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FUZZY STRONGLY NODEC SPACES

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Abstract. In this paper is an attempt to study and introduce the notion of fuzzy strongly boundary set and fuzzy strongly nodec spaces in fuzzy topological spaces. Several examples are given to illustrate the concepts introduced in this paper. By means of fuzzy strongly boundary set, the notion of fuzzy strongly nodec spaces defined and several characterizations of fuzzy strongly boundary set and fuzzy strongly nodec spaces are obtained.

Keywords: Fuzzy strongly boundary set, fuzzy strongly nowhere dense set, fuzzy strongly first and second category, fuzzy strongly residual set and fuzzy strongly nodec spaces.

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1. **Introduction**

The concepts of Fuzzy sets and fuzzy set operations were first introduced by L.A.Zadeh in his classical paper [11] in the year 1965. Thereafter the paper of C.L.Chang [4] in 1968 paved the way for the subsequent tremendous growth of the numerous Fuzzy topological concepts. Since then much alteration has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed in X.Tang [6] used a slightly changed version of Chang's fuzzy topological spaces to model spatial objects for GIS data bases and Structured Query Language (SQL) for GIS.

Through Pu and Liu [5] defined the notion of fuzzy boundary in fuzzy topological spaces in 1980. Ahmed and Athar [1] defined the concepts of fuzzy semi boundary and characterized fuzzy semi continuous functions. In terms of fuzzy semi boundary.

In this paper, we present several properties of fuzzy strongly boundary and fuzzy strongly nodec spaces which have been supported by examples. In section 4, the concepts of fuzzy strongly nodec spaces and several characterization of inter relationships between fuzzy strongly boundary set and fuzzy strongly nodec spaces are investigated.

2. **Preliminaries**

First, we briefly recall certain definitions and results are described.
A fuzzy set λ in a set X is a function from X to [0,1], that is, $\lambda: X \to [0, 1]$.

Definition 2.1. [4] Let λ and μ be any two fuzzy sets in (X,T). Then we define $\lambda \vee \mu : X \to [0,1]$ as follows: $(\lambda \vee \mu)(x) = Max\{\lambda(x), \mu(x)\}$. Also we define $\lambda \wedge \mu : X \to [0, 1]$ as follows: $(\lambda \wedge \mu)(x) = Min\{\lambda(x), \mu(x)\}$.

Definition 2.2. [4] Let (X,T) be a fuzzy topological space and λ be any fuzzy set in (X,T). We define int(λ) = $\vee \{\mu/\mu \leq \lambda, \mu \in T\}$ and cl(λ) = $\wedge \{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.
For any fuzzy set in a fuzzy topological space (X,T), it is easy to see that
$1 -$ cl(λ) =int ($1 - \lambda$) and $1 -$ int(λ) =cl ($1 - \lambda$).

Lemma 2.3. *[2] For a fuzzy set λ of a fuzzy topological space (X,T),*
(i)1 − int (λ) =cl (1 − λ),
(ii) 1 − cl(λ) =int (1 − λ).

Definition 2.4. [8] A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy dense if there exists no fuzzy closed set μ in (X,T) such that $\lambda < \mu < 1$. That is, cl(λ)=1 .

Definition 2.5. [7] A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X,T) such that $\mu <$ cl(λ). That is, int cl (λ) = 0.

Definition 2.6. [3] A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy G_δ-set in (X,T) if $\lambda = \bigwedge_{i=1}^{\infty}(\lambda_i)$ where 1 - $\lambda_i \in$ T for i $\in$ I.

Definition 2.7. [3] A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy F_σ-set in (X,T) if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$ where 1 - $\lambda_i \in$ T for i $\in$ I.

Definition 2.8. [7] A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy first category set if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Any other fuzzy set in (X, T) is said to be of second category.

Definition 2.9. [8] Let λ be a fuzzy first category set in a fuzzy topological space in (X,T). Then 1 - λ is called a fuzzy residual set in (X,T).

Definition 2.10. [8] A fuzzy topological space (X,T) is called fuzzy first category if the fuzzy set 1_X is a fuzzy first category space in (X,T). That is $1_x = \bigvee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X T). Otherwise (X,T) will be called a fuzzy second category.

Lemma 2.11. *[5] For a family of $\{\lambda_\alpha\}$ of fuzzy sets of a fuzzy topological space (X,T), $\vee cl(\lambda_\alpha) \leq cl(\vee\lambda_\alpha)$. In case A is a finite set, $\vee cl(\lambda_\alpha) = cl(\vee\lambda_\alpha)$*
Also $\vee int(\lambda_\alpha) \leq int(\vee\lambda_\alpha)$

Definition 2.12. [10] Let (X,T) be a fuzzy topological spaces. A fuzzy set λ defined on X is called a fuzzy strongly nowhere dense set, if $\lambda \wedge (1-\lambda)$ is a fuzzy nowhere dense set in (X,T). That is, λ is a fuzzy nowhere dense set in (X,T), if $intcl[\lambda \wedge (1-\lambda)] = 0$ in (X,T).

Definition 2.13. [10] A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy strongly first category if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy strongly nowhere dense sets in (X,T). Any other fuzzy set in (X, T) is said to be of fuzzy strongly second category.

Definition 2.14. [10] Let λ be a fuzzy strongly first category set in(X,T). Then 1 - λ is called a fuzzy residual set in (X,T).

3. **Fuzzy Strongly Boundary Set**

Definition 3.1. Let λ be a fuzzy set in a fuzzy topological space (X,T). Then the fuzzy strongly boundary of λ is defined as $Bd(\lambda) = cl(\lambda) \wedge cl(1-\lambda)$, $Bd(\lambda)$ is a fuzzy closed set in (X,T) .

Example 3.2. Let X={a, b, c} and λ, μ, γ be the fuzzy sets defined on X as follows:
λ: X → [0,1] is defined by λ(a)=0.7, λ(b)=0.4, λ(c)=0.6
μ: X → [0,1] is defined by μ(a)=0.6, μ(b)=0.6, μ(c)=0.5
γ: X → [0,1] is defined by γ(a)=0.5, γ(b)=0.6, γ(c)=0.7
Then $T=\{0, \lambda, \mu, \gamma, \lambda\vee\mu, \mu\vee\gamma, \lambda\vee\gamma, \lambda\wedge\mu, \mu\wedge\gamma, \lambda\wedge\gamma, \mu\vee(\lambda\wedge\gamma), \lambda\wedge(\mu\vee\gamma), \mu\wedge(\lambda\vee\gamma), \gamma\wedge(\lambda\vee\mu), 1\}$ is a fuzzy topology on I. Now $1-\lambda$, $1-\mu$, $1-\lambda\vee\mu$, $1-\lambda\wedge\mu$ and 1 are fuzzy closed sets in (X,T).On computation, we see that $cl(\lambda) \wedge cl(1-\lambda) = 1-\lambda\vee\mu$; $cl(\mu) \wedge cl(1-\mu) = 1-\lambda\vee\mu$; $cl(\lambda\vee\mu)\wedge cl(1-\lambda\vee\mu) = 1-\lambda\vee\mu$. $Bd(\lambda) = 1-\lambda\vee\mu \implies intcl[(\lambda\vee\mu)\wedge(1-\lambda\vee\mu)] = intcl(1-\lambda\vee\mu) = int(1-\lambda\vee\mu) = 0$ and hence $Bd(\lambda)$ is a fuzzy strongly nowhere dense set in (X,T). Also, $Bd(\mu)$ and $Bd(\lambda\vee\mu)$ is a fuzzy strongly nowhere dense sets in (X,T). Hence $Bd(\lambda)$, $Bd(\mu)$ and $Bd(\lambda\vee\mu)$ is a fuzzy strongly boundary sets in (X,T).

Proposition 3.3. *If λ is a fuzzy strongly nowhere dense set in (X,T), then $Bd(\lambda)$ is a fuzzy strongly nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy strongly nowhere dense set in (X,T). Then $intcl[\lambda \wedge (1-\lambda)] = 0$ in (X,T). Since $\lambda \wedge (1-\lambda) \leq cl(\lambda) \wedge cl(1-\lambda)$ in (X,T), $intcl[(\lambda) \wedge (1-\lambda)] \leq intcl[cl(\lambda) \wedge cl(1-\lambda)]$

and hence $0 \leq intcl[cl(\lambda) \wedge cl(1-\lambda)]$. That is, $intcl[cl(\lambda) \wedge cl(1-\lambda)] = 0$. This implies that $intcl[Bd(\lambda)] = 0$. Hence Bd(λ) is a fuzzy strongly nowhere dense set in (X,T). □

Proposition 3.4. *If Bd(λ) is a fuzzy strongly nowhere dense set in (X,T) then Bd(int λ) is a fuzzy strongly nowhere dense set in (X,T).*

Proof. Let Bd(λ) is a fuzzy strongly nowhere dense set in (X,T). Then $intcl[Bd(\lambda)] = 0$. This implies that $intcl[cl(\lambda) \wedge cl(1-\lambda)] = 0$. Now, $Bd(int\lambda) = cl(int\lambda) \wedge cl(1-int\lambda)$. This implies that $Bd(int\lambda) = clint\lambda \wedge cl(cl(1-\lambda)) = clint\lambda \wedge cl(1-\lambda)$ Now $Bd(int\lambda) \subset cl(\lambda) \wedge cl(1-\lambda) \subset Bd(\lambda)$. Thus $Bd(int\lambda) \subset Bd(\lambda)$. By Proposition 3.3, Bd(λ) is afuzzy strongly nowhere dense set in (X,T). Also Bd(int λ) is a subset of Bd(λ). Therefore Bd($int\lambda$) is also a fuzzy strongly nowhere dense set in (X,T). □

Proposition 3.5. *If Bd(λ) is a fuzzy strongly nowhere dense set in (X,T), then Bd(cl λ) is a fuzzy strongly nowhere dense set in (X,T).*

Proof. Let Bd(λ) is a fuzzy strongly nowhere dense set in (X,T). Then $intcl[Bd(\lambda)] = 0$. This implies that $intcl[cl(\lambda) \wedge cl(1-\lambda)] = 0$. Now, $Bd(cl\lambda) = cl(cl\lambda) \wedge cl(1-cl\lambda)$. This implies that $Bd(cl\lambda) = cl(\lambda) \wedge cl(int(1-\lambda)) = cl(\lambda) \wedge clint(1-\lambda)$. Now $Bd(cl\lambda) \subset cl(\lambda) \wedge cl(1-\lambda) \subset Bd(\lambda)$. Thus $Bd(cl\lambda) \subset Bd(\lambda)$. By Proposition 3.3, Bd(λ) is a fuzzy strongly nowhere dense set in (X,T). Also Bd(cl λ) is a subset of Bd(λ). Therefore Bd($cl\lambda$) is also a fuzzy strongly nowhere dense set in (X,T). □

Proposition 3.6. *If Bd(λ) and Bd(μ) are fuzzy strongly nowhere dense set in a fuzzy topological space (X,T), then Bd(int λ ∧ μ) is a fuzzy strongly nowhere dense set in (X,T).*

Proof. Let Bd(λ) and Bd(μ) are fuzzy strongly nowhere dense set in (X,T). Then $intcl[Bd(\lambda)] = 0$ *and* $intcl[Bd(\mu)] = 0$. This implies that $intcl[cl(\lambda) \wedge cl(1-\lambda)] = 0$ *and* $intcl[cl(\mu) \wedge cl(1-\mu)] = 0$in (X,T). Now, $intcl[Bd(\lambda \wedge \mu)] = intcl[cl(\lambda \wedge \mu) \wedge cl(1-\lambda \wedge \mu)] = 0$. Then By Proposition 3.3, λ is a fuzzy strongly nowhere dense set in (X,T), then Bd(λ) is a fuzzy strongly nowhere dense set in (X,T) which implies that $\lambda \wedge \mu$ is a subset of λ and μ, λ and μ is a fuzzy strongly nowhere dense set in (X,T) and Bd($\lambda \wedge \mu$) is a fuzzy strongly nowhere dense sets in (X,T). □

Example 3.7. We follow the Ex.3.2, Bd(λ) and Bd(μ) is a fuzzy strongly nowhere dense set in (X,T) which implies that $Bd(\lambda \wedge \mu) = intcl[(\lambda \wedge \mu) \wedge (1-\lambda \wedge \mu)] = intcl(1-\mu) = int(1-\mu) = 0$. And hence Bd($\lambda \wedge \mu$) is a fuzzy strongly nowhere dense sets in (X,T).

4. **Fuzzy Strongly Nodec Space**

Definition 4.1. A fuzzy topological space (X,T) is called a fuzzy strongly nodec space if every fuzzy strongly nowhere dense set in (X,T) is a fuzzy closed set in (X,T).

Example 4.2. Let X={a, b, c}. Let I=[0,1]. Then the fuzzy sets λ, μ, γ and δ are defined on X as follows:

λ: X → [0,1] is defined by λ(a)=0.5, λ(b)=0.6, λ(c)=0.7
μ: X → [0,1] is defined by μ(a)=0.4, μ(b)=0.8, μ(c)=0.3
γ: X → [0,1] is defined by γ(a)=0.5, γ(b)=0.6, γ(c)=0.6
δ: X → [0,1] is defined by δ(a)=0.4, δ(b)=0.6, δ(c)=0.3 Then, T = { 0, λ, μ, γ, 1 } is a fuzzy topology on X. On computation we see that cl(λ) = 1; cl(μ) = 1, cl(γ)=1, cl($\lambda \vee \mu$) = 1, cl($\mu \vee \gamma$) = 1, cl($\lambda \wedge \mu$) = 1, int($1-\lambda$)=0, int($1-\mu$)=0, int($1-\gamma$)=0, int($1-\lambda \vee \mu$)=0, int($1-\mu \vee \gamma$)=0, int($1-\lambda \vee \mu$)=0 in (X,T).

Now, int[cl(δ)] = int(1-$\lambda \wedge \mu$)=0 and hence δ is a fuzzy nowhere dense set in (X,T). Also int[$\delta \wedge (1-\delta)$] = $int(1-\lambda)$ = 0 and hence δ is a fuzzy strongly nowhere dense set in (X,T).

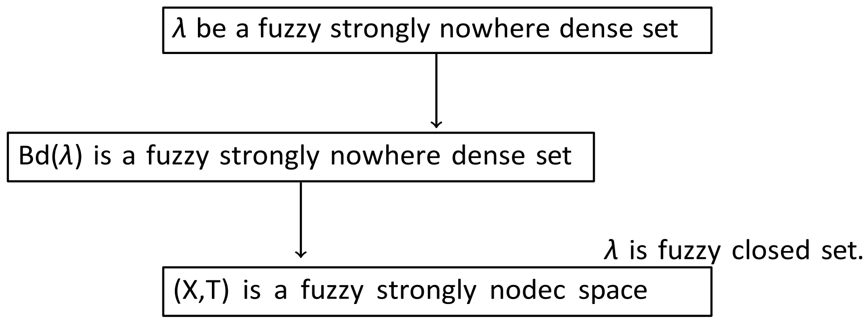
Now, int[cl(λ)] = int(1) $\neq$ 0 and hence λ is not a fuzzy nowhere dense set in (X,T). Also int cl[$\lambda \wedge (1-\lambda)$] = $int[cl(1-\lambda)]$ = $int(1-\lambda)$ = 0 and hence λ is a fuzzy strongly nowhere dense set in (X,T). Which implies that fuzzy strongly nowhere dense sets is a fuzzy closed set in (X,T). Hence (X,T) is fuzzy strongly nodec space.

Proposition 4.3. *If λ is a fuzzy strongly nowhere dense set in a fuzzy strongly nodec space then Bd(λ) is a fuzzy strongly nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy strongly nowhere dense set in (X,T). Then $intcl[\lambda \wedge (1-\lambda)]$ = 0 in (X,T). Since (X,T) is a fuzzy strongly nodec space. By hypothesis, λ is a fuzzy closed set in (X,T). Now Bd(λ) = cl(λ) $\wedge$ $(1-\lambda)$ = $\lambda \wedge 1-\lambda$. Therefore Bd(λ) $\subset \lambda$, λ is a fuzzy strongly nowhere dense set in (X,T) and hence Bd(λ) is a fuzzy strongly nowhere dense set in (X,T).□

Proposition 4.4. *Let (X,T) be a fuzzy topological space. If Bd(λ) is a fuzzy strongly nowhere dense set λ is a fuzzy closed set in (X,T), then (X,T) is a fuzzy strongly nodec space.*

Proof



Proposition 4.5. *Let (X,T) be a fuzzy topological space. Then the following are equivalent*

(i) (X,T) is a fuzzy strongly nodec space

(ii) For every fuzzy strongly first category set λ is a fuzzy $F_\sigma-$ set in (X,T).

(iii) For every fuzzy strongly residual category set μ is a fuzzy $G_\delta-$ set in (X,T).

Proof. (1) ⇒ (2). Let λ be a fuzzy first category set in (X,T). Then $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy strongly nowhere dense sets in (X,T). since (X,T) is a fuzzy strongly nodec space, every fuzzy strongly nowhere dense sets in (X,T) is a fuzzy closed sets in (X,T). This implies that $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where $(1-\lambda_i) \in T$ and hence λ is a fuzzy $F_\sigma-$ set in (X,T).

(2) ⇒ (3). Let μ be a fuzzy residual category set in (X,T). Then $1-\mu$ is a fuzzy strongly first category set in (X,T). Now $1-\mu = \bigvee_{i=1}^{\infty}(1-\mu_i)$,$(1-\mu_i)$'s are fuzzy strongly nowhere dense sets in (X,T) and by hypothesis $1-\mu_i$ is a fuzzy closed sets in (X,T). Then $1-\mu = 1-\bigvee_{i=1}^{\infty}(\mu_i) \Longrightarrow \mu = \bigvee_{i=1}^{\infty}\mu_i$, $\mu_i \in T$ and hence μ is a fuzzy $G_\delta-$ set in (X,T).

(3) ⇒ (1).Let λ be a fuzzy first category set in (X,T). Then $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy strongly nowhere dense sets in (X,T). Now λ is fuzzy strongly first category set in (X,T) implies that $1-\lambda$ is a fuzzy strongly residual category set in (X,T). By hypothesis $1-\lambda$ is a fuzzy closed sets in (X,T). Then (λ_i)'s are fuzzy strongly nowhere dense sets and fuzzy closed sets in (X,T). Hence (X,T) is a fuzzy strongly nodec space. □

5. **Conclusion**

In this paper the concepts of fuzzy strongly boundary set and fuzzy strongly nodec spaces are discussed. Some of its characteristics and examples are established. This shall be extended in the future research studies.

References

1. Ahmed B and Athar M, ”Fuzzy sets, fuzzy s-open and s-closed functions,” Hindawi Publishing Corporation Advances in Fuzzy Systems, vol.2009,article id-303042. .
2. Azad K. K.., On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl., 82 (1)(1981) 14 – 32.
3. Balasubramanian.G, Maximal fuzzy topologies, Kybernetika, 31(5)(1995),459 – 464.
4. Chang C. L, Fuzzy Topological Spaces, J. Math. Anal. Appl. 24,(1) (1968),182-190.
5. Pu P. M. and Liu Y. M., ”Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence”, Journal of Mathematical Analysis and Applications,Vol.76 no.2 PP.571-599,1980.
6. Tang X, Spatial Object modeling in fuzzy topological spaces withapplicationsto land cover change in China, Ph. D. Dissertation, Universityof Twente, Enschede, The Netherlands, 2004 ITC Dissertation No. 108.
7. Thangaraj.G and Anjalmose.S, Fuzzy Baire Spaces and Functions” International Journal of Fuzzy Mathematics and system Spaces, vol.3, no.4(2013),pp.285-293.
8. Thangaraj.G and Balasubramanian.G, On somewhat fuzzy continuous functions, The Journal of . fuzzy Mathematics, 11(3)(2003)725 – 736.
9. Thangaraj.G and Poongothai.E, On fuzzy σ - Baire spaces. Inter.J.fuzzy Math. Sys., 3(4)(2013),275-283.
10. Thangaraj.G and Palani.R, Fuzzy Strongly Baire Spaces, Journal of the International Mathematical Virtual Institute Vol.8(2018),35-51.
11. Zadeh L. A, Fuzzy Sets, Information and Control, Vol. 8 (1965), 338-353.