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Penta Graph Topological Space: A New Perspective in Graph Induced Topology

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Abstract: The primary objective of this paper is to introduce the concepts of lower penta subgraph, upper penta subgraph, and boundary penta subgraph. Based on these foundational structures, the author defines a new topological framework called the Penta graph topological space. Furthermore, several fundamental properties of this new topological space are discussed and established.

Keywords: Penta graph topological space, Lower penta subgraph, Upper penta subgraph, Boundary penta subgraph, Penta open subgraph, Penta closed subgraph, Penta graph interior of a subgraph, Penta graph closure of a subgraph.

1. Introduction and preliminaries

M. Lellis Thivagar, S. P. R. Priyalatha, and D. Evangeline Christina Lily [9] proposed an alternative formulation of nano topological spaces by leveraging concepts from graph theory, particularly focusing on the neighborhood structure of vertices to establish bigraphs in nano topology and Binoy Balan [3] introduced new Graph Topology on decomposition of graph. Inspired by their work we have developed a new concept called the penta graph topological space which extends their ideas by incorporating specialized graph-based structures to explore deeper topological properties. In this paper we use the multigranulation method to introduce a new definition of the penta graph topological space. The idea of the multigranulation is based on using multirelation instead of a single relation to obtain better approximation. Thus we start by giving the definition of multigranular rough sets based on equivalence relations.

Definition 1.1. [3] The upper approximation of a subgraph H, denoted as $N^*(H)$, captures the subgraphs that overlap with H in some way but may not be fully contained within it. The approximations are based on the collection $R(G)$, which consists of distinct edge induced sub graphs of G. These sub graphs are derived from the edge sets of G. Formally, we define the upper approximation as:

$$N^*(H) = \{\cup H_i : H_i \in R(G) \ and \ H_i \cap H \neq \phi\}$$

This definition suggests that $N^*(H)$ includes all the edge-induced subgraphs H_i from $R(G)$, that share at least one edge with H. Intuitively, this represents the idea of a loose or larger approximation of H, where the graph can extend beyond the boundaries of H but still retains some overlap.

Definition 1.2. [3] The lower approximation of a subgraph H, denoted as $N_*(H)$, is more restrictive. It includes only those edge-induced subgraphs that are completely contained within H.

Formally, the lower approximation is defined as:

$$N_*(H) = \{\cup H_i : H_i \in R(G) and\ H_i \subseteq H\}$$

Thus, $N_*(H)$ consists of all the subgraphs in $R(G)$ that are subsets of H, meaning they fit entirely within the boundaries of H. This approximation can be thought of as the tightest or smallest approximation of H, where the graph cannot extend beyond H.

Definition 1.3. [3] The boundary region of H in G is defined as the difference between the upper and lower approximations:

$$B_N(H) = N^*(H) - N_*(H)$$

The boundary region captures the elements that are in the upper approximation but not in the lower approximation. In other words, it consists of subgraphs from $R(G)$ that partially overlap with H but are not fully contained within it. This can be seen as the border or boundary between H and the rest of the graph, highlighting areas where H is not completely represented.

Definition 1.4. (3) Consider a non-empty simple graph $G = (V, E)$ and a collection of distinct edge-induced subgraphs of G generated by the subsets of E. For any subgraph H of G define

$$\Gamma_R(H) = \{G, \phi, N^*(H), N_*(H)\ , B_N(H)\}$$

Where $\Gamma_R(H)$ satisfies the following axioms

- The graph G (the full graph) and the empty graph ϕ belong to $\Gamma_R(H)$
- An arbitrary union of members of $\Gamma_R(H)$ is in $\Gamma_R(H)$
- A finite intersection of members of $\Gamma_R(H)$ is in $\Gamma_R(H)$

The collection $\Gamma_R(H)$is termed a N −graph topology on G. The pair $(G, \Gamma_R(H))$ is designated as the N −Graph topological space. The elements of $\Gamma_R(H)$ are referred to as N −open subgraphs in G, and the complement of an N −open subgraph is termed a N-closed subgraph of $\Gamma_R(H)$. A subgraph that is both a N −open subgraph and a N −closed subgraph is designated as a N −clopen subgraph.

Definition 1.5. (3) For a subgraph H of a non-empty simple graph G, the Indiscrete N-graph topology is defined as

$$\Gamma_R(H) = \{\ \phi, G\}$$

That is, $\Gamma_R(H)$ is a collection consisting only of the trivial subgraphs of G, the empty graph and the full graph itself.

This is the coarsest possible N −graph topology on G containing the least number of N −open subgraphs.

Definition 1.6. (3) Let $G = (V, E)$ be a graph and let $\Gamma_R(H)$ be a N −graph topological space for any subgraph H of G.

Let S be a subgraph of G and consider an edge $e \in E(S)$. We say that S is a neighborhood graph of e if there exists a subgraph $H_i \in \Gamma_R(H)$ such that:

$$e \in \mathrm{E}(H_{\mathrm{i}}) \quad \text{and} \quad H_{\mathrm{i}} \subset S$$

In this case, the edge e is called an edge interior of S.

The collection of all edge interiors of S is denoted by S'

The N-interior subgraph of S is the subgraph generated by S', and is defined as:

$$N_H Int(S) = \langle \mathrm{S}' \rangle$$

Definition 1.7. (3) $G = (V, E)$ be a graph, and let $\Gamma_R(H)$ be an N −graph topology on G.

Let K be an edge-induced subgraph of G. An edge $e \in E(G) \setminus E(K)$ is called an edge limit of K if for all open subgraphs $K' \in \Gamma_R(H)$ such that $e \in E(K')$ we have

$$E(K) \cap \mathrm{E}(\mathrm{K}') \neq \phi$$

The collection of all edge limits of an edge induced subgraph K is denoted by K^*.

The N-closure of the subgraph K is the edge-induced subgraph generated by the edge set $E(K) \cup \mathrm{K}^*$and is denoted by

$$N_H Cl(S) = \langle E(K) \cup \mathrm{K}^* \rangle$$

- $N_H Int(S)$is the largest N −open subgraph of S
- $N_H Cl(S)$ is the smallest N −closed super graph of S

Definition 1.8. (3) A subgraph S of a N −graph topology on G is called N −dense subgraph if

$$N_H Cl(S) = G$$

That is the N −closure of S equals the full graph G

Definition 1.9. (5) Let $G = (V, E)$ be a graph, $v \in V(G)$. Then the neighborhood of v denote that $\mathcal{N}(\mathrm{v})$ were defined by $\mathcal{N}(\mathrm{v}) = \{v\} \cup \{u \in V(G) : \overline{vu} \in E(G)\}$

2. Lower Penta and Upper Penta subgraph

In this section, a novel method for constructing new types of graphs based on the concepts of lower, upper, and boundary subgraphs are introduced. Subsequently, the definitions of Lower Penta Subgraphs, Upper Penta Subgraphs, and Boundary Penta Subgraphs are defined.

Definition 2.1. Let $G = (V, E)$ be a non-empty graph with minimum 3 vertices and $\mathcal{N}(v)$ be the neighborhood of a vertex $v \in V(G)$. Consider D_i be any different subgraphs of G, then a lower approximations and upper approximations are defined by

$$\mathcal{P}_{-}(D_i) = \{v : \mathcal{N}(v) \subseteq D_i\} \; and \; \mathcal{P}^{-}(D_i) = \{v : \mathcal{N}(v) \cap D_i \neq \phi\}$$

Definition 2.2. Let $G = (V, E)$ be a non-empty graph with minimum 3 vertices and $N(v)$ be the neighborhood of a vertex $v \in V(G)$. Consider D_i, i= 1, 2, 3, 4, 5 be five different subgraphs of G.

A lower Penta subgraph $\mathcal{P}_*(Z)$ is defined as the union of $\mathcal{P}_{-}(D_i)$. Formally,

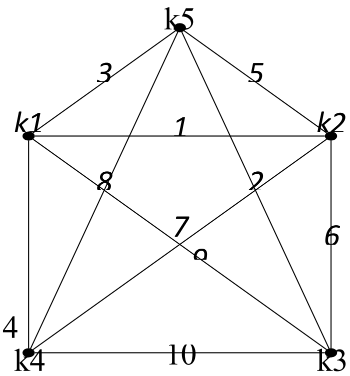
$$\mathcal{P}_*(Z) = \cup_{v\in V(G)}\{v: (\mathcal{N}(v) \subseteq D_1) \cup (\mathcal{N}(v) \subseteq D_2) \cup (\mathcal{N}(v) \subseteq D_3) \cup (\mathcal{N}(v) \subseteq D_4) \cup (N(v) \subseteq D_5)\}$$

An upper Penta subgraph $\mathcal{P}^*(Z)$ is defined as the intersection of $\mathcal{P}^{-}(D_i)$. Formally,

$$\mathcal{P}^*(Z) = \cup_{v\in V(G)}\{v: (\mathcal{N}(v) \cap D_1 \neq \phi) \cap (\mathcal{N}(v) \cap D_2 \neq \phi) \cap (\mathcal{N}(v) \cap D_3 \neq \phi) \cap (\mathcal{N}(v) \cap D_4 \neq \phi) \cap (N(v) \cap D_5 \neq \phi)\}$$

The boundary penta subgraph is defined as $B_{\mathcal{P}}(Z) = \mathcal{P}^*(Z) - \mathcal{P}_*(Z)$

Example 2.3. Consider an undirected graph G with 10 edges and 5 vertices



D_i, i = 1, 2, 3, 4, 5	$\mathcal{P}_{-}(D_i)$	$\mathcal{P}^{-}(D_i)$
{k1, k2}	ϕ	G
{k1, k4, k5}	ϕ	G
{k1, k2, k3}	ϕ	G
{k2, k3, k4}	ϕ	G
{k1, k4}	ϕ	G

Hence the Lower penta subgraph and the Upper penta subgraph and Boundary Penta subgraph is $\mathcal{P}_*(Z)$= ϕ, $\mathcal{P}^*(Z)$= $\{G\}$, $B_{\mathcal{P}}(Z)$=$\{G\}$

Property 2.4. Let $G = (V, E)$ be a graph with minimum 3 vertices and $\mathcal{N}(v)$ be a neighborhood of a vertex $v \in V(G)$ of G. Consider D_i, be different subgraphs of G, then we have

(1) $\mathcal{P}_{-}(D_i) \subseteq D_i \subseteq \mathcal{P}^{-}(D_i)$
(2) $\mathcal{P}_{-}(G) = \text{G} = \mathcal{P}^{-}(\text{G})$

Proof:

(1) Let $v \in \mathcal{P}\_(D_i)$. By the definition, $\mathcal{P}\_(D_i)$ is a collection of a vertices which neighborhood belongs to D_i. Obviously $\mathcal{P}\_(D_i) \subseteq (D_i)$. Thus $\mathcal{P}\_(D_i) \subseteq (D_i) \neq \phi$
From this $\mathcal{N}(v) \cap (D_i) \neq \phi$. Therefore $v \in \mathcal{P}^-(D_i)$.
Which implies that $\mathcal{P}\_(D_i) \subseteq D_i \subseteq \mathcal{P}^-(D_i)$

(2) The graph G is both a subset of itself and intersects with itself, so applying the lower and upper approximation operators to yields $\mathcal{P}\_(G) = G = \mathcal{P}^-(G)$

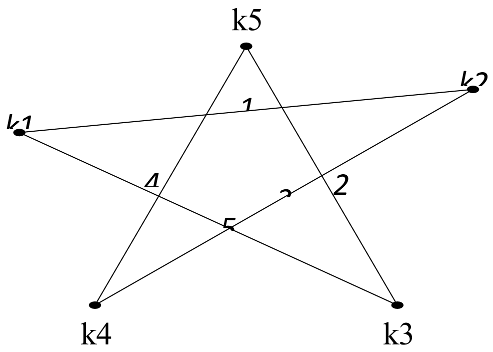
3. Penta Graph Topological space

Definition 3.1. Let $G = (V, E)$ be a graph with minimum 3 vertices. Consider $D_i, i = 1,2,3,4,5$ be five different subgraphs of G, then $\eta\mathcal{P}(Z) = \{G, \phi, \mathcal{P}_*(Z), \mathcal{P}^*(Z), B_{\mathcal{P}}(Z)\}$
Let $D_i \subseteq G$, $\eta\mathcal{P}(Z)$ satisfies the following axioms

(1) $G, \phi \in \eta\mathcal{P}(Z)$, where $G = V(G)$ = full graph, ϕ = empty graph
(2) Any union of elements of $\eta\mathcal{P}(Z)$ is in $\eta\mathcal{P}(Z)$
(3) A finite intersection of the elements of $\eta\mathcal{P}(Z)$ is in $\eta\mathcal{P}(Z)$

Then a pair $\eta\mathcal{P}(Z)$ is known as penta graph topology on G with regard to a subgraph D_i of G. Call the pair $(G, \eta\mathcal{P}(Z))$ as the penta graph topological space on G. The elements of penta graph topological spaces are regard as penta open subgraph of G and the complement of the penta open subgraph of G is called penta closed subgraph of G.

Example 3.2. Consider an undirected graph G with 5 edges and 5 vertices.



D_i, i = 1, 2, 3, 4, 5	$\mathcal{P}\_(D_i)$	$\mathcal{P}^-(D_i)$
{k1, k2}	ϕ	{k1, k2, k3, k4}
{k1, k4, k5}	ϕ	G
{k1, k2, k3}	{k1}	G
{k1, k2, k3, k4}	{k1, k2}	G
{k1, k4}	ϕ	G

Hence the Lower penta subgraph and the Upper penta subgraph and Boundary Penta subgraph is $\mathcal{P}_*(Z)$ ={k1, k2}, $\mathcal{P}^*(Z)$= {k1, k2, k3, k4}, $B_{\mathcal{P}}(Z)$= {k3, k4} then the Penta graph topology is defined as $\eta\mathcal{P}(Z)$={G,ϕ, {k1, k2}, {k1, k2, k3, k4}, {k3, k4}}

Definition 3.3. Let G be a graph with minimum 3 vertices and $\mathcal{N}(v)$ be the neighborhood of a vertex $v \in V(G)$. Consider D_i, $i = 1, 2, 3, 4, 5$ be five different subgraphs of G, Some forms of penta graph topology defined as follows:

PG-Form I	$\mathcal{P}_*(Z) \neq \mathcal{P}^*(Z)$, where $\mathcal{P}_*(Z) \neq \phi, \mathcal{P}^*(Z) \neq \mathrm{G}$
PG-Form II	$\mathcal{P}_*(Z) = \phi, \mathcal{P}^*(Z) \neq \mathrm{G}$
PG-Form III	$\mathcal{P}_*(Z) \neq \phi, \mathcal{P}^*(Z) = \mathrm{G}$
PG-Form IV	$\mathcal{P}_*(Z) = \phi, \mathcal{P}^*(Z) = \mathrm{G}$

Result of PG-Form I	$\eta\mathcal{P}(Z)$= { G, $\phi, \mathcal{P}_*(Z), \mathcal{P}^*(Z), B_{\mathcal{P}}(Z)$}
Result of PG-Form II	$\eta\mathcal{P}(Z)$= { G, $\phi, \mathcal{P}^*(Z)$}
Result of PG-Form III	$\eta\mathcal{P}(Z)$= { G, $\phi, \mathcal{P}_*(Z), B_{\mathcal{P}}(Z)$}
Result of PG-Form IV	$\eta\mathcal{P}(Z)$= { G, ϕ}

Definition 3.4. Let $\eta\mathcal{P}(Z)$ be a Penta graph topological space concerning to five different subgraph D_i of G. The Penta graph interior of a subgraph Q of G is described as the union of all Penta open subgraph which is an subgraph of Q and it is identified by $\mathfrak{N}_p Int$(Q). That is $\mathfrak{N}_p Int$(Q) is the greatest Penta open subgraph of Q. The Penta graph closure of Q is specific as the intersection of all Penta closed subgraph which is a super graph of Q and it is identified by $\mathfrak{N}_p Cl$ (Q). Therefore, $\mathfrak{N}_p Cl$(Q) is the lowest Penta closed subgraph of Q.

Example 3.5. Consider example 3.2 with 5 edges, 5 vertices and D_i be five different subgraph then its Penta graph topology is defined as
η$\mathcal{P}(Z)$= {G, ϕ, {k1, k2}, {k1, k2, k3, k4}, {k3, k4}}} and
η$\mathcal{P}(Z)^c$ = {G, ϕ, {k3, k4, k5}, {k5}, {k1, k2, k5}. For a subgraph Q = {k1}then its
$\mathfrak{N}_p$Cl (Q)={k1, k2, k5} and $\mathfrak{N}_p$Int(Q) = ϕ

Property 3.6. Let G be the undirected graph with penta graph topological space (G, η$\mathcal{P}$(Z)) and $T, P \subseteq G$ then the following properties holds:

(1) T is penta open subgraph $\Leftrightarrow \mathfrak{N}_p\mathrm{Int}[T] = T$
(2) $\mathfrak{N}_p\mathrm{Int}[\phi] = \phi$
(3) $\mathfrak{N}_p\mathrm{Int}[T] = \mathfrak{N}_p\mathrm{Int}[\mathfrak{N}_p\mathrm{Int}[T]]$
(4) $\mathfrak{N}_p\mathrm{Int}[T \cup P] \subseteq \mathfrak{N}_p\mathrm{Int}[T] \cup \mathfrak{N}_p\mathrm{Int}[P]$
(5) $\mathfrak{N}_p\mathrm{Int}[T \cap P] = \mathfrak{N}_p\mathrm{Int}[T] \cap \mathfrak{N}_p\mathrm{Int}[P]$
(6) $\mathfrak{N}_p\mathrm{Cl}[T] = \mathfrak{N}_p\mathrm{Cl}[\mathfrak{N}_p\mathrm{Cl}[T]]$

Theorem 3.7. Let G be the undirected graph with penta graph topological space $(G, \eta\mathcal{P}(Z))$ and for a subgraph T, P of G and $T \subseteq P$ then the following conditions are hold:

(1) $T \subseteq P \Rightarrow \mathfrak{N}_p\mathrm{Int}[T] \subseteq \mathfrak{N}_p\mathrm{Int}[P]$

(2) $\mathfrak{N}_p\text{Int}[\mathfrak{N}_p\text{Int}\,[\,T]] \subseteq \mathfrak{N}_p\text{Int}\,[\mathfrak{N}_p\text{Int}[P]]$

proof: Obvious.

4. CONCLUTION

A new form of topological space termed the Penta Graph Topological Space is introduced. This space is induced by the vertices of a graph G and constructed using five distinct sub-graphs of G. The framework aims to deepen our understanding of graph properties and their corresponding topological interpretations. This study opens new avenues for future research by highlighting the potential of graph-theoretical methods to enrich topological theory, and conversely, how topological insights can enhance the analysis of graph structures.

FUTURE WORK

In future work, the concept of penta graph topology could be extended to explore notions such as penta graph continuity, penta graph homeomorphism, and penta graph Hausdorff properties. This extension could provide deeper insights into the relationships between graph- theoretical structures and topological concepts, enriching the understanding of both fields.

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