Batch arrival Queue with dual-stage service with an additional service and multi-phase Bernoulli governed vacations under N-Policy

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Abstract

We consider a batch arrival queueing system where the server provides dual stages of heterogeneous service with an optional stage under a modified Bernoulli schedule and N-policy. The server stays idle until the queue size reaches N (≥1). As soon as the queue size becomes at least N, it begins serving each customer through primary and secondary stages of service. After the dual stages of service, the customer leaves with probability (1−α) or receives optional service with probability α. Upon completing all service stages, the server enters a vacation mode governed by a Bernoulli schedule: with probability (1−β), it serves the next customer; with probability β, it takes a k-phases of vacation. After this, the server may take an optional K+1 phase vacation with probability or resume service with (1−θ). We derive queue size distribution and performance measures using generating functions, and propose an optimal threshold N to minimize total cost.

Keywords: Batch arrivals, Dual-stage service, Optional service, Bernoulli vacation, k-phases of vacation, N-policy, Queue length characteristics, Performance analysis, Cost optimization.

1 Introduction

The study of queueing system is a powerful tool used to model service systems in diverse areas such as manufacturing, computer networks, and healthcare. A key extension of traditional models involves server vacations, where servers may take breaks according to specific rules, improving realism and flexibility. One notable policy, the N-policy, activates service only when the queue reaching a specified threshold, first developed by Yadin and Naor [25]. Since then, extensive research has evolved around vacation queues. For instance, Levy and Yechiali [20] provided a foundational survey of vacation models and Doshi [8] provided a queueing systems with vacations survey. The classical Bernoulli schedule vacation discipline was developed by Keilson and Servi [14]. Batch arrival

$M^{[X]}/G/1$ queueing system with multiple vacations were first studied by Baba [3]. The first study of batch arrival queue with control policy (N-policy) by Kella [15], Lee and Srinivasan [17], and Takagi [24] was first proposed the concept of a variant vacation for the single arrival M/G/1 regular system. Krishna and Lee [16], Doshi [9], Selvam and Sivasankaran [23] and Lee et al. [18], those papers mentioned above are characterized by a common feature; the second stage of service is provided only to a proportion of the original incoming customers, motivation for this type of model also comes from some communication and computer networks where messages are processed in two stages by a single server. These vacation model variations with N-policy were expanded upon by Lee et al [19]. Madan [21] investigated Bernoulli vacation models for queueing system involving two heterogeneous service stages. Anabosi and Madan [1], Artalejo and Choudhury [2], along with many others have made notable contributions to the study of vacation models. Choudhury and Madan [4] addressed batch arrivals with a vacation time under Bernoulli schedule. Ke [11] derived the system characteristics of the model and determined the optimal N-policy threshold that minimizes cost; in this direction, several researchers have made notable contributions.

Choudhury and Madhuchanda [6] analyzed a batch arrival queueing system with an additional service channel operating under an N-policy. Choudhury and Madan [5] investigated a batch arrival queueing system with heterogeneous service governed by a modified Bernoulli schedule under N-policy. Later, Choudhury et al. [7] carried out a steady-state analysis of an $M^{[X]}/G/1$ queue with two-phase service and a Bernoulli vacation schedule under a multiple vacation policy. In the same year, Ke [12] applied the supplementary variable technique to study an $M^{[X]}/G/1$ queue with balking under a variant vacation policy. Ke et al. [13] examined the operating characteristics of an $M^{[X]}/G/1$ queueing system governed by an N-policy with a maximum of J vacations. In their model, the server is allowed to take up to consecutive vacations and resumes service only when, upon returning from vacation, there are at least customers waiting in the queue. Manoharan and Sankara Sasi [22] studied an M/G/1 reneging queueing system with second optional service and with second optional vacation. Kalyanaraman and Shanthi [10] investigated an M/G/1 queueing system incorporating k-phase vacations and a state-dependent arrival rate. In their model, the server initiates a vacation comprising k-phases whenever the system becomes empty following a service completion.

This paper presents a generalization of the batch arrival two-stage service queueing model with a modified Bernoulli vacation schedule under N-policy, previously analyzed by Choudhury and Madan [5], through the inclusion of k-phase vacations, optional service, and optional vacation.

For example, in a manufacturing plant, where a machine processes items in batches and stays idle until items accumulate (N-policy). Each item undergoes two stages of processing, with a probability of needing optional rework. After processing, the machine may begin another job or enter multi-phase maintenance (vacation), potentially followed by optional extra maintenance. This practical scenario motivates the analysis of such queueing models, where determining the optimal threshold helps minimize operational cost while accounting for realistic service behaviours like rework and maintenance.

The structure of the paper is as follows. Section 2 outlines the system model, including the necessary assumptions and notation. Sections 3, 4, and 5 present the steady-state equations, developed using the supplementary variable technique. In Section 6, the probability generating functions for the queue length distributions at both arbitrary and departure epochs are derived. Section 7 discusses the average queue lengths at these epochs, derives results from earlier models as special cases, and establishes key performance measures. Section 8 introduces the cost optimization function and determines the optimal threshold value. Section 9 presents a discussion of the special cases. Numerical results supporting the analysis are provided in Section 10. Finally, Section 11 concludes the paper with a summary of findings.

2 System Model

An optimal batch arrival queueing system with dual-stage compulsory service with an additional optional service under modified Bernoulli schedule k-phases of vacation with an extended optional k+1 vacation is considered. To formulate the system model, the following assumption are made.

- The First-Come, First-Served (FCFS) discipline governs the service order.
- Customers arrive in batches of size `X_i′ according to compound Poisson process, where the batch size follows a probability mass function $Pr[X_i = m] = c_m$ m=1,2,3, ... with $\sum_{m=1}^{\infty} c_m = 1$. The PGF of X_i is denoted by $Y(z) = \sum_{m=1}^{\infty} z^m c_m$. The m^{th} factorial moment of X_i, given by $E[X_i^m] = E[X_i(X_i - 1)\ldots(X_i - m + 1)]$ is assumed to the infinite, i,e., $E[X_i^m] < \infty$,
- The server does not begin service until at least N (with $N \geq 1$) customers have accumulated in the queue. Once the queue length meets or exceeds this threshold, the server immediately begins operation, providing a dual-stage compulsory service to each customer. Specifically, every customer first undergoes the Primary Stage Service (PSS), with service time denoted by B_1, followed by the Secondary Stage Service (SSS), with service time denoted by B_2. After completing the SSS, a customer may either leave the system with probability (1-α), or receive an Optional Stage Service (OSS) with probability α, where ($0 \leq \alpha \leq 1$). The service times B_1, B_2, and B_3 are generally distributed with distribution functions (DF), $B_i(x)$, i=1,2,3 respectively. Their Laplace-Stieltjes transforms are given by, $B_i^*(s)$, and each of these service times is assumed to have finite moments $E[B_i^a]$, for $a \leq 1$, i=1,2,3.
- Once a customer completes all stages of service, the server may start the next service with probability (1-β) or, with probability β ($0 \leq \beta \leq 1$), it initiates a vacation consisting of k-phases, denoted by V_j, for j=1,2, …., k. After completing all k-phases of vacation, the server may take an additional vacation V_{j+1}, for j=1,2,…,k with probability θ ($0 \leq \theta \leq 1$), or return to the system with probability 1- θ. Upon returning, the server remains idle until the queue size reaches the threshold N (with $N \geq 1$). Each vacation time V_i (where i=j, j+1 and j=1,2, …., k) is generally distributed with distribution function (DF), and its corresponding Laplace-Stieltjes Transform (LST) is denoted by $V_i^*(s)$. All vacation times are assumed to have finite moments $E[V_i^a]$, for $a \leq 1$, i=j, j+1 and j=1,2, …., k and are independent of both the service times B_i and the arrival process.

Now, the modified vacation period is,

$$V=\begin{cases}\sum_{j=1}^{k} V_j & with\ probability\ \beta,\\ \sum_{j=1}^{k} V_{j+1} & with\ probability\ \theta.\end{cases}$$

3 Queue Length Characteristics

In this section, we formulate the system of state equations for the queue length characteristics at an arbitrary time point by incorporating supplementary variables. These variables represent the elapsed service times for the three service stages and the elapsed vacation times for the vacation k-phases as well as the optional k+1 phase. We then solve these equations and derive the PGFs of the queue length characteristics. The analysis is carried out under the assumption that the system has reached a steady-state. Let $\delta(t)$ denote the number of customers in the queue at time t. Define B_i^0 as the elapsed service time at time t for the i^{th} service stage—namely, the Primary Stage Service (PSS), Secondary Stage Service (SSS), and Optional Stage Service (OSS) for i=1,2,3, respectively. Similarly, let V_j^0 and V_{j+1}^0 denote the elapsed times of the j^{th} phase of vacation and the optional $(j+1)^{th}$ vacation phase at time t, for j=1,2, …., k .

Let the random variable $\gamma(t)$ is defined as,

$$\gamma(t)=\begin{cases}0, & if\ the\ server\ is\ idle\ at\ time\ `t',\\ 1, & if\ the\ server\ is\ busy\ with\ PSS\ at\ time\ `t',\\ 2, & if\ the\ server\ is\ busy\ with\ SSS\ at\ time\ `t',\\ 3, & if\ the\ server\ is\ busy\ with\ OSS\ at\ time\ `t',\\ 4, & if\ the\ server\ is\ on\ k-phases\ of\ vacation\ at\ time\ `t',\\ 5, & if\ the\ server\ is\ on\ k+1\ optional\ vacation\ at\ time\ `t',\end{cases}$$

Let the random variable $\omega(t)$ is defined as

$$\omega(t)=\begin{cases}0, & if\ \gamma(t)=0\\ B_1^0(t), & if\ \gamma(t)=1\\ B_2^0(t), & if\ \gamma(t)=2\\ B_3^0(t), & if\ \gamma(t)=3\\ V_j^0(t), & if\ \gamma(t)=4\\ V_{j+1}^0(t), & if\ \gamma(t)=5\end{cases}$$

Let the random variable $\delta(t)$ represent the number of customers in the queue at time t. To proceed with the analysis, the corresponding limiting probabilities have been introduced:

$$Q_n=\lim_{t\to\infty}\Pr ob\ [\delta(t)=n,\ \omega(t)=0],\ n=0,1,2,....,N-1,$$

$$P_{i,n}(x)\,dx=\lim_{t\to\infty}\Pr ob\ [\delta(t)=n,\ \omega(t)=B_i^0(t);x<B_i^0(t)\le x+dx],$$

$$x>0,\ n\ge 1, i=1,2,3.$$

$$R_{j,n}(x)\,dx = \lim_{t\to\infty} \Pr ob\,[\delta(t) = n,\ \omega(t) = V_j^0(t); x < V_j^0(t) \le x + dx],$$
$$x > 0,\ n \ge 0, 1 \le j \le k.$$
$$R_{j+1,n}(x)\,dx = \lim_{t\to\infty} \Pr ob\,[\delta(t) = n,\ \omega(t) = V_{j+1}^0(t); x < V_{j+1}^0(t) \le x + dx],$$
$$x > 0,\ n \ge 0, 1 \le j \le k.$$

Further, it is assumed that $B_i(0) = 0$, $B_i(\infty) = 1$, $V_j(0) = 0$, $V_j(\infty) = 1$, $V_{j+1}(0) = 0$, $V_{j+1}(\infty) = 1$ for i=1,2,3 and j=1,2,3,….,k and that $B_i(x)$, $V_j(x)$ and $V_{j+1}(x)$ are continuous at $x = 0$, so that

$$\mu_i(x)dx = \frac{dB_i(x)}{1 - B_i(x)};\ \upsilon_j(x)dx = \frac{dV_j(x)}{1 - V_j(x)} \text{ and } \upsilon_{j+1}(x)dx = \frac{dV_{j+1}(x)}{1 - V_{j+1}(x)}$$

for *i*=1,2,3 and 1≤ *j* ≤ *k*;

are the first order differential (hazard rate) functions of B_i, V_j and V_{j+1} respectively for *i*=1,2,3 and *j*=1,2,3,….,k.

The long-term behaviour of the queueing process at an arbitrary epoch can now be analyzed using the Kolmogorov forward equations.

$$\frac{d}{dx}P_{n,i}(x) + [\lambda + \mu_i(x)]P_{n,i}(x) = \lambda\sum_{r=1}^{n} c_r P_{n-r,i}(x), \quad x > 0,\ n \ge 1, \tag{3.1}$$

$$\frac{d}{dx}R_{n,j}(x) + [\lambda + \upsilon_j(x)]R_{n,j}(x) = \lambda\sum_{r=1}^{n} c_r R_{n-r,j}(x), \quad x > 0,\ n \ge 1,\ 1 \le j \le k, \tag{3.2}$$

$$\frac{d}{dx}R_{0,j}(x) + [\lambda + \upsilon_j(x)]R_{0,j}(x) = 0, \quad x > 0,\ 1 \le j \le k, \tag{3.2a}$$

$$\frac{d}{dx}R_{n,j+1}(x) + [\lambda + \upsilon_{j+1}(x)]R_{n,j+1}(x) = \lambda\sum_{r=1}^{n} c_r R_{n-r,j+1}(x), \quad x > 0,\ n \ge 1,\ 1 \le j \le k, \tag{3.3}$$

$$\frac{d}{dx}R_{0,j+1}(x) + [\lambda + \upsilon_{j+1}(x)]R_{0,j+1}(x) = 0, \quad x > 0,\ 1 \le j \le k, \tag{3.3a}$$

$$\lambda Q_0 = (1-\theta)\int_0^\infty \upsilon_j(x)R_{0,j}(x)dx + \int_0^\infty \upsilon_{j+1}(x)R_{0,j+1}(x)dx + (1-\beta)\int_0^\infty \mu_3(x)P_{1,3}(x)\,dx + (1-q)\int_0^\infty \mu_2(x)P_{1,2}(x)dx, \tag{3.4}$$

$$\lambda Q_n = \lambda\sum_{r=1}^{n} c_r\, Q_{n-r}, \quad n = 1, 2, 3, \ldots\ldots\ldots, N-1. \tag{3.5}$$

where $P_{0,i}(x) = 0$ for *i*=1,2,3 occurring in above equation (3.1) respectively.

The steady-state equations (3.1)–(3.5) must be solved under the following boundary conditions at *x*=0, specified at:

$$P_{n,1}(0)=(1-\beta)\int_0^\infty \mu_3(x)\,P_{n+1,3}(x)\,dx+(1-\theta)\int_0^\infty \upsilon_j(x)\,R_{n,j}(x)\,dx$$
$$+(1-\alpha)\int_0^\infty \mu_2(x)\,P_{n+1,2}(x)\,dx+\int_0^\infty \upsilon_{j+1}(x)\,R_{n,j+1}(x)\,dx, \quad n=1,2,....,N-1,\ 1\le j\le k, \tag{3.6}$$

$$P_{n,1}(0)=(1-\beta)\int_0^\infty \mu_3(x)\,P_{n+1,3}(x)\,dx+(1-\theta)\int_0^\infty \upsilon_j(x)\,R_{n,j}(x)\,dx$$
$$+(1-\alpha)\int_0^\infty \mu_2(x)\,P_{n+1,2}(x)\,dx+\int_0^\infty \upsilon_{j+1}(x)\,R_{n,j+1}(x)\,dx+\lambda\sum_{r=1}^{n} c_r\,Q_{n-r}, \quad n\ge N,\ 1\le j\le k, \tag{3.7}$$

$$P_{n,2}(0)=\int_0^\infty \mu_1(x)\,P_{n,1}(x)\,dx, \quad n\ge 1, \tag{3.8}$$

$$P_{n,3}(0)=\alpha\int_0^\infty \mu_2(x)\,P_{n,2}(x)\,dx, \quad n\ge 1, \tag{3.9}$$

$$R_{n,1}(0)=\beta\int_0^\infty \mu_3(x)\,P_{n+1,3}(x)\,dx, \quad n\ge 0, \tag{3.10}$$

$$R_{n,j}(0)=\int_0^\infty \upsilon_{j-1}(x)R_{n,j-1}(x)\ dx, \quad n=0,1,2,....,\ 1\le j\le k, \tag{3.11}$$

$$R_{n,j+1}(0)=\theta\int_0^\infty \upsilon_j(x)R_{n,j}(x)\ dx, \quad n=0,1,2,....,\ 1\le j\le k, \tag{3.12}$$

and the normalizing conditions,

$$\sum_{n=0}^{N-1}Q_n+\sum_{n=1}^{\infty}\sum_{i=1}^{3}\int_0^\infty P_{n,i}(x)\,dx+\sum_{n=0}^{\infty}\sum_{j=1}^{k}\int_0^\infty R_{n,j}(x)\,dx$$
$$+\sum_{n=0}^{\infty}\sum_{j=1}^{k}\int_0^\infty R_{n,j+1}(x)\,dx=1 \tag{3.13}$$

To facilitate the solution of the system of equations (3.1)–(3.5) along with the boundary conditions (3.6)–(3.12), we introduce the following probability generating functions:

$$P_i(z;x)=\sum_{n=1}^{\infty}P_{n,i}(x)\,z^n, \quad i=1,2,3,\ x>0,\ |z|<1,$$

$$P_i(z;0)=\sum_{n=1}^{\infty}P_{n,i}(0)\,z^n, \quad i=1,2,3,\ |z|<1,$$

$$R_j(z;x)=\sum_{n=0}^{\infty}R_{n,j}(x)\,z^n, \quad 1\le j\le k,\ x>0,\ |z|<1,$$

$$R_j(z;0)=\sum_{n=0}^{\infty}R_{n,j}(0)\,z^n, \quad 1\le j\le k,\ |z|<1,$$

$$R_{j+1}(z;x)=\sum_{n=0}^{\infty}R_{n,j+1}(x)\,z^{n},\ 1\le j\le k,\ x>0,\ |z|<1,$$

$$R_{j+1}(z;0)=\sum_{n=0}^{\infty}R_{n,j+1}(0)\,z^{n},\ \ 1\le j\le k,\ |z|<1,$$

$$Q(z)=\sum_{n=0}^{N-1}Q_{n}\,z^{n},\ |z|<1.$$

Proceeding as usual with equations (3.1), (3.2), and (3.3) we derive the following results.

$$P_1(z;x)=P_1(z;0)[1-B_1(x)]e^{-\phi x},\qquad x>0 \tag{3.14}$$

$$P_2(z;x)=P_2(z;0)[1-B_2(x)]e^{-\phi x},\qquad x>0 \tag{3.15}$$

$$P_3(z;x)=P_3(z;0)[1-B_3(x)]e^{-\phi x},\qquad x>0 \tag{3.16}$$

$$R_j(z;x)=R_j(z;0)[1-V_j(x)]e^{-\phi x},\quad x>0\ ,\ j=1,2,\ldots,k,\ 1\le j\le k, \tag{3.17}$$

Similarly,

$$R_{j+1}(z;x)=R_{j+1}(z;0)[1-V_{j+1}(x)]e^{-\phi x},\quad x>0\ ,\ j=1,2,\ldots,k,\ 1\le j\le k, \tag{3.18}$$

By multiplying equations (3.6) and (3.7) by corresponding powers of z, summing across all values of n, and applying equation (3.4), we obtain the following simplification:

$$\begin{aligned}P_1(z;0)=&(1-\alpha)B_2^*(\phi)P_2(z;0)z^{-1}+(1-\beta)B_3^*(\phi)P_3(z;0)z^{-1}\\&+(1-\theta)V_j^*(\phi)R_j(z;0)+V_{j+1}^*(\phi)R_{j+1}(z;0)\\&+\lambda\sum_{n=N}^{\infty}z^n\sum_{r=0}^{N-1}c_{n-r}Q_r-\lambda Q_{0,}\end{aligned} \tag{3.19}$$

where, $B_i^*(\phi)=\int_0^{\infty}e^{-\phi x}dB_i(x)$ is the Z-transform of B_i, for i=1,2,3 and $V_j^*(\phi)=\int_0^{\infty}e^{-\phi x}dV_j(x)$, $V_{j+1}^*(\phi)=\int_0^{\infty}e^{-\phi x}dV_{j+1}(x)$ is the Z-transform of V_j and V_{j+1}.

Now, by applying equation (3.5) and performing a double summation on the term $\lambda\sum_{n=N}^{\infty}z^n\sum_{r=0}^{N-1}c_{n-r}Q_r$ in equation (3.19), followed by a reordering of the summation, we obtain the simplified expression.

$$\lambda\sum_{n=N}^{\infty}z^n\sum_{r=0}^{N-1}c_{n-r}Q_r=\lambda Q(z)[Y(z)-1]+\lambda Q_0. \tag{3.20}$$

Therefore, equations (3.19) and (3.20), yield:

$$P_1(z;0)=(1-\alpha)B_2^*(\phi)P_2(z;0)z^{-1}+(1-\beta)B_3^*(\phi)P_3(z;0)z^{-1} \\ +(1-\theta)V_j^*(\phi)R_j(z;0)+V_{j+1}^*(\phi)R_{j+1}(z;0) \\ +\lambda Q(z)[Y(z)-1]. \tag{3.21}$$

Following a similar approach on equations (3.8) through (3.12), the following results are derived.

$$P_2(z;0)=P_1(z;0)B_1^*(\phi), \tag{3.22}$$

$$P_3(z;0)=\alpha P_2(z;0)B_2^*(\phi), \tag{3.23}$$

$$R_1(z;0)=\frac{\beta P_3(z;0)B_3^*(\phi)}{z}, \tag{3.24}$$

$$R_j(z;0)=\prod_{m=1}^{j-1}V_m^*(\phi)\left[\frac{\beta P_3(z;0)B_3^*(\phi)}{z}\right], \tag{3.25}$$

Similarly,

$$R_{j+1}(z;0)=\theta\prod_{m=1}^{j}V_m^*(\phi)\left[\frac{\beta P_3(z;0)B_3^*(\phi)}{z}\right]. \tag{3.26}$$

Now, using equations (3.22) to (3.26) sequentially in equation (3.21), we derive the simplified expression.

$$P_1(z;0)=\frac{zQ(z)[\phi]}{\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta) \\ +\theta V_{j+1}^*(\phi))\prod_{m=1}^{j}V_m^*(\phi)]B_3^*(\phi)\end{bmatrix}B_2^*(\phi)B_1^*(\phi)-z}, \tag{3.27}$$

So, that

$$\tau_1(z)=\int_0^\infty P_1(z,0)\,dx \\ =\frac{zQ(z)[1-B_1^*(\phi)]}{\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta) \\ +\theta V_{j+1}^*(\phi))\prod_{m=1}^{j}V_m^*(\phi)]B_3^*(\phi)\end{bmatrix}B_2^*(\phi)B_1^*(\phi)-z}, \tag{3.28}$$

In a similar manner, equations (3.15), (3.22), and (3.27), we derive

$$\tau_2(z)=\int_0^\infty P_2(z,0)\,dx \\ =\frac{zQ(z)B_1^*(\phi)[1-B_2^*(\phi)]}{\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta) \\ +\theta V_{j+1}^*(\phi))\prod_{m=1}^{j}V_m^*(\phi)]B_3^*(\phi)\end{bmatrix}B_2^*(\phi)B_1^*(\phi)-z}, \tag{3.29}$$

Similarly, applying equations (3.16), (3.23), and (3.27), we derive

$$\tau_3(z)=\int_0^\infty P_3(z,0)\,dx$$
$$=\frac{z\alpha Q(z)B_1^*(\phi)B_2^*(\phi)[1-B_3^*(\phi)]}{\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta)\\ +\theta V_{j+1}^*(\phi))\prod_{m=1}^{j}V_m^*(\phi)]B_3^*(\phi)\end{bmatrix}B_2^*(\phi)B_1^*(\phi)-z}, \tag{3.30}$$

Likewise, applying equations (3.17), (3.22)–(3.24), and (3.27), we derive

$$\psi_j(z)=\int_0^\infty R_j(z,0)\,dx$$
$$=\frac{\alpha\beta Q(z)B_1^*(\phi)B_2^*(\phi)B_3^*(\phi)[1-V_j^*(\phi)]\prod_{m=1}^{j-1}V_m^*(\phi)}{\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta)\\ +\theta V_{j+1}^*(\phi))\prod_{m=1}^{j}V_m^*(\phi)]B_3^*(\phi)\end{bmatrix}B_2^*(\phi)B_1^*(\phi)-z}, \quad 1\le j\le k, \tag{3.31}$$

Finally, using the same approach with equations (3.18), (3.22), (3.23), (3.25) and (3.27), the following result is obtained

$$\psi_{j+1}(z)=\int_0^\infty R_{j+1}(z,0)\,dx$$
$$=\frac{\alpha\beta\theta Q(z)B_1^*(\phi)B_2^*(\phi)B_3^*(\phi)[1-V_{j+1}^*(\phi)]\prod_{m=1}^{j}V_m^*(\phi)}{\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta)\\ +\theta V_{j+1}^*(\phi))\prod_{m=1}^{j}V_m^*(\phi)]B_3^*(\phi)\end{bmatrix}B_2^*(\phi)B_1^*(\phi)-z}, \quad 1\le j\le k, \tag{3.32}$$

Now, let $\pi(z)=Q(z)+\tau_1(z)+\tau_2(z)+\tau_3(z)+z\sum_{j=1}^{k}\psi_j(z)+z\psi_{j+1}(z)$ represent the probability generating function of the stationary queue length characteristics at an arbitrary epoch; then

$$\pi(z)=\frac{(1-z)Q(z)\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta)\\ +\theta V_{j+1}^*(\phi))\sum_{j=1}^{k}\prod_{m=1}^{j}V_m^*(\phi)]B_3^*(\phi)\end{bmatrix}B_1^*(\phi)B_2^*(\phi)}{\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta)\\ +\theta V_{j+1}^*(\phi))\prod_{m=1}^{j}V_m^*(\phi)]B_3^*(\phi)\end{bmatrix}B_2^*(\phi)B_1^*(\phi)-z}, \tag{3.33}$$

where, $\phi=(\lambda-\lambda X(z))$.

4 Evaluation of the Queue Length Characteristics

This section focuses on deriving the system state probabilities and examining the probability generating function *P(z)* to facilitate their proper interpretation. To interpret *Q(z)*, we define $\xi_n\ (n=1,2,3,\ldots\ldots,N-1)$ as the probability that a batch of customers encounters at least 'n' tasks awaiting service during a server idle time. This probability follows the recursive equation:

$$\xi_n = \sum_{k=1}^{n} c_k \xi_{n-k} \qquad (n = 0,1,2,......, N-1) \text{ and } \xi_0 = 1.$$

Now, by applying this reasoning, it is straightforward to show that Q_n $(for\, n = 0,1,2,....., N-1)$ satisfies the following expression,

$$Q_n = \psi_0 \xi_n \qquad (n = 0,1,2,....., N-1), \tag{4.1}$$

where ψ_0 is the constant ensuring normalisation.
Accordingly,

$$Q(z) = \psi_0 \sum_{n=0}^{n-1} \xi_n z^n. \tag{4.2}$$

To evaluate ψ_0, applying the normalizing condition (3.13), which corresponds to $\pi(1) = 1$, and obtain the following

$$\psi_0 = \frac{(1-\rho)}{\sum_{n=0}^{N-1} \xi_n}. \tag{4.3}$$

Now, using (4.3) into (4.2), we have

$$Q(z) = \frac{(1-\rho)\sum_{n=0}^{N-1} \xi_n z^n}{\sum_{n=0}^{N-1} \xi_n}, \tag{4.4}$$

where $\rho = \lambda E(X)[E(B_1) + E(B_2) + \alpha(E(B_3) + \beta(\sum_{j=1}^{k} E(V_j) + \theta E(V_{j+1})))]$ is the server utilization of this system and $\sum_{n=0}^{N-1} \xi_n$ is the average number of batches arriving while the system is idle. Also, from equations (4.3) and (4.4) we have $\rho < 1$, which is the necessary stability condition ensuring the existence of steady state solutions.

Define g_n as the probability that the system is in an idle period given n customers are present. Using equation (4.1) and conditioning on the arrival counts, we derive:

$$g_n = \frac{Q_n}{\sum_{n=0}^{N-1} Q_n} = \frac{\xi_n}{\sum_{n=0}^{N-1} \xi_n}, \quad n = 0,1,2,....., N-1, \tag{4.5}$$

Accordingly, the PGF G(z) representing the number of customers in the system during an idle time $\{g_n; n = 0,1,2,....., N-1\}$ can be written as

$$G(z) = \sum_{n=0}^{N-1} g_n z^n = \frac{\sum_{n=0}^{N-1} \xi_n z^n}{\sum_{n=0}^{N-1} \xi_n}. \tag{4.6}$$

Applying equations (4.6) and (4.4), we derive

$$Q(z) = (1-\rho)\, G(z) \tag{4.7}$$

By substituting equation (4.7) into equation (3.33), the stochastic decomposition property of this model can be derived as follows,

$$\pi(z) = \frac{Q(z)(1-z)\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta) \\ +\theta V_{j+1}^{*}(\phi))\sum_{j=1}^{k}\prod_{m=1}^{j}V_m^{*}(\phi)]B_3^{*}(\phi)\end{bmatrix}B_1^{*}(\phi)B_2^{*}(\phi)}{\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta)+ \\ \theta V_{j+1}^{*}(\phi))\prod_{m=1}^{j}V_m^{*}(\phi)]B_3^{*}(\phi)\end{bmatrix}B_2^{*}(\phi)B_1^{*}(\phi)-z}$$

$$= \left(\frac{(1-\rho)(1-z)\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta) \\ +\theta V_{j+1}^{*}(\phi))\sum_{j=1}^{k}\prod_{m=1}^{j}V_m^{*}(\phi)]B_3^{*}(\phi)\end{bmatrix}B_1^{*}(\phi)B_2^{*}(\phi)}{\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta) \\ +\theta V_{j+1}^{*}(\phi))\prod_{m=1}^{j}V_m^{*}(\phi)]B_3^{*}(\phi)\end{bmatrix}B_1^{*}(\phi)B_2^{*}(\phi)-z}\right)\left(\frac{\sum_{n=0}^{N-1}\xi_n z^n}{\sum_{n=0}^{N-1}\xi_n}\right) \tag{4.8}$$

$$\pi(z) = A(z)G(z),$$

Here, $A(z)$ denotes the PGF of the stationary queue length behavior of an $M^X/(G_1,G_2,G_3)/(V_j,V_{j+1})/1(BS)$ queue. This expression is readily derived by substituting the original service time distribution with the modified one i.e.,

$$T^{*}(S) = [(1-\alpha)+\alpha((1-\beta)+\beta((1-\theta) + \theta V_j^{*}(\phi))\sum_{j=1}^{k}\prod_{m=1}^{j}V_m^{*}(\phi))B_3^{*}(\phi)]B_1^{*}(\phi)B_2^{*}(\phi)$$

in the well-known Pollaczek-khinchine formula.

Define $\eta(z)$ as the PGF corresponding to the number of customers in a batch arriving during the modified service time s, then

$$\eta(z) = [(1-\alpha)+\alpha((1-\beta)+\beta((1-\theta) + \theta V_j^{*}(\phi))\sum_{j=1}^{k}\prod_{m=1}^{j}V_m^{*}(\phi))B_3^{*}(\phi)]\, B_1^{*}(\phi)B_2^{*}(\phi)$$

and

$$\eta'(1) = \lambda E(X)\begin{bmatrix}E(B_1)+E(B_2)+\alpha((E(B_3) \\ +\beta(\sum_{j=1}^{k}E(V_j)+\theta E(V_{j+1})))\end{bmatrix} = \rho$$

Now, by applying $\eta(z)$ in the Pollaczek-Khinchine formula, we can express it as:

$$P_0(z)=\frac{(1-\eta'(1))(1-z)\eta(z)}{\eta(z)-z}$$

$$=\left(\frac{(1-\rho)(1-z)\begin{bmatrix}(1-\alpha)+\alpha((1-\beta)+\beta((1-\theta)\\ +\theta V_{j+1}^*(\phi))\sum_{j=1}^{k}\prod_{m=1}^{j}V_m^*(\phi))B_3^*(\phi)\end{bmatrix}B_1^*(\phi)B_2^*(\phi)}{\begin{bmatrix}(1-\alpha)+\alpha((1-\beta)+\beta((1-\theta)\\ +\theta V_{j+1}^*(\phi))\prod_{m=1}^{j}V_m^*(\phi))B_3^*(\phi)\end{bmatrix}B_1^*(\phi)B_2^*(\phi)-z}\right),$$

where $\phi=(\lambda-\lambda Y(z))$,
This represents the first factor on the right-hand side of equation (4.8), and

$$G(z)=\frac{\sum_{n=0}^{N-1}\xi_n z^n}{\sum_{n=0}^{N-1}\xi_n}$$

Here, G(z) represents the PGF of the number of customers present in the system (before reaching or exceeding N) during an idle period. More specifically, $\eta(z)$ characterizes the additional queue length distribution due to N-policy.

5 Evaluation of the Queue Length Characteristics at a Departure Time

The probability generating function (PGF) of the queue length characteristics at a departure time for the $M^X/G/1$ queue was previously obtained through various approaches. In this section, we extend those results to our $M^X/(G_1,G_2,G_3)/(V_j,V_{j+1})/1(BS)/N-$policy queue, which has not been studied in the context of either the $M^X/G/1/N-$policy queues or the *(BS)/N*-policy queue. Based on the "Poisson Arrivals See Time Averages" (PASTA) principle, we assert that the customer upon departure observes `$i$′ customers in the queue immediately after departure if and only if there were $(i+1)$ customers in the queue just before the departure. Let $\{P_i^+;i\geq 0\}$ denote the probability of observing `i′ customers in the queue at a departure time. Then, we can express this as

$$\begin{aligned}P_i^+&=\Omega_0(1-\alpha)\int_0^\infty\mu_2(x)P_{i+1,2}(x)dx+\Omega_0(1-\beta)\int_0^\infty\mu_3(x)P_{i+1,3}(x)dx\\&+\Omega_0(1-\theta)\int_0^\infty\eta_j(x)R_{i+1,j}(x)dx+\Omega_0\int_0^\infty\eta_{j+1}(x)R_{i,j+1}(x)dx,\qquad i\geq 0\end{aligned}\tag{5.1}$$

where Ω_0 is the constant ensuring normalisation.
We define the probability generating function (PGF) as follows

$$\pi^+(z)=\sum_{i=o}^{\infty}P_i^+z^i\tag{5.2}$$

Now, by multiplying equation (5.1) by corresponding powers of z and summing across all values of n, and applying equation (5.2), we derive

$$\pi^{+}(z) = \Omega_0(1-\alpha)z^{-1}\int_0^\infty \mu_2(x)P_2(x;z)dx + \Omega_0(1-\beta)z^{-1}\int_0^\infty \mu_3(x)P_3(x;z)dx$$
$$+\Omega_0(1-\theta)z^{-1}\int_0^\infty \eta_j(x)R_j(x;z)dx + \Omega_0\int_0^\infty \eta_{j+1}(x)R_{j+1}(x;z)dx, \quad 1\le j\le k, \qquad (5.3)$$

Using equations (3.15) - (3.17) and (3.18), we get after simplification

$$\pi^{+}(z) = \frac{\lambda\Omega_0(1-\rho)\left[\sum_{n=0}^{N-1}\xi_n z^n\right][1-Y(z)]\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta)\\ +\theta V_{j+1}^*(\phi))\prod_{m=1}^{j} V_m^*(\phi)]B_3^*(\phi)\end{bmatrix}B_1^*(\phi)B_2^*(\phi)}{\left[\sum_{n=0}^{N-1}\xi_n\right]\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta)+\\ \theta V_{j+1}^*(\phi))\prod_{m=1}^{j} V_m^*(\phi)]B_3^*(\phi)\end{bmatrix}B_1^*(\phi)B_2^*(\phi)-z}. \qquad (5.4)$$

Now, applying the normalization condition $\pi^{+}(1)=1$ given in equation (5.4), we obtain:

$$\Omega_0 = [\lambda E(X)]^{-1}.$$

Thus, we have

$$\pi^{+}(z) = \frac{(1-\rho)\left[\sum_{n=0}^{N-1}\xi_n z^n\right][1-Y(z)]\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta)+\\ \theta V_{j+1}^*(\phi))\prod_{m=1}^{j} V_m^*(\phi)]B_3^*(\phi)\end{bmatrix}B_1^*(\phi)B_2^*(\phi)}{E(X)\left[\sum_{n=0}^{N-1}\xi_n\right]\begin{bmatrix}(1-\alpha)+\alpha[(1-\beta)+\beta((1-\theta)+\\ \theta V_{j+1}^*(\phi))\prod_{m=1}^{j} V_m^*(\phi)]B_3^*(\phi)\end{bmatrix}B_1^*(\phi)B_2^*(\phi)]-z}. \qquad (5.5)$$

Hence, the connection between $\pi(z)$ and $\pi^{+}(z)$ is established as follows

$$\pi^{+}(z) = \frac{[1-Y(z)]}{E(X)(1-z)}\pi(z) = F(z)A(z)G(z), \qquad (5.6)$$

where,

$$F(z) = \frac{[1-Y(z)]}{E(X)(1-z)},$$

This represents the probability generating function of the number of customers positioned ahead of an arbitrary (tagged) customer within the same arriving batch. This quantity corresponds to the backward recurrence time in a discrete-time renewal process, where the renewal events are determined by the batch arrival size. This arises due to the randomness inherent in the arrival size distribution.

In the special case where $\alpha,\beta,\theta=1$ equation (5.5), the result becomes

$$\pi^{+}(z)=\frac{(1-\rho)\left[\sum_{n=0}^{N-1}\xi_n z^n\right][1-Y(z)]B_1^*(\phi)B_2^*(\phi)B_3^*(\phi)\prod_{m=1}^{j}V_m^*(\phi)V_{j+1}^*(\phi)}{E(X)\left[\sum_{n=0}^{N-1}\xi_n\right][B_1^*(\phi)B_2^*(\phi)B_3^*(\phi)\prod_{m=1}^{j}V_m^*(\phi)V_{j+1}^*(\phi)-z]}, \tag{5.7}$$

where now we have

$$\rho=\lambda E(X)[E(B_1)+E(B_2)+E(B_3)+\sum_{j=1}^{k}E(V_j)+E(V_{j+1})].$$

Note that equation (5.7) represents the PGF of the queue length characteristics at departure time for an $M^X/(G_1,G_2,G_3)/1$ limited service queueing system, which includes a *k*-phase vacation followed by an optional *k*+1 vacation, operating under an *N*-policy. In this model, if the system contains at least one customer when a vacation ends, service resumes without delay; otherwise, the server remains idle until precisely *N* customers are present.

Additionally, by taking *N*=1 in equation (5.7), the expression becomes

$$\pi^{+}(z)=\frac{(1-\rho)[1-Y(z)]B_1^*(\phi)B_2^*(\phi)B_3^*(\phi)\prod_{m=1}^{j}V_m^*(\phi)V_{j+1}^*(\phi)}{E(X)[B_1^*(\phi)B_2^*(\phi)B_3^*(\phi)\prod_{m=1}^{j}V_m^*(\phi)V_{j+1}^*(\phi)-z]}, \tag{5.8}$$

which is the probability generating function (PGF) $M^X/(G_1,G_2,G_3)/1$ for a limited service queueing system with a *k*-phase of vacation and an additional *k*+1 of vacation is of interest. Notably, certain aspects of the standard $M^X/G/1$ limited service queue with a single vacation under the N-policy have been previously explored by Chowdhury and Madan [5].

Upon substituting z = 0 into equation (5.5), the result is

$\pi^{+}(0)$ = Prob [No waiting units in the system when a departure occurs]

$$=\frac{(1-\rho)\xi_0}{E(X)\left[\sum_{n=0}^{N-1}\xi_n\right]}=\frac{\psi_0\xi_0}{E(X)}=P_0^{+},$$

So that

$$Q_0=E(X)P_0^{+}.$$

It illustrates that a random observer is more likely to encounter an empty system than a customer departing from it.

6 Average Queue Length

Let L_q be the average queue length $M^X/(G_1,G_2,G_3)/(V_j,V_{j+1})/1(BS)/N-$policy queue at a random epoch. Then,

$$L_q=\left.\frac{d\pi(z)}{dz}\right|_{z=1}$$

$$= \rho + \frac{\lambda^2 E^2(X)[E(B_1^2) + E(B_2^2) + \alpha \begin{bmatrix} E(B_3^2) + \\ \beta(\prod_{m=1}^{j} E(V_m^2) + \theta E(V_{j+1}^2)) \end{bmatrix}]}{2(1-\rho)}$$

$$+ \frac{\lambda^2 E^2(X) \begin{bmatrix} 2[E(B_1)E(B_2) + \alpha(E(B_3)[E(B_1) + E(B_2)] \\ +\beta([E(B_1) + E(B_2) + E(B_3)]\prod_{m=1}^{j} E(V_m) \\ +\theta[E(B_1) + E(B_2) + E(B_3) + \prod_{m=1}^{j} E(V_m)]E(V_{j+1})))] \end{bmatrix}}{2(1-\rho)}$$

$$+ \frac{\lambda[E(B_1) + E(B_2) + \alpha \begin{bmatrix} E(B_3) + \\ \beta(\prod_{m=1}^{j} E(V_m) + \theta E(V_{j+1})) \end{bmatrix}]E(X(X-1))}{2(1-\rho)} \quad (6.1)$$

$$+ \frac{\sum_{n=0}^{N-1} n\xi_n}{\sum_{n=0}^{N-1} \xi_n}$$

Moreover, if we define L_d^+ as the average queue length at a departure epoch for the $M^X/(G_1, G_2, G_3)/(V_j, V_{j+1})/1(BS)/N-$ policy queue, then it is given by

$$L_d^+ = \left.\frac{d\pi^+(z)}{dz}\right|_{z=1} = L_q + E(X_R), \quad (6.6)$$

where, $E(X_R) = \frac{E(X(X-1))}{2E(X)}$ is the average residual batch size. Equation (6.5) shows that $L_d^+ > L_q$ and equality holds iff $E(X_R) = 0$.

Let W_q represent the average waiting time of an arbitrary customer in the $M^X/(G_1, G_2, G_3)/(V_j, V_{j+1})/1/(BS)/N-$ policy queue, Then, using Little's formula and equation (6.1), W_q is given by $W_q = \lambda E(X)L_q$.

7 Performance Characteristics

We use PGF functions in this section to compute various performance indicators as follows:

7.1. State of the server and their corresponding probabilities

(i). Prob [L_0]= Prob [No service is being performed] = $\sum_{n=0}^{N-1} Q_n = (1-\rho)$.

(ii). Prob [The service unit is active in PSS] $= \tau_1(1) = \lambda E(X)E(B_1)$.

(iii). Prob [The service unit is active in SSS] $= \tau_2(1) = \lambda E(X)E(B_2)$.

(iv). Prob [The service unit is active in OSS] = $\tau_3(1) = \alpha\lambda E(X)E(B_3)$.

(v). Prob [The service unit is active in $k-$phase of vacation]

$$=\psi_j(1)=\alpha\beta\lambda E(X)E(V_j).$$

(vi). Prob [The service unit is active in $k+1$ optional vacation]

$$=\psi_{j+1}(1)=\alpha\beta\theta\lambda E(X)E(V_{j+1}).$$

8 Cost Optimization Function

In this section, we aim to determine the optimal choice of N that minimizes the total cost by formulating a cost function for the average cost in an $M^X/(G_1,G_2,G_3)/(V_j,V_{j+1})/1/(BS)/N-$ policy queue.

Let C_s, C_h, C_o, and T_c denote the start-up cost per cycle, holding cost per unit time, operating cost per unit time, and cycle length, respectively.

To identify the optimal choice of N, we utilize the following form of a linear cost function, similar to the one proposed in Lee and Srinivasan [17]. If $TC(N)$ represents the total average cost per unit time, then

$$TC(N)=\rho C_o+C_hL_q+\frac{C_s}{E(T_c)} \tag{8.1}$$

To evaluate $E(T_c)$, we begin by defining the following events.

L_0 be the length of the idle period, and L_b be the length of the busy period.

So that $E(T_c)=E(L_0)+E(L_b)$.

Now, since $\sum_{k=0}^{N-1}\xi_k$ represents the average number of batches arriving during an idle period and λ is the arrival rate, the mean duration of that number of batches is $\sum_{k=0}^{N-1}\xi_k/\lambda$, which gives the average idle period as

$$E(L_0)=\frac{\sum_{k=0}^{N-1}\xi_k}{\lambda}.$$

Furthermore, since $E(L_0)\lambda E(X)$ denotes the average number of arrivals during an idle period, the corresponding average busy period is given by"

$$E(L_b)=\frac{\rho}{1-\rho}E(L_0).$$

Thus, we have

$$E(T_c)=\frac{\sum_{k=0}^{N-1}\xi_k}{\lambda(1-\rho)}. \tag{8.2}$$

By substituting equations (6.1) and (7.2) into (7.1), we obtain the average cost incurred per unit of time as

$$TC(N) = \frac{\lambda C_s(1-\rho) + C_h \sum_{k=0}^{N-1} k\xi_k}{\sum_{k=0}^{N-1} \xi_k} + C_h L_q + \rho C_o, \quad (8.3)$$

Now, in order to identify the optimal choice of N , the differences as shown below

$$\Delta TC(N) = TC(m+1) - TC(m) = \frac{\xi_m I(m)}{H(m-1)H(m)},$$

Here, 'Δ' is the difference operator and

$$I(m) = C_h[mH(m) - U(m)] - \lambda C_s(1-\rho),$$

where, $H(m) = \sum_{k=0}^{m} \xi_k$ and $U(m) = \sum_{k=0}^{m} k\xi_k$.

$$I(m) = C_h \left[\sum_{k=0}^{m-1} (m-k)\xi_k \right] - \lambda C_s \left[\begin{array}{l} 1 - \lambda E(X)\{E(B_1) + E(B_2) + \\ \alpha(E(B_3) + \beta(\sum_{j=1}^{k} E(V_j) + \theta E(V_{j+1})))\} \end{array} \right] \quad (8.4)$$

we note that

$$C_h \left[\sum_{k=0}^{m-1} (m-k)\xi_k \right] > 0 \text{ and } \frac{\xi_m}{H(m)H(m-1)} > 0 \quad (8.5\ )$$

By using equation (8.4), we have $\Delta TC(m+1) > 0$. Let 'k' be the first 'm' such that $J(m) > 0,$ now setting $m = k,$ we see that $I(k+1) = C_h[(k+1)H(k+1) - U(k+1)] - \lambda C_s(1-\rho) = I(k) + C_h U(k),$ such that $I(k+1) > I(k),$ hence for some $n > k,$ we have $TC(n) > TC(k).$

Let N° be the optimal value of 'N', which minimizes (8.3), then from equation (8.4), we have

$$N^\circ = \min \left\{ m \geq 1 \middle| \sum_{k=0}^{m-1} (m-k)\xi_k > \frac{\lambda C_s[1-\rho]}{C_h} \right\} \quad (8.6)$$

Denoting $W(m) = \sum_{k=0}^{m-1} (m-k)\xi_k$ and $A = \lambda C_s[1-\rho]$, equation (8.6) can be written as

$$N^\circ = \min_{1 \leq m \leq N} \left\{ m : W(m) \geq \frac{A}{C_h} \right\}, \quad (8.7)$$

Therefore, equation (8.7) can be used to determine the optimal value of 'N' by identifying the best positive integer 'm' in the vicinity of 'N'.

Furthermore, it should be noted that if

$$\frac{C_h}{C_s} > \frac{\lambda[1-\rho]}{W(m)},$$

where $\rho = \lambda E(X)[E(B_1) + E(B_2) + \alpha(E(B_3) + \beta(\sum_{j=1}^{k} E(V_j) + \theta E(V_{j+1})))]$.

Therefore, under these conditions the optimal threshold value is 1.

9 Specific Case

In Case 1, assuming no additional k+1 vacation and with θ = 0 in equation (3.33), the expression simplifies to:

$$\pi(z) = \frac{(1-z)Q(z)\begin{bmatrix}(1-\alpha)+\alpha((1-\beta)+ \\ \beta\sum_{j=1}^{k}\prod_{m=1}^{j}V_m^*(\phi))B_3^*(\phi)\end{bmatrix}B_1^*(\phi)B_2^*(\phi)}{\begin{bmatrix}(1-\alpha)+\alpha((1-\beta)+ \\ \beta\sum_{j=1}^{k}\prod_{m=1}^{j}V_m^*(\phi))B_3^*(\phi)\end{bmatrix}B_1^*(\phi)B_2^*(\phi)-z}$$

In Case 2, assuming no optional service and with α = 0 in equation (3.33), the expression simplifies to:

$$\pi(z) = \frac{(1-z)Q(z)\begin{bmatrix}(1-\beta)+\beta((1-\theta)+ \\ \theta V_{j+1}^*(\phi))\sum_{j=1}^{k}\prod_{m=1}^{j}V_m^*(\phi))\end{bmatrix}B_1^*(\phi)B_2^*(\phi)}{\begin{bmatrix}(1-\beta)+\beta((1-\theta)+ \\ \theta V_{j+1}^*(\phi))\sum_{j=1}^{k}\prod_{m=1}^{j}V_m^*(\phi))\end{bmatrix}B_1^*(\phi)B_2^*(\phi)-z}$$

In Case 2, assuming no optional service or optional vacation, and the system consists of only one vacation composed of K-phases. By setting α = 0 and θ = 0 in equation (3.33), the expression simplifies to:

$$\pi(z) = \frac{(1-z)Q(z)\left[(1-\beta)+\beta V^*(\phi)\right]B_1^*(\phi)B_2^*(\phi)}{\left[(1-\beta)+\beta V^*(\phi)\right]B_1^*(\phi)B_2^*(\phi)-z}$$

where $\phi = (\lambda - \lambda Y(z))$,
This result aligns with the PGF derived by Chowdhury and Madan [5].

10 Numerical Analysis

A numerical analysis is examined in this section by using MATLAB to illustrate how the cost structure can be employed to determine the optimal choice of N° that minimize the average cost for varying parameters values. For simplicity, we set $\lambda = 1, E(B_1) = 0.02, E(B_2) = 0.03$ and $E(B_3) = 0.1$ and vacation k-phase here k=1,2,3,4,5 then $E(V_1)$ to $E(V_5)$ values equal to 0.03 and $E(V_6) = 0.1$.

We assume without loss of generality, that the batch size follows a displaced geometric distribution characterized by parameter 'α ', 'β' and 'θ'. The probability mass function of the random variable 'X_i' is expressed as

$$c_m = (1-p)p^{m-1}, \qquad m = 1, 2, 3, \ldots.$$

Then, we have

$$Y(z) = (1-p)z / (1-pz)$$

and $E(X) = (1-p)^{-1}$.

Now, taking $p = 0.5$, $E(X) = 2$ and we present numerical results for the cost ratio A / C_h, shown in Table 1. These results are obtained for a fixed value of C_h=250 and various values of parameters q, r, θ and C_s in Table 1.
To determine the optimal choice of N°, let us now calculate $W(m) = \sum_{k=0}^{m-1}(m-k)\xi_k$. This can be calculated with the help of recursive equation $\xi_k = \sum_{i=1}^{k} a_i \xi_{k-i}$ and $\xi_0 = 1$ as shown in Table 2.

Table 1 Cost ratio A / C_h for varying values.

α, β, θ=q	*Cs*=1000	*Cs*=2000	*Cs*=4000
0	3.6	7.2	14.4
0.1	3.3	6.6	13.3
0.2	3.0	6.1	12.2
0.3	2.8	5.5	11.0
0.4	2.5	4.9	9.9
0.5	2.2	4.4	8.8
0.6	1.9	3.8	7.7
0.7	1.6	3.3	6.6
0.8	1.4	2.7	5.4
0.9	1.3	2.6	5.3

Table 2 Different values of $W(m) = \sum_{k=0}^{m-1}(m-k)\xi_k$.

m	ξ_m	$W(m)$
1	0.5	1
2	0.5	2.5
3	0.5	4.5
4	0.5	7.0
5	0.5	10.0

6	0.5	13.5
7	0.5	17.5
8	0.5	22.0
9	0.5	27.0
10	0.5	32.5

Table 3 Optimal N° values across varying cost level.

Values of C_s an C_h	Optimal values of N°
C_s =1000 C_h =250	$N^{\circ} = 3$; for $0 \le \alpha \le 0.3$ $N^{\circ} = 2$; for $0.4 \le \alpha \le 0.9$
C_s =2000 C_h =250	$N^{\circ} = 5$; for $\alpha = 0$ $N^{\circ} = 4$; for $0.1 \le \alpha \le 0.4$ $N^{\circ} = 3$; for $0.5 \le \alpha \le 0.9$
C_s =4000 C_h =250	$N^{\circ} = 7$; for $\alpha = 0$ $N^{\circ} = 6$; for $0.1 \le \alpha \le 0.3$ $N^{\circ} = 5$; for $0.4 \le \alpha \le 0.6$ $N^{\circ} = 4$; for $0.7 \le \alpha \le 0.9$

Based on the result derived from equation (8.4), the optimal choice of N° is determined by comparing the values of $W(m)$ from Table 2 with cost ratio A / C_h. Consequently, the optimal N° is identified and presented in Table 3.

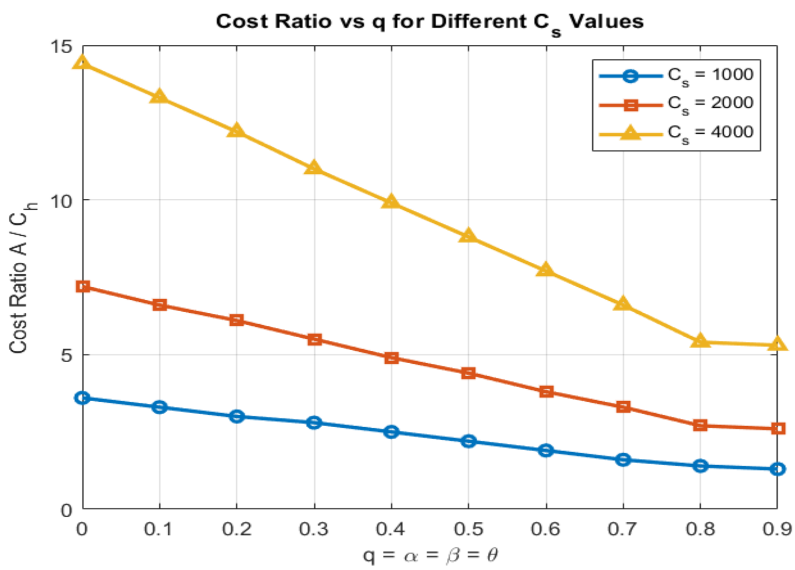

Figure 1 Shows how the cost ratio A / C_h decreases with increasing q for different values of start-up cost C_s.

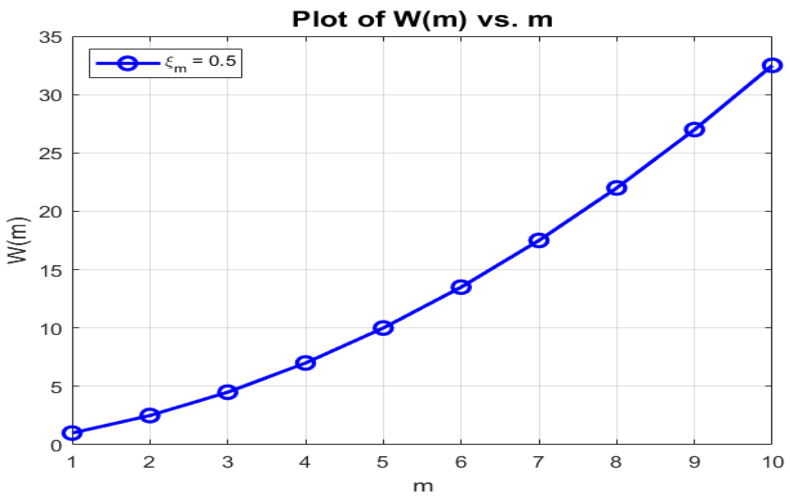

Figure 2 Presents the growth of $W(m)$ when m values increases.

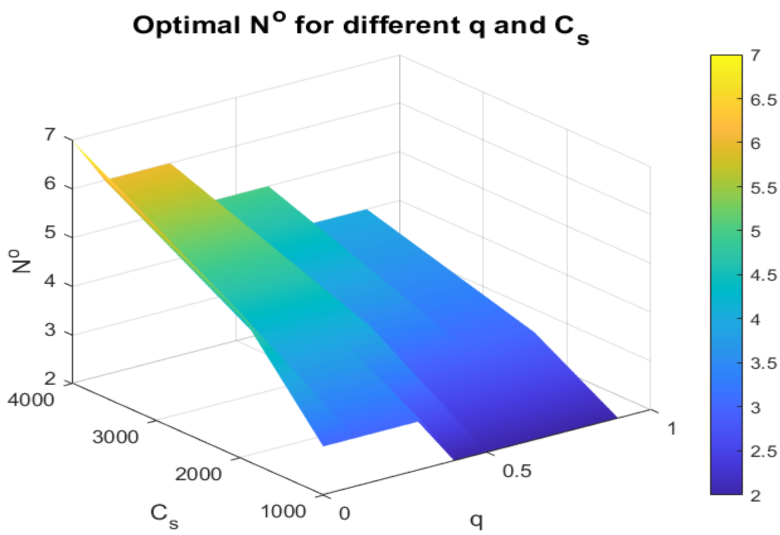

Figure 3 Illustrates the optimal threshold level N° as a function of the parameter q and start-up cost C_s. It demonstrates that N° increases with higher values of C_s, while it decreases as q increases.

11 Conclusion

This study analyzes an M/G/1 queuing system with batch arrivals governed by an N-policy and subject to k-phase Bernoulli-scheduled vacations. Utilizing the supplementary variable technique and probability generating functions, we derived performance measures related to the number of customers in the system under various server states. The analysis yielded multiple key performance metrics and identified the optimal threshold value across a range of cost structures. These findings provide a basis for determining efficient system configurations under different cost considerations.

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