\xi)$ (resp. fuzzy $\xi = c_\mu(\xi)$). Also for a fuzzy subset $\xi$ of X. we have
$c_\mu(\xi) = 1_X - i_\mu(\xi)$. The aim of this paper is to define a novel class of fuzzy $\mu$-open sets, referred to as fuzzy maximal $\mu$ -open sets, along with the notion of fuzzy minimal $\mu$ -closed sets, and to investigate their basic structural properties.
Throughout the paper X stands for generalized fuzzy topological space (X, $\mu$) or GFTS.

## 2. Fuzzy Maximal μ-open and Fuzzy Minimal μ-closed Sets

**Definition 2.1.** A proper nonempty fuzzy μ-open set ξ of a GFTS (X,μ) is called a fuzzy maximal μ-open set if there is no fuzzy μ-open set strictly between ξ and X.

**Definition 2.2.** A proper nonempty fuzzy μ-closed set γ of a GFTS (X,μ) is called a fuzzy minimal μ-closed set if there is no fuzzy μ-closed set strictly between 0 and γ.

**Theorem 2.1.** Let ξ be a proper nonempty fuzzy set of a GFTS (X,μ). Then ξ is fuzzy maximal μ-open if and only if X\ξ is fuzzy minimal μ-closed.
**Proof.** Assume that ξ be fuzzy maximal μ-open set and γ be a fuzzy μ-closed set such that γ ⫅ X\ξ. Hence ξ ⫅ X \ γ ∈ μ. Since ξ is fuzzy maximal μ-open, then X\ γ = X, then is either ξ or X. X\ γ=ξ, then γ=X\ξ and if X\ γ = X then γ =∅. Thus in any case X\ξ becomes fuzzy minimal μ-closed set.

Conversely let ζ be a fuzzy minimal μ-closed set and ρ be a fuzzy μ-open set containing X\B. So X\ρ ⫅ ζ. Now ζ being fuzzy minimal μ-closed, X\ρ is either 0 or ζ. If X\ρ=0, then ρ=X and if X\ρ = ζ, then ρ = X\ζ. Hence in either case, X\ζ becomes a fuzzy maximal μ-open set.

**Example 2.2.** Let X = {a, b, c}. Then fuzzy sets $\gamma_1 = \frac{1}{a} + \frac{1}{b} + \frac{0}{c}$ ; $\gamma_2 = \frac{0}{a} + \frac{1}{b} + \frac{1}{c}$ and $\gamma_3 = \frac{0}{a} + \frac{0}{b} + \frac{1}{c}$ are defined as follows: Consider the fuzzy generalized topology $\mu = \{0_X, \gamma_1, \gamma_2, \gamma_3, 1_X\}$. Hence $\gamma_3$ is not fuzzy maximal μ-open set but it is a fuzzy μ-open set.

**Theorem 2.3.** Let ξ be a fuzzy maximal μ-open set and ζ be a fuzzy μ-open sets and let γ and λ be two fuzzy minmal μ-closed sets in X. The subsequent assertions are equivalent:

*(i)* Then ξ ∪ζ = X or ζ ⫅ ξ.
*(ii)* Either ξ ∪ζ = X or ξ = ζ.
*(iii)* Then γ ∩ λ=0 or γ ⫅ λ for any fuzzy μ-closed set λ.
*(iv)* Either γ ∩λ = 0 or γ=λ.

**Proof:** (i) In either case ξ ∪ ζ = X or ξ ∪ ζ ≠ X. If ξ ∪ ζ ≠ X, then ξ ∪ ζ is a fuzzy μ-open set such that ξ ⫅ ξ ∪ζ which gives ξ ∪ζ = ξ (since ξ ∪ζ ≠ X). Hence ζ ⫅ ξ..

(ii) Case is over if ξ ∪ ζ =X. If ξ ∪ ζ ≠ X , then ξ ∪ ζ is a fuzzy μ-open set such that ξ, ζ ⫅ ξ ∪ ζ implies ξ ∪ζ =ξ and ξ ∪ζ=ζ. Hence ξ = ζ.
(iii) Analogous to (i).
(iv) Analogous to (ii).

**Theorem 2.4.** Let ξ be a fuzzy maximal μ-open set and ζ be a fuzzy μ-open set in X. The subsequent assertions are equivalent.
*(i)* Then X \ ξ ⫅ ζ and $x_\alpha$ ∈ X \ ξ for any fuzzy μ-open set ζ containing $x_\alpha$.
*(ii)* Then one of the subsequent (a) or (b) holds:
*(a)* For each fuzzy μ-open set ζ containing $x_\alpha$ and $x_\alpha$ ∈ X \ ξ, ζ = X.
*(b)* There exists a fuzzy μ-open set ζ such that X \ ξ ⫅ ζ and ζ ⫅ X.
*(iii)* Then one of the subsequent (a) or (b) preserves:
*(a)* For each fuzzy μ-open set ζ containing $x_\alpha$ and each $x_\alpha$ ∈ ξ, X \ ξ ⊋ ζ.
*(b)* There exists a fuzzy μ-open set ζ such that X \ ξ = ζ (≠ X).
**Proof:** (i) As $x_\alpha$ ∈ X \ ξ, we acquire ζ ⊈ ξ for any fuzzy μ-open set ζ containing $x_\alpha$. In accordance with the Theorem 2.3 (i), ξ ∪ ζ = X or ζ ⊆ ξ ⟹ X \ ξ ⊆ ζ.

(ii) Case is over if (a) preserves. Let (a) don't hold. Then there is an existing element $x_\alpha \in X \setminus \xi$ and a fuzzy μ-open set ζ containing $x_\alpha$ such that $\zeta \nsubseteq X$. In accordance with the Theorem 2.5 (i), $\xi \cup \zeta = X$ or $\zeta \subseteqq \xi$. $Z \not\subset \xi \Longrightarrow \xi \cup \zeta = X \Longrightarrow X \setminus \xi \subseteqq \zeta$.

(iii) Case is over if (b) preserves. Let (b) don't bear.Then (by (i)) for each $x_\alpha \in X \setminus \xi$ and each fuzzy μ-open set ζ containing $y_\beta$ $X \setminus \xi \subseteq \zeta$ .Therefore by assumption $X \setminus \xi \subsetneqq \zeta$ .

**Theorem 2.5.** Let ξ, ζ, η be fuzzy maximal μ-open sets in a GFTS (X,μ) such that $\xi \neq \zeta$ . If $\xi \cap \zeta \subseteqq \eta$, then either $\xi = \eta$ o r $\zeta = \eta$.

**Proof:** Case is over if $\xi = \eta$. If $\xi \neq \eta$, then it is necessary to show that $\zeta = \eta$. Let $\zeta \cap \eta = \zeta \cap (\eta \cap X) = \zeta \cap (\eta \cap (\xi \cup \zeta))$ (by Theorem 2.3 (ii)) $= \zeta \cap [(\eta \cap \xi) \cup (\eta \cap \zeta)] = (\zeta \cap \eta \cap \xi) \cup (\zeta \cap \eta) = (\xi \cap \zeta) \cup (\zeta \cap \eta) = \zeta \cap (\xi \cup \eta) = \zeta \cap X = \zeta \Longrightarrow \zeta \subseteqq \eta$. So in light of definition of fuzzy maximal μ-open set it follows that $\zeta = \eta$.

**Theorem 2.6.** Let ξ, ζ and η be three distinct fuzzy maximal μ-open sets in X. Then $\xi \cap \zeta \subsetneqq \xi \cap \eta$.

**Proof:** If possible, let $\xi \cap \zeta \subseteqq \xi \cap \eta$. Then $(\xi \cap \zeta) \cup (\zeta \cap \eta) \subseteqq (\xi \cap \eta) \cup (\zeta \cap \eta)$. So $\zeta \cap (\xi \cup \eta) \subseteqq \eta \cap (\xi \cup \zeta)$. In accordance with the Theorem 2.3 (ii), $\xi \cup \eta = X = \xi \cup \zeta \Rightarrow \zeta \subseteqq \eta$. So in light of definition of fuzzy maximal μ-open set it follows that $\zeta = \eta$, a contradiction.

**Theorem 2.7.** (I) Let γ be a fuzzy minimal μ-closed set of X. If $x_\alpha \in \gamma$, then $\gamma \subseteqq \lambda$ for any fuzzy μ-closed set λ containing $x_\alpha$.
(ii) Let γ be a fuzzy minimal μ-closed set of X. Then $\gamma = \cap \{\lambda : x_\alpha \in \lambda$ and λ is fuzzy μ–closed}.

**Proof:** (i) Let $x_\alpha \in \gamma$ and λ be a fuzzy μ-closed set such that $x_\alpha \in \lambda$. Then $\gamma \cap \lambda \neq 0$. In accordance with the Theorem 2.3(iii), $\gamma \subseteqq \lambda$ .

(ii) By (i) it is clear that $\gamma \subseteqq \cap \{\lambda : x_\alpha \in \lambda$ and λ is fuzzy μ-closed}. Let $x_\alpha \in \cap \{\lambda : x_\alpha \in \lambda$ and λ is fuzzy-closed} $\Longrightarrow x_\alpha \in \lambda$ for all fuzzy μ-closed set $\lambda \Longrightarrow x_\alpha \in \gamma$ (as γ is fuzzy μ-closed) $\Longrightarrow \cap \{\lambda : x_\alpha \in \lambda$ and λ is fuzzy μ-closed} =γ. It could be a co-finite set if its complement is finite.

**Theorem 2.8.** Let ξ be a proper nonempty co-finite fuzzy μ-open subset of X. Then there exists at least one (co-finite) fuzzy maximal μ-open set ζ such that $\xi \subseteqq \zeta$.

**Proof:** Suppose ξ is a fuzzy maximal μ-open set, then $\xi = \zeta$. If ξ is not a fuzzy maximal μ-open set, then we can acquire a co-finite fuzzy μ-open set $\xi_1$ such that $\xi \subseteqq \xi_1$ ($\neq X$). If $\xi_1$ is a fuzzy maximal μ-open set, then we acquire $\zeta = \xi_1$. If $\xi_1$ is not a fuzzy maximal μ-open set, then we can acquire a fuzzy maximal μ-open set $\xi_2$ ($\neq X$) such that $\xi \subsetneqq \xi_1 \subsetneqq \xi_2$. As we proceed with this method, we eventually acquire a sequence of fuzzy μ-open sets such that $\xi \subsetneqq \xi_1 \subsetneqq \xi_2 \subsetneqq \xi_3 \subsetneqq \ldots \subsetneqq \xi_n \subsetneqq \ldots$.Since ξ is a co-finite fuzzy set, only finite number of occurence of the process and eventually we acquire a fuzzy maximal μ-open set $\zeta = \xi_k$ for one positive integer $k$.

**Theorem 2.9.** (i) Let $\{\Omega_\kappa : \kappa \in \Lambda\}$, Ω be a fuzzy minimal μ-closed sets in X. If $\Omega \subseteqq$

$\bigcup_{k\in\Lambda}\Omega_k$. Then there exists $k_0 \in \Lambda$ such that $\Omega = \Omega_{k_0}$.

(ii) Let $\{\Omega_\kappa: \kappa \in \Lambda\}$, $\Omega$ be a fuzzy minimal μ-closed sets. If $\Omega \neq \Omega_\kappa$ for any $k \in \Lambda$, then

$$\left(\bigcup_{k\in\Lambda}\Omega_k\right) \cap \Omega = 0$$

**Proof:**(i) We need to show that $\Omega \cap \Omega_{k_0} \neq 0$, for at least one $\kappa_0 \in \Lambda$. If $\Omega \cap \Omega_\kappa = 0$ for each $\kappa \in \Lambda$, then $\Omega_\kappa \subseteqq X \setminus \Omega$ for each $\kappa \in \Lambda$. Hence $\Omega \subseteqq \bigcup_{k\in\Lambda}\Omega_k \subseteqq X \setminus \Omega$, a contradiction. Hence $\Omega \cap \Omega_{k_0} \neq 0$ for some $\kappa_0 \in \Lambda$. Now $\Omega \cap \Omega_{k_0} \subseteqq \Omega, \Omega_{x_0}$. Since $\Omega$ is fuzzy minimal μ-closed set, $\Omega \cap \Omega_{k_0} = \Omega \Longrightarrow \Omega \subseteqq \Omega_{k_0}$. Analogously $\Omega_{k_0} \subseteqq \Omega \Longrightarrow \Omega = \Omega_{k_0}$.

(ii) Suppose that $(\bigcup_{k\in\Lambda}\Omega_k) \cap \Omega \neq 0$. Then we could have $\kappa \in \Lambda$ such that $\Omega_\kappa \cap \Omega \neq 0$. In accordance with the Theorem 2.3(iv), $\Omega = \Omega_\kappa$ for any κ, a contradiction.

# 3. Fuzzy μ-closure, μ-interior and maximal μ-open sets

**Theorem 3.1.** Let ξ be a fuzzy maximal μ-open set in X. Then either $c_\mu(\xi) = X$ or $c_\mu(\xi) = \xi$ .

**Proof :** Merely the succeeding cases (a) and (b) arise using Theorem 2.4(iii) as ξ is a fuzzy maximal μ -open set.

(a) For each fuzzy μ-open set ζ containing $x_\alpha$ and each $x_\alpha \in X \setminus \xi$, we acquire $X \setminus \xi \subsetneqq \zeta$.. Let ζ be any fuzzy μ-open set containing $y_\beta$ and $x_\alpha \in X \setminus \xi$. Since $X \setminus \xi \neq \zeta$, we acquire $\zeta \cap \xi \neq 0$ and hence $X \setminus \xi \subseteqq c_\mu(\xi)$. Since $X = \xi \cup (X \setminus \xi) \subseteqq \xi \cup c_\mu(\xi) = c_\mu(\xi) \subseteqq X$, $X = c_\mu(\xi)$.

(b) There is a fuzzy μ-open set ζ such that $X \setminus \xi = \zeta\ (\neq X)$. Since $X \setminus \xi = \zeta$, a fuzzy μ-open, ξ is a fuzzy μ-closed set $\Longrightarrow \xi = c_\mu(\xi)$.

**Theorem 3.2.** Let ξ be a fuzzy maximal μ-open set in X. Then either $i_\mu(X\setminus\xi) = X\setminus\xi$ or $i_\mu(X\setminus\xi) = 0$ .

**Proof :** By using Theorem 3.1, we acquire $c_\mu(\xi) = \xi$ or $c_\mu(\xi) = X$, i.e., $i_\mu(X\setminus\xi) = X \setminus \xi$, or $i_\mu(X\setminus\xi) = 0$.

**Theorem 3.3.** Let ξ be a fuzzy maximalμ-open set and ζ be a nonzero fuzzy subset of $X\setminus\xi$ in X . Then $c_\mu(\zeta) = X\setminus\xi$

**Proof :** As $0 \neq \zeta \subseteqq X\setminus\xi$, $\rho \cap \zeta \neq 0$ for any element $x_\alpha \in X \setminus \xi$ and any μ-open set ρ containing $x_\alpha$ in accordance with the Theorem 2.4 (i). Likewise $X \setminus \xi \subseteqq c_\mu(\zeta)$. Since $X\setminus\xi$ is fuzzy μ-closed and $\zeta \subseteqq X\setminus\xi$, we acquire $c_\mu(\zeta) \subseteqq X\setminus\xi$.

**Corollary 3.4.** Let ξ be a fuzzy maximal μ-open set and $\xi \subsetneqq \zeta$ . Then $c_\mu(\zeta) = X$ .

**Proof :** Since $\xi \subsetneqq \zeta$ c X, there exists a nonzero fuzzy subset γ of $X \setminus \xi$ such that $\zeta = \xi \cup \gamma$. Therefore we acquire, $c_\mu(\zeta) = c_\mu(\xi \cup \gamma) \supseteqq c_\mu(\xi) \cup c_\mu(\gamma) \supseteqq \xi \cup (X\setminus\xi)$ (by Theorem 3.1) = X .

**Theorem 3.5.** Let ξ be a fuzzy maximal μ-open set and let $X \setminus \xi$ have at least two elements. Then $c_\mu(X\setminus \{x_\alpha\}) = X$ for any element a of $X \setminus \xi$ .

**Proof:** As $\xi \subsetneqq X \setminus \{x_\alpha\}$ and by means of Corollary 3.4, $c_\mu(X \setminus \{x_\alpha\}) = X$.

**Theorem 3.6.** Let ξ be a fuzzy maximal μ-open set and β be a subset of (X,μ) with $\xi \subsetneqq \beta$. Then $i_\mu(\beta) = \xi$.

**Proof :** If $\lambda = \xi$, then $i_\mu(\lambda) = i_\mu(\xi) = \xi$. If $\lambda \neq \xi$, then we acquire $\xi \subsetneqq \lambda$. In this way, $\xi \subseteqq i_\mu(\lambda)$. Since $\xi$ is fuzzy maximal μ-open, we acquire $i_\mu(\lambda) \subseteqq \xi$. Hence $i_\mu(\lambda) = \xi$.

**Theorem 3.7.** Let $\xi$ be a fuzzy maximal μ-open set and $\gamma$ be a nonzero fuzzy subset of $X \backslash \xi$ . Then $X \backslash c_\mu(\gamma) = \xi$

**Proof :** Since $\xi \subseteqq X \backslash \gamma \subsetneqq X$, based on assumption and by using Theorem 3.3, $X \backslash c_\mu(\gamma) = \xi$.

## 4. Fuzzy Almost Minimal μ-closed and Fuzzy Almost Minimal μ-open Sets

**Definition 4.1.** A fuzzy subset $\vartheta \subset X$ is said to be fuzzy almost maximal μ-open if there exists a fuzzy maximal μ-open set $\rho$ such that $\rho \subseteqq \vartheta \subseteqq c_\mu(\rho)$.

**Remark 4.1.** Every fuzzy maximal μ-open set (fuzzy minimal μ-closed set) is a fuzzy maximal μ-open set (resp. an fuzzy almost minimal μ-closed set).

**Example 4.2.** Let X = { a, b, c, d}. Then fuzzy sets $\beta_1 = \frac{1}{a} + \frac{0}{b} + \frac{1}{c} + \frac{1}{d}$; $\beta_2 = \frac{1}{a} + \frac{1}{b} + \frac{0}{c} + \frac{0}{d}$; $\beta_3 = \frac{1}{a} + \frac{0}{b} + \frac{0}{c} + \frac{0}{d}$ and $\beta_4 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{0}{d}$ are defined as follows: Consider the fuzzy generalized topology $\mu = \{0_X, \beta_1, \beta_2, \beta_3, 1_X\}$. Here $\gamma_4$ is not fuzzy maximal μ-open Set (since it is not fuzzy μ-open set) but it is almost fuzzy maximal μ-open set.

**Theorem 4.3.** If $\vartheta$ is a fuzzy maximal μ-open set and $\vartheta \subseteqq \zeta \subseteqq c_\mu(\vartheta)$, then $\zeta$ is also a fuzzy almost maximal μ-open set.

**Proof:** As $\vartheta$ is an fuzzy almost maximal μ-open set, there exists a fuzzy maximal μ-open set $\rho$ such that $\rho \subseteqq \vartheta \subseteqq c_\mu(\rho)$ and $\rho \subseteqq \vartheta \subseteqq \zeta \subseteqq c_\mu(\vartheta) \subseteqq c_\mu(\rho) \Longrightarrow \zeta$ is also a fuzzy almost maximal μ-open set.

**Theorem 4.4.** A subset $\gamma$ of a GFTS X is fuzzy almost minimal μ-closed iff there exists a fuzzy minimal μ-closed set $\lambda \subseteqq X$ such that $i_\mu(\lambda) \subseteqq \gamma \subseteqq \lambda$.

**Proof :** Let $\gamma$ be a fuzzy almost minimal μ-closed set. Hence a fuzzy maximal μ-open set $\rho$ such that $\rho \subseteqq X \backslash \gamma \subseteqq c_\mu(\rho) \Longrightarrow i_\mu(X \backslash \rho) = X \backslash c_\mu(\rho) \subseteqq \gamma \subseteqq X \backslash \rho$. Put $X \backslash \rho = \lambda$. Then $\lambda$ is a fuzzy minimal μ-closed set such that $i_\mu(\lambda) \subseteqq \gamma \subseteqq \lambda$.

Conversely let for $\gamma \subseteqq X$, there exist a fuzzy minimal μ-closed set $\lambda$ such that $i_\mu(\lambda) \subseteqq \gamma \subseteqq \lambda$. Then $X \backslash \lambda \subseteqq X \backslash \gamma \subseteqq X \backslash i_\mu(\lambda) = c_\mu(X \backslash \lambda)$. So a fuzzy maximal μ-open set $\rho = X \backslash \lambda$ such that $\rho \subseteqq X \backslash \gamma \subseteqq c_\mu(\rho) \Longrightarrow X \backslash \gamma$ is an fuzzy almost maximal μ-open set. So $\gamma$ is an fuzzy almost minimal μ-closed set.

**Theorem 4.5.** Let $\beta$ be a fuzzy almost minimal μ-closed set. If $i_\mu(\beta) \subseteqq \gamma \subseteqq \beta$, then $\gamma$ is also a fuzzy almost minimal μ-closed set.

**Proof :** As $\beta$ is a fuzzy almost minimal μ-closed set, there exists a fuzzy minimal μ-closed set $\lambda$ such that $i_\mu(\gamma) \subseteqq \lambda \subseteqq \gamma$ (using Theorem 4.4). So $i_\mu(\gamma) \subseteqq i_\mu(\lambda) \subseteqq \gamma \subseteqq \lambda$. Therefore $i_\mu(\lambda) \subseteqq \gamma \subseteqq \lambda \Longrightarrow \gamma$ is a fuzzy almost minimal μ-closed set.

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