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Weyl's Theorem for Algebraically Ɱ* Quasi Paranormal Operators

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Abstract:

In this paper, a new operator Ɱ*quasi paranormal operator is introduced and studied. A bounded linear operator $\mathcal{J}$ in a Hilbert space H is said to be a Ɱ* quasi paranormal operator if it satisfies $\|\mathcal{J}^{*2}x\|^2 \leq Ɱ\ \|\mathcal{J}^{3}x\|.\|\mathcal{J}\,x\|$, for each $x\epsilon\, H$, where Ɱ is a real positive number. We shall deal with the characteristics of Ɱ* quasi paranormal operator & establish the result as $Ɱ^2\mathcal{J}^{*3}\mathcal{J}^{3} - 2\lambda\mathcal{J}^{2}\mathcal{J}^{*2} + \lambda^2\mathcal{J}^{*}\mathcal{J} \geq 0,$ for each $\lambda > 0$. Furthermore, we discuss algebraically Ɱ* quasi paranormal operator and its properties. It is also demonstrated that Weyl & Weyl type theorems hold for algebraically Ɱ* quasi paranormal operators.

Key Words:

Ɱ* quasi paranormal operator, Algebraically Ɱ*quasi paranormal operator, Weyl's theorem.

Ɱ* Quasi Paranormal Operator

Main Result: 1.1

An operator $\mathcal{J}\epsilon$ B(H) is Ɱ* *q*uasi *p*aranormal iff

$$Ɱ^2\mathcal{J}^{*3}\mathcal{J}^{3} - 2\lambda\mathcal{J}^{2}\mathcal{J}^{*2} + \lambda^2\mathcal{J}^{*}\mathcal{J} \geq 0\,, \forall\lambda > 0$$

Proof

Let $\mathcal{J}$ be Ɱ* *q*uasi *p*aranormal, for a fixed real +ve Ɱ,

$$\|\mathcal{J}^{*2}x\|^2 \leq Ɱ\ \|\mathcal{J}^{3}x\|\,.\,\|\mathcal{J}x\|,\ \forall\ x\epsilon\, H$$

Then

$$\|\mathcal{J}^{*2}x\|^2 - Ɱ\ \|\mathcal{J}^{3}x\|\,.\|\mathcal{J}\,x\| \leq 0$$

$4\,(||\mathcal{J}^{*2}x||^2)^2 - 4Ɱ^2||\mathcal{J}^{3}x||^2\,||\mathcal{J}x||^2 \le 0$

From the properties of real quadratic forms,

$\lambda^2\big||\mathcal{J}x||^2 - 2\lambda\big|\mathcal{J}^{*2}x||^2 + Ɱ^2||\mathcal{J}^{3}x||^2 \ge 0, \forall x\epsilon H, \forall\lambda > 0$

$\lambda^2\langle\mathcal{J}^{*}\mathcal{J}x/x\rangle - 2\lambda < \langle\mathcal{J}^{2}\mathcal{J}^{*2}x/x\rangle + Ɱ^2\langle\mathcal{J}^{*3}\mathcal{J}^{3}x/x\rangle \ge 0, \forall x\epsilon H, \forall\lambda > 0$

Hence,

$Ɱ^2\,\mathcal{J}^{*3}\mathcal{J}^{3} - 2\lambda\mathcal{J}^{2}\mathcal{J}^{*2} + \lambda^2\mathcal{J}^{*}\mathcal{J} \ge 0, \forall x\epsilon H, \forall\lambda > 0$

Theorem:1.2

Suppose that $\mathcal{J} \in B(H)$ is a Ɱ* quasi paranormal operator. If $\mathcal{J}$ is unitarily equivalent to Ʀ, then Ʀ is known as Ɱ* quasi paranormal.

Proof:

Let $\mathcal{J}$ be unitarily equivalent to Ʀ, ∃ an unitarily operator U such that Ʀ= U*$\mathcal{J}$U. We must show that $Ɱ^2Ʀ^{*3}Ʀ^3 - 2\lambda Ʀ^2Ʀ^{*2} + \lambda^2Ʀ^*Ʀ \ge 0$

∵ $\mathcal{J}$ is of Ɱ* quasi paranormal, we have $Ɱ^2\mathcal{J}^{*3}\mathcal{J}^{3} - 2\lambda\mathcal{J}^2\mathcal{J}^{*2} + \lambda^2\mathcal{J}^*\mathcal{J} \ge 0$

So, $Ɱ^2Ʀ^{*3}Ʀ^3 - 2\lambda Ʀ^2Ʀ^{*2} + \lambda^2Ʀ^*Ʀ \ge 0$

$\Rightarrow Ɱ^2(U^*\mathcal{J}^*U)(U^*\mathcal{J}^*U)(U^*\mathcal{J}^*U)(U^*\mathcal{J}U)(U^*\mathcal{J}U)(U^*\mathcal{J}U) - 2\lambda(U^*\mathcal{J}U)(U^*\mathcal{J}U)(U^*\mathcal{J}^*U)(U^*\mathcal{J}^*U)$
$+ \lambda^2(U^*\mathcal{J}^*U)(U^*\mathcal{J}U) \ge 0$

$\Rightarrow Ɱ^2[U^*\mathcal{J}^*\mathcal{J}^*\mathcal{J}^*\mathcal{J}\mathcal{J}\mathcal{J}U] - 2\lambda[U\mathcal{J}\mathcal{J}\mathcal{J}^*\mathcal{J}^*U^*] + \lambda^2[U^*\mathcal{J}^*\mathcal{J}U] \ge 0$

Hence, $U^*[Ɱ^2U^*\mathcal{J}^{*3}\mathcal{J}^{3} - 2\lambda\mathcal{J}^2\mathcal{J}^{*2} + \lambda^2\mathcal{J}^*\mathcal{J}]U \ge 0$

∴ Ʀ is Ɱ*quasi paranormal.

Theorem:1.3

If $\mathcal{J} \in B(H)$ is a weighted shift operator on H with non zero weight $\delta_ƙ(ƙ = 0, 1, 2, \ldots)$. When $\mathcal{J}$ is Ɱ* quasi paranormal operator iff $|\delta_ƙ|^2 \le Ɱ|\delta_{ƙ+1}||\delta_{ƙ+2}|$ where Ɱ is a real positive and ƙ=1, 2, …

Proof:

Take $\{\delta_ƙ\}_{ƙ=0}^{\infty}$ is an orthonormal basis of H.

Considering, $\mathcal{J}e_ƙ = Ɱ\delta_ƙ e_{ƙ+1}$ & $\mathcal{J}^{*2}e_ƙ = Ɱ\bar{\delta}_{ƙ-1}\,e_{ƙ-1}$, then

$$||\mathcal{J}^{*2}e_ƙ||^2 = ||\,\mathcal{J}(Ɱ\delta_ƙ e_{ƙ+1})||^2$$
$$= Ɱ|\delta_ƙ|^2\;||\mathcal{J}^{*2}e_{ƙ+1}||^2$$
$$= Ɱ|\delta_ƙ|^2\,\left|\bar{\delta}_{ƙ-1}\right|^2\,||e_ƙ||^2$$

$= \text{Ɱ}|\delta_{ƙ}|^4$

And $\|\mathcal{J}^3 e_{ƙ}\|\|\mathcal{J} e_{ƙ}\| = \text{Ɱ}|\delta_{ƙ}|^2|\delta_{ƙ+1}||\delta_{ƙ+2}|$

Thus, $\|\mathcal{J}^{*2}x\|^2 \leq \text{Ɱ}\|\mathcal{J}^3 x\|\|\mathcal{J}x\|$ for each vector $x \in H$ iff $\|\mathcal{J}^{*2}e_{ƙ}\|^2 \leq \text{Ɱ}\|\mathcal{J}^3 e_{ƙ}\|\|\mathcal{J}e_{ƙ}\|$ for every ƙ = 1, 2, ….

Hence, $\mathcal{J}$ is of Ɱ* quasi paranormal operator iff $|\delta_{ƙ}|^2 \leq \text{Ɱ}|\delta_{ƙ+1}||\delta_{ƙ+2}|$, where ƙ = 1, 2, ….

2.Weyl's Theorem for Algebraically Ɱ* Quasi Paranormal Operators

Weyl's theorem is most important property studied by number of Mathematicians around the world. In this section it is shown that Weyl & Weyl type theorem holds for algebraically Ɱ* quasi paranormal operators. If $\mathcal{J}$ is an algebraically Ɱ* quasi paranormal operator for real +ve integer Ɱ, then $\mathcal{J}$ is nilpotent, Polaroid, Reguloid, Isoloid and single valued extension property are discussed.

Introduction:2.1

- Let N($\mathcal{J}$) & *R*($\mathcal{J}$) be the *N*ull space & *R*ange of $\mathcal{J}$, where $\mathcal{J} \epsilon$ B(H)
- Let $\alpha(\mathcal{J}) = dimN(\mathcal{J}), \beta(\mathcal{J}) = dimN(\mathcal{J}^*)$
- Let $\sigma(\mathcal{J}), \sigma_P(\mathcal{J})$ and $\sigma_A(\mathcal{J})$ be the *S*pectrum, *P*oint *S*pectrum & *a*pprox *P*oint *S*pectrum of $\mathcal{J}$.
- If $\mathcal{J}\epsilon$B(H) has closed range, not an infinite dimensional null space & its range has not an infinite co dimension then it is known as *F*redholm.
- The index of a *F*redholm is $i(\mathcal{J}) = \alpha(\mathcal{J})$ - $\beta(\mathcal{J})$. $\mathcal{J}$ is known as Weyl when it is *F*redholm of index 0 & *B*rowder when it is *F*redholm of not infinite descent & ascent.
- Weyl spectrum $\sigma_w(\mathcal{J}) = \{ \lambda \epsilon C: \mathcal{J} - \lambda$ is not Weyl}

  Let $\Pi_{00}(\mathcal{J})$ = { $\lambda \epsilon$ iso$\sigma(\mathcal{J})$: $0 < \alpha(\mathcal{J} - \lambda) < \infty$

Theorem:2.2

If $\mathcal{J}$ be a M* Quasi Paranormal, $\lambda \epsilon C$, when $\sigma(\mathcal{J}) = \{\lambda\}$, then $\mathcal{J} = \lambda$.

Proof:

Let us consider the 2 cases as follows:

Case:(1): λ $equal$ to 0

If $\mathcal{J}$ is paranormal, then $\mathcal{J}$ is *n*ormaloid,

∴ $\mathcal{J}$ = 0

Case:(2): λ not $equal$ to 0

Here $\mathcal{J}$ is invertible & since $\mathcal{J}$ is paranormal, $\mathcal{J}^{-1}$ is also paranormal.

∴ $\mathcal{J}^{-1}$ is *n*ormaloid.

On the contrary, $\sigma(\mathcal{J}^{-1}) = \left\{\frac{1}{\lambda}\right\}$

Thus, $\| \mathcal{J} \| \| \mathcal{J}^{-1} \| = |\lambda| \left|\frac{1}{\lambda}\right| = 1$

$\mathcal{J}$ is convexoid using [1, Lemma 3]

So, $W(\mathcal{J}) = \{\lambda\}$,

$\therefore \mathcal{J} - \lambda$.

Hence, $\mathcal{J} = \lambda$

Lemma:2.3

Let $\mathcal{J} \in B(H)$ be Ɱ* quasi paranormal.

When $\lambda \in \sigma_p(\mathcal{J}) - \{0\}$, then $\lambda \in \sigma_p(\mathcal{J}^*)$

Thus, $\mathcal{J}$ be an *a*lgebraically Ɱ* quasi *p*aranormal when ∃ not a constant complex *p*olynomial p, ∋ p($\mathcal{J}$) is also Ɱ* quasi paranormal operator.

Lemma:2.4

If $\mathcal{J}$ is a Ɱ* quasi paranormal & P is an invariant subspace of $\mathcal{J}$, then G/P is Ɱ* quasi paranormal where Ɱ is a positive real number.

Theorem:2.5

Let $\mathcal{J}$ be algebraically Ɱ* quasi paranormal where Ɱ is positive real value and $\sigma(\mathcal{J}) = \{\lambda\}$, then $\mathcal{J} - \lambda I$ be *n*ilpotent.

Proof

When $\mathcal{J}$ is *a*n *a*lgebraically Ɱ* *q*uasi *p*aranormal ∃ not a constant polynomial *p*(g) ∋ $p(\mathcal{J})$be Ɱ* quasi paranormal for non negative real Ɱ,

By *Theorem 2.2*

$\sigma(p(\mathcal{J})) = p(\sigma(\mathcal{J})) = \{p(\lambda)\} \Rightarrow p(\mathcal{J}) - p(\lambda)I$ *be n*ilpotent.

Let $p(z) - p(\lambda) = a(Z - \lambda_0)^{k_0}(Z - \lambda_1)^{k_1} \ldots (Z - \lambda_t)^{k_t}$ where $\lambda_j \neq \lambda_s$ when $j \neq s$.

For some non negative integer l,

$$0 = (p(\mathcal{J}) - p(\lambda_0)I)^l = a(\mathcal{J} - \lambda_0 I)^{lk_0}(\mathcal{J} - \lambda_1 I)^{lk_1} \ldots (\mathcal{J} - \lambda_t I)^{lk_t}$$

Since $\mathcal{J} - \lambda_1 I, \mathcal{J} - \lambda_2 I, \ldots \mathcal{J} - \lambda_t I$ are invertible, $(\mathcal{J} - \lambda_0 I)^{lk_0} = 0$.

Hence, $\mathcal{J} - \lambda I$ is nilpotent.

Lemma:2.6

Let $\mathcal{J}$ be a quasi nilpotent algebraically Ɱ* quasi paranormal, then $\mathcal{J}$ is nilpotent.

Proof

Let **P**$(\mathcal{J})$ be a Ɱ* *q*uasi *p*aranormal for few non constant polynomial **P**.

When $\sigma(\mathrm{p}(\mathcal{J})) = \mathrm{p}(\sigma(\mathcal{J}))$, then $\mathrm{p}(\mathcal{J})$ - $\mathrm{p}(\sigma)$ be a *q*uasi *n*ilpotent.

Then *Theorem 2.2* would imply that

$C\mathcal{J}^m(\mathcal{J}-\lambda_1), (\mathcal{J}-\lambda_2), \ldots (\mathcal{J}-\lambda_n) = \mathrm{p}(\mathcal{J}) - \mathrm{p}(\sigma) = 0$ for $m \geq 1$.

As $\mathcal{J}-\lambda_i$ is invertible $\forall\lambda$ not equal to *0* implies that $\mathcal{J}^m = 0$

Hence, $\mathcal{J}$ is nilpotent.

Lemma:2.7

If $\mathcal{J}$ is invertible & quasi nilpotent algebraically Ɱ* quasi paranormal, then $\mathcal{J}$ is nilpotent.

Proof

Let $p(\mathcal{J})$ be a Ɱ* quasi paranormal for few non constant polynomial p.

As $\sigma(\mathrm{p}(\mathcal{J})) = \mathrm{p}(\sigma(\mathcal{J}))$, then $\mathrm{p}(\mathcal{J}) - \mathrm{p}(0)$ be *q*uasi *n*ilpotent.

By *Theorem 2.2*,

$C\mathcal{J}^m(\mathcal{J}-\lambda_1)\ (\mathcal{J}-\lambda_2)\ (\mathcal{J}-\lambda_3) \ldots (\mathcal{J}-\lambda_n) = \mathrm{p}(\mathcal{J}) - \mathrm{p}(0) = 0$ for $m \geq 1$

Thus $\mathcal{J}-\lambda_i$ is invertible $\forall\lambda_i \neq 0$ & hence $\mathcal{J}^m = 0$

Theorem:2.8

If $\mathcal{J}$ is an algebraically Ɱ*quasi paranormal for non negative real number Ɱ, then

$\sigma_{ea}(\mathrm{f}(\mathcal{J})) = \mathrm{f}(\sigma_{ea}(\mathcal{J}))\ \forall\ f \epsilon \mathrm{H}(\sigma(\mathcal{J}))$

Proof:

By [2], the inclusion $\sigma_{ea}(\mathrm{f}(\mathcal{J})) \subseteq \mathrm{f}(\sigma_{ea}(\mathcal{J}))$ holds $\forall\ f \epsilon \mathrm{H}(\alpha(\mathcal{J}))$ without conditions on $\mathcal{J}$.

We have to prove $f(\sigma_{ea}(\mathcal{J})) \subseteq \sigma_{ea}(f(\mathcal{J}))$.

When $\lambda \epsilon f(\sigma_{ea}(\mathcal{J}))$ then $f(\mathcal{J}) - \lambda I$ is *u*pper *s*emi *F*redholm with index ≤ 0

Also, $f(\mathcal{J}) - \lambda I = c\ (\mathcal{J} - \Psi_1 I)(\mathcal{J} - \Psi_2 I) \ldots (\mathcal{J} - \Psi_n I)\ g(\mathcal{J})$, where $g(\mathcal{J})$ be *i*nvertible &

$c, \Psi_1, \Psi_2, \ldots \Psi_n \epsilon\ \mathbb{C}$ [3]

Let $\mathcal{J}$ be an algebraically Ɱ* *q*uasi *p*aranormal, when ∃ a non constant polynomial $p(g) \ni p(\mathcal{J})$ is Ɱ* *q*uasi *p*aranormal. Since $p(\mathcal{J})$ has *SVEP* & thus $\mathcal{J}$ has *SVEP*.

$\therefore \lambda = f(\Psi_i) \notin f((\sigma_{ea}(\mathcal{J}))$.

Hence, $f(\sigma_{ea}(\mathcal{J})) = \sigma_{ea}(\mathrm{f}(\mathcal{J}))$

Theorem:2.9

Let $\mathcal{J}$ be an algebraically Ɱ* quasi paranormal for non negative real Ɱ, then $\mathcal{J}$ is *i*soloid.

Proof:

Assume that $\lambda \epsilon \; iso \; \sigma(\mathcal{J})$ & associated *R*iesz Idempotent as $P = \frac{1}{2\pi i}\int_{\partial D}(\mu - \mathcal{J})^{-1} d\mu$ here *D* is a closed disc with λ as center it does not contain other points of $\sigma(\mathcal{J})$.

Let us denote $\mathcal{J}$ as, $\mathcal{J} = \begin{pmatrix} \mathcal{J}_1 & 0 \\ 0 & \mathcal{J}_2 \end{pmatrix}$, for $\sigma(\mathcal{J}_1) = \{\lambda\}$ and $\sigma(\mathcal{J}_2) = \sigma(\mathcal{J}) / \{\lambda\}$

As $\mathcal{J}$ be *a*lgebraically Ɱ* *q*uasi *p*aranormal, $p(\mathcal{J})$ be Ɱ* *q*uasi *p*aranormal for some non constant polynomial p.

When $\sigma(\mathcal{J}_1) = \{\lambda\}$, then $\sigma(\mathrm{p}(\mathcal{J}_1)) = \mathrm{p}(\sigma(\mathcal{J}_1)) = \mathrm{p}\{(\lambda)\}$.

$\therefore \mathrm{p}(\mathcal{J}_1) - \mathrm{p}(\lambda)$ is quasi nilpotent.

As $\mathrm{p}(\mathcal{J}_1)$ be Ɱ* *q*uasi *p*aranormal, by theorem 2.2 implies $\mathrm{p}(\mathcal{J}_1) - \mathrm{p}(\lambda) = 0$

Let $q(z) = \mathrm{p}(z) - \mathrm{p}(\lambda)$, when $q(\mathcal{J}_1) = 0$ & thus $\mathcal{J}_1$ be *a*lgebraically Ɱ* *q*uasi *p*aranormal operator.

When $\mathcal{J}_1 - \lambda$ be quasi nilpotent & *a*lgebraically Ɱ* *q*uasi *p*aranormal, then from theorem 2.5, $\mathcal{J}_1 - \lambda$ is *n*ilpotent.

$\therefore \lambda \in \Pi_0(\mathcal{J}_1)$ & thus $\lambda \in \Pi_0(\mathcal{J})$

Hence, $\mathcal{J}$ is isoloid.

Theorem: 2.10

Let $\mathcal{J}$ be an algebraically Ɱ* quasi paranormal for non negative real Ɱ, then $\mathcal{J}$ is *P*olaroid.

Proof:

Let $\mathcal{J}$ be *a*lgebraically Ɱ* *q*uasi *p*aranormal. When $P(\mathcal{J})$ is Ɱ* quasi paranormal for non constant polynomial *P*.

As $\lambda \epsilon \sigma(\mathcal{J})$, we can represent $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_2$ where $\sigma(\mathcal{J}_1) = \{\lambda\}$ and $\sigma(\mathcal{J}_2) = \sigma(\mathcal{J}_1)/\{\lambda\}$

Since $\mathcal{J}_1$ is *a*lgebraically Ɱ* *q*uasi *p*aranormal, $\mathcal{J}_1 - \lambda$ is *a*lgebraically Ɱ* *q*uasi *p*aranormal.

But $\sigma(\mathcal{J}_1 - \lambda) = \{0\}$, from theorem 2.5 $\mathcal{J}_1 - \lambda$ is *n*ilpotent, $\mathcal{J}_1 - \lambda$ has finite descent & ascent.

Since $\mathcal{J}_2 - \lambda$ is invertible and hence it has finite descent & ascent.

∴ $\mathcal{J} - \lambda$ has finite descent & ascent and thus λ corresponds to a pole in the resolvent operator of $\mathcal{J}$.

Thus, $\lambda \in \text{iso}\, \sigma(\mathcal{J})$ gives $\lambda \in P_0(\mathcal{J})$ & then iso $\sigma(\mathcal{J}) \leq P(\mathcal{J})$

Hence, $\mathcal{J}$ is *P*olaroid.

Corollary:2.11

Let $\mathcal{J}$ be an *a*lgebraically Ɱ* *q*uasi *p*aranormal for non negative real Ɱ, then $\mathcal{J}$ is *R*eguloid.

Theorem:2.12

If $\mathcal{J}$ is an invertible *a*lgebraically Ɱ* *q*uasi *p*aranormal, then $\mathcal{J}$ is isoloid.

Proof:

Let $\lambda \in iso\ \sigma(\mathcal{J})$ & the associated *R*iesz *i*dempotent $P = \frac{1}{2\pi i}\int_{\partial D}(\mu - \mathcal{J})^{-1} d\mu$, here D is a closed disc with λ as center which does not contain other points of $\sigma(\mathcal{J})$.

Let us denote $\mathcal{J}$ as, $\mathcal{J} = \begin{pmatrix} \mathcal{J}_1 & 0 \\ 0 & \mathcal{J}_2 \end{pmatrix}$ where $\sigma(\mathcal{J}_1) = \{\lambda\}$ & $\sigma(\mathcal{J}_2) = \sigma(\mathcal{J}) / \{\lambda\}$

As $\mathcal{J}$ is algebraically Ɱ*quasi paranormal, $p(\mathcal{J})$is Ɱ**q*uasi *p*aranormal for not a constant polynomial p.

As $\sigma(\mathcal{J}_1) = \{\lambda\}$, $\sigma(p(\mathcal{J}_1) = p(\sigma(\mathcal{J}_1)) = \{p(\lambda)\}$.

Thus, $p(\mathcal{J}_1)$ - $p(\lambda)$ will quasi nilpotent.

Hence, $p(\mathcal{J}_1)$ is Ɱ* quasi paranormal, using Theorem 2.2, $p(\mathcal{J}_1) - p(\lambda) = 0$

Let $q(z) = p(z) - p(\lambda)$

When $q(\mathcal{J}_1) = 0$ & thus $\mathcal{J}_1$ is *a*lgebraically Ɱ* *q*uasi *p*aranormal.

∵ $\mathcal{J}_1 - \lambda$ is quasi nilpotent & *a*lgebraically Ɱ* *q*uasi paranormal, from Theorem 2.5 implies that $\mathcal{J}_1 - \lambda$ is *n*ilpotent.

∴ $\lambda \in \Pi_0(\mathcal{J}_1)$ & thus $\lambda \in \Pi_0(\mathcal{J})$

Hence, $\mathcal{J}$ *i*s *i*soloid.

Theorem:2.13

If $\mathcal{J}$ is an algebraically Ɱ* *q*uasi *p*aranormal then $\mathcal{J}$ has SVEP.

Proof:

Let $\mathcal{J}$ be Ɱ* *q*uasi *p*aranormal.

When $\Pi_{00}(\mathcal{J}) = \phi$, then $\mathcal{J}$ has *SVEP*.

Assume, $\Pi_{00}(\mathcal{J}) \neq \phi$.

If $\Delta(\mathcal{J}) = \{ \lambda \in \Pi_{00}(\mathcal{J}: N(\mathcal{J} - \lambda) \subseteq N(\mathcal{J}^* - \bar{\lambda}\} \Rightarrow \Delta(\mathcal{J}) \neq \phi$.

Let us take Ɱ (i.e. the closure of the linear span of the subspaces), $N(\mathcal{J} - \lambda)$ for $\lambda \in \Delta(\mathcal{J})$.

When Ɱ reduced as $\mathcal{J}$ & then $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_2$.

Clearly, $\mathcal{J}_1$ is normal and $\Pi_{00}(\mathcal{J}_2) = \phi$.

Since, both $\mathcal{J}_1$ *and* $\mathcal{J}_2$ have *SVEP*, then $\mathcal{J}$ has *SVEP*.

If $\mathcal{J}$ is *a*lgebraically Ɱ* *q*uasi *p*aranormal, then $p(\mathcal{J})$ is Ɱ* quasi paranormal for few non constant polynomial p.

As, $p(\mathcal{J})$ has single valued extension property, by [6, Theorem 3.3.9] $\mathcal{J}$ has single valued extension property.

Theorem:2.14

Let $\mathcal{J}$ be algebraically Ɱ* quasi paranormal then Weyl's theorem holds for $\mathcal{J}$.

Proof:

Let $\lambda \in \sigma(\mathcal{J})\backslash w(\mathcal{J})$. Thus $\mathcal{J} - \lambda$ is Weyl & not an invertible,

To prove $\lambda \in \partial_\sigma(\mathcal{J})$.

As, λ is an interior pt of $\sigma(\mathcal{J})$, Then $\exists$ a nbd U of λ, $\ni$ dimn$N(\mathcal{J} - \mu) > 0$, for every $\mu \in U$.

Using [7, *Theorem* 10], $\mathcal{J}$ does not have *SVEP*.

Furthermore, $\because$ $p(\mathcal{J})$ is Ɱ* *q*uasi *p*aranormal for few nonconstant polynomial p, from theorem 2.13, $p(\mathcal{J})$ has *SVEP* which is contradictory.

$\therefore$, $\lambda \in \partial_\sigma(\mathcal{J})\backslash w(\mathcal{J})$ & using the punctured neighbourhood theorem $\lambda \in \pi_{00}(\mathcal{J})$.

On the contrary, let $\lambda \in \pi_{00}(\mathcal{J})$.

From $E = \frac{1}{2\pi i}\int_{\partial D}(\mu - \mathcal{J})^{-1}d\mu$ for λ, denote $\mathcal{J}$ as $\mathcal{J} = \begin{pmatrix} \mathcal{J}_1 & 0 \\ 0 & \mathcal{J}_2 \end{pmatrix}$

where $\sigma(\mathcal{J}_1) = \{\lambda\}$ & $\sigma(\mathcal{J}_2) = \sigma(\mathcal{J})\backslash\{\lambda\}$.

Let us consider the following 2 cases.

*C*ase: (i) λ equal to 0

Then $\mathcal{J}_1$ is *a*lgebraically Ɱ* quasi paranormal & quasi nilpotent. $\mathcal{J}_1$ is *n*ilpotent using (*Lemma 2.6)*

To prove dimn R(E) < ∞.

Let N($\mathcal{J}_1$) be not a finite dimensional, then $0 \notin \pi_{00}(\mathcal{J})$ which is contradictory.

∴ $\mathcal{J}_1$ is not an infinite dimensional space R(E) operator. From this $\mathcal{J}_1$ is *W*eyl.

But, as $\mathcal{J}_2$ is invertible, consider as $\mathcal{J}$ is *W*eyl.

∴, 0ϵ σ$(\mathcal{J})\backslash w(\mathcal{J})$.

*C*ase: (ii) λ is not equal to *0*

From 2.5, $\mathcal{J}_1 - \lambda$ is *n*ilpotent.

∵ $\lambda \epsilon \pi_{00}(\mathcal{J})$, $\mathcal{J}_1 - \lambda$ is operator on not an infinite dimensional space R(E).

Thus, $\mathcal{J}_1 - \lambda$ is *W*eyl.

∵ $\mathcal{J}_2 - \lambda$ is invertible, $\mathcal{J} - \lambda$ is *W*eyl.

From both the cases, *W*eyl's theorem holds for $\mathcal{J}$.

Theorem:2.15

If $\mathcal{J}$ is an *a*lgebraically Ɱ* *q*uasi *p*aranormal then *W*eyl's theorem holds for $f(\mathcal{J})$ $\forall f \epsilon H(\sigma(\mathcal{J}))$.

Proof:

Let $f \epsilon H(\sigma(\mathcal{J}))$. As it generally holds Ɱ$(f(\mathcal{J})) \subseteq f($Ɱ$(\mathcal{J}))$, it is sufficient to prove $f($Ɱ$(\mathcal{J})) \subseteq$ Ɱ$(f(\mathcal{J}))$ for all $f \epsilon H(\sigma(\mathcal{J}))$

If $\lambda \notin$ Ɱ$(f(\mathcal{J}))$, then $f(\mathcal{J}) - \lambda$ is Weyl &

$f(\mathcal{J}) - \lambda = c(\mathcal{J} - \alpha_1)(\mathcal{J} - \alpha_2)(\mathcal{J} - \alpha_3) \ldots\ldots (\mathcal{J} - \alpha_n)\, g(\mathcal{J})$ -----------(1)

where c, $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n \epsilon C$ & $g(\mathcal{J})$ is invertible, as the RHS of the above are commute, for all $\mathcal{J} - \alpha_i$ is *F*redholm.

As $\mathcal{J}$ is algebraically Ɱ*quasi paranormal then $\mathcal{J}$ has SVEP by lemma 2.7.

From [5, *Theorem* 2.6], ind$(\mathcal{J} - \alpha_i) \leq 0$, ∀ i = 1, 2, …n.

Thus $\lambda \notin f(Ɱ(\mathcal{J}))$ & so $f(Ɱ(\mathcal{J})) = Ɱ(f(\mathcal{J}))$

Using [8], that if $\mathcal{J}$ is isoloid, then

$f(\sigma(\mathcal{J})) \setminus \Pi_{00}(\mathcal{J}) = \sigma(f(\mathcal{J})) \setminus \Pi_{00} f(\mathcal{J})$ for every $f \epsilon H(\alpha(\mathcal{J}))$

As, $\mathcal{J}$ is isoloid from 2.12 & Weyl's theorem holds for $\mathcal{J}$ from 2.14,

$\sigma(f(\mathcal{J})) \setminus \Pi_{00} f(\mathcal{J}) = f(\sigma(\mathcal{J})) \setminus \Pi_{00}(\mathcal{J}) = f(Ɱ(\mathcal{J})) = Ɱ(f(\mathcal{J}))$

Hence, proved the theorem.

Corollary:2.16

Let $\mathcal{J}$ be an *a*lgebraically Ɱ* quasi *p*aranormal for non negative real Ɱ, then $Ɱ(f(\mathcal{J})) = f(Ɱ(\mathcal{J}))$ for every $f \epsilon H(\sigma(\mathcal{J}))$

Corollary:2.17

Suppose $\mathcal{J}$ is algebraically Ɱ* quasi paranormal then $\sigma_{Ɱ}(f(\mathcal{J})) = f\left(\sigma_{Ɱ}(\mathcal{J})\right)$ for every $f \epsilon H(\sigma(\mathcal{J}))$.

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