UtilitasMathematica
ISSN 0315-3681 Volume 122, 2025

Existence and Attractivity Results for First-Order Random Differential Equations via Fixed Point Theorems

P. D. Bhosale[1], **S. S. Bellale**[2] and **S. V. Badgire**[3]
[1]Research Scholar, Mathematics Research Centre, Dayanand Science College, Latur
[2]Head, Department of Mathematics, Dayanand Science College, Latur
[3]Head, Department of Mathematics, Azad Mahavidyalaya, Ausa
Email: pavanrajed80@gmail.com, sidhesh.bellale@gmail.com, sanjayvbadgire@gmail.com

Abstract: In this paper, we investigate the existence and attractivity of solutions to a class of first-order random differential equations by using the well-known Random Banach Fixed Point Theorem. The stochastic nature of the system is modeled via measurable random operators and integrability conditions. Using Carathéodory-type assumptions and compactness arguments, we establish sufficient conditions under which a unique random solution exists. A lemma is provided to convert the random differential equation into an equivalent random integral equation. Furthermore, we construct a verified example satisfying all conditions of the main theorem. This study contributes to the growing body of literature on stochastic analysis and fixed point theory in random metric spaces.

Keywords: Random differential equation, random operator, Random fixed point, Locally attractive, measurability, Carathéodory conditions.

1. Introduction

Random differential equations (RDEs) are used to simulate a wide range of uncertain events, such as processes in economics, engineering, physics, and medicine. The study of existence, uniqueness, and qualitative properties of their solutions plays a significant role in the mathematical theory of stochastic systems.Many real-world systems are affected by random influences, either from the environment or internal fluctuations. To handle such uncertainty, traditional deterministic models are extended to incorporate randomness, leading to the formulation of random differential equations. In recent years, various methods have been developed for analyzing RDEs, such as stochastic calculus, random integral equations, and fixed point theory. Among them, fixed point theorems in random metric and normed spaces have proven to be powerful tools for establishing the existence and uniqueness of solutions [6][1][2]have extensively studied random equations using probabilistic and analytical approaches.

In this paper, we focus on the application of a well-known fixed point theorem, namely the Random Banach Fixed Point Theorem, to study the existence and attractivity of solutions to the following first-order random differential equation:

$$\frac{dx(t,\omega)}{dt} = f(t, x(t,\omega), \omega), \quad t \in [0,T], \quad \omega \in \Omega \tag{1}$$

with initial condition

$x(0,\omega) = x_0(\omega)$ (2)

where:

- $f: [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a jointly measurable random function,
- $x_0: \Omega \to \mathbb{R}$ is a given random variable (initial condition),
- $x(t,\omega)$ is the unknown random function (solution).

2. Preliminaries

Definition 2.1. *A random variable is a measurable function $x: \Omega \to \mathbb{R}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. [6]*

Definition 2.2. *A function $f: [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is called a random function if for each $(t,x) \in [0,T] \times \mathbb{R}$, the map $\omega \mapsto f(t,x,\omega)$ is $\mathcal{F}$-measurable .[6][1]*

Definition 2.3. *A mapping $A: X \times \Omega \to X$ is called a random operator if:*

- *For each $\omega \in \Omega$, $A(\cdot,\omega): X \to X$ is a deterministic operator;*
- *For each $x \in X$, the map $\omega \mapsto A(x,\omega)$ is $\mathcal{F}$-measurable .[2][6]*

Definition 2.4. *A function $f: [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is said to satisfy Carathéodory conditions if:*

- *For each fixed $\omega \in \Omega$, the mapping $(t,x) \mapsto f(t,x,\omega)$ is continuous;*
- *For each fixed $(t,x) \in [0,T] \times \mathbb{R}$, the mapping $\omega \mapsto f(t,x,\omega)$ is $\mathcal{F}$-measurable .[1][2]*

Definition 2.5. *Let $C([0,T],\mathbb{R})$ be the space of all continuous real-valued functions on $[0,T]$, equipped with the supremum norm. A function $x: [0,T] \times \Omega \to \mathbb{R}$ is said to be jointly measurable if the mapping $(t,\omega) \mapsto x(t,\omega)$ is measurable with respect to the product σ-algebra $\mathcal{B}([0,T]) \times \mathcal{F}$. [6]*

A family $\{x_n\}$ of functions from $[0,T]$ to $\mathbb{R}$ is said to be:

- *Uniformly bounded if there exists* $M > 0$ *such that* $|x_n(t)| \leq M$ *for all* n *and* $t \in [0, T]$;
- *Equicontinuous if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $|x_n(t_1) - x_n(t_2)| < \varepsilon$ *whenever* $|t_1 - t_2| < \delta$, *for all* n .[1][6]

3. Conversion to Integral and Operator Equation

To apply fixed point theorems, we first convert the given first-order random differential equation into an equivalent random integral equation.

Lemma 3.1 (Conversion Lemma). *Let* $f: [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ *be a jointly measurable function satisfying the Carathéodory conditions and suppose that for each* $\omega \in \Omega$, $f(\cdot,\cdot,\omega)$ *is continuous in* (t, x). *Then the initial value problem:* $\frac{dx(t,\omega)}{dt} = f(t, x(t, \omega), \omega)$, $\quad t \in [0, T]$; $\quad x(0, \omega) = x_0(\omega)$ *is equivalent to the integral equation:* $x(t, \omega) = x_0(\omega) + \int_0^t f(s, x(s, \omega), \omega)\, ds, \quad \forall t \in [0, T],\ \omega \in \Omega$.

Now, define the operator $\mathcal{A}$ on the space $\mathcal{X} := C([0, T], \mathbb{R})$ of continuous functions (with values depending on ω) by:

$$(\mathcal{A}x)(t, \omega) := x_0(\omega) + \int_0^t f(s, x(s, \omega), \omega)\, ds \tag{3}$$

Thus, a function $x(t, \omega)$ is a solution of the random differential equation if and only if it is a fixed point of the operator $\mathcal{A}$, that is,

$$x(t, \omega) = (\mathcal{A}x)(t, \omega) \tag{4}$$

4. Theorem Used and Hypotheses

To establish the existence and uniqueness of the solution to the random integral equation, we employ the following well-known fixed point theorem adapted for random operators.

Theorem 4.1 (Random Banach Fixed Point Theorem).[6] *Let* (X, d) *be a complete metric space and* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. Let* $T: X \times \Omega \to X$ *be a random operator such that for each* $\omega \in \Omega$, *the mapping* $T(\cdot, \omega): X \to X$ *is a contraction; i.e., there exists* $0 < k < 1$ *such* *that:* $d(T(x,\omega),T(y,\omega)) \leq k \cdot d(x,y), \quad \forall x,y \in X$ *and for each* $x \in X$*, the map* $\omega \mapsto T(x,\omega)$ *is* $\mathcal{F}$ *-measurable.*

Then there exists a unique random fixed point $x^*(\omega) \in X$ *such that:* $T(x^*(\omega),\omega) = x^*(\omega), \quad$ for almost all $\omega \in \Omega$.

We consider the following set of hypotheses under which we will establish the main result.

H1- Measurability and Continuity:

The function $f:[0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is measurable in ω for each (t,x) and continuous in (t,x) for each ω.

H2- Lipschitz Condition:

There exists a constant $L > 0$ such that for all $t \in [0,T]$, all $x_1, x_2 \in \mathbb{R}$, and all $\omega \in \Omega$:

$$|f(t,x_1,\omega) - f(t,x_2,\omega)| \leq L \cdot |x_1 - x_2| \tag{5}$$

H3- Bounded Initial Value:

The initial function $x_0 : \Omega \to \mathbb{R}$ is a measurable and bounded random variable.

H4- Integrability Condition:

The function $f(t,0,\omega)$ is integrable with respect to t on $[0,T]$ for almost every $\omega \in \Omega$, i.e.,

$$\int_0^T |f(t,0,\omega)| \, dt < \infty, \quad \text{for almost every } \omega \in \Omega. \tag{6}$$

5. Main Theorem

Theorem 5.1 (Main Theorem). *Let* $f:[0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ *satisfy the hypotheses (H1)–(H4). Then the random integral equation* $x(t,\omega) = x_0(\omega) + \int_0^t f(s,x(s,\omega),\omega)\, ds$ *has a unique continuous random solution* $x(t,\omega)$*, and this solution is attractive in the sense that iterative sequences converge to it almost surely.*

Proof:

Step 1: Define the Operator

Let $\mathcal{X} := C([0,T],\mathbb{R})$, the Banach space of all continuous functions on $[0,T]$, equipped with the supremum norm:

$$\| x \| := \sup_{t\in[0,T]} |x(t)| \tag{7}$$

Define the operator $\mathcal{A}$ on $\mathcal{X}$ by:

$$(\mathcal{A}x)(t,\omega) := x_0(\omega) + \int_0^t f\,(s, x(s,\omega), \omega)\, ds \tag{8}$$

Step 2: Show that $\mathcal{A}$ maps $\mathcal{X}$ into itself

Let $x \in \mathcal{X}$. By assumptions (H1) and (H4), the mapping $f(s, x(s,\omega), \omega)$ is measurable in ω and integrable in s. Since f is continuous in x and s, the integral defines a continuous function in t. Therefore, $\mathcal{A}x \in \mathcal{X}$.

Step 3: Show that $\mathcal{A}$ is a Contraction

Let $x_1, x_2 \in \mathcal{X}$. Then:

$$|(\mathcal{A}x_1)(t,\omega) - (\mathcal{A}x_2)(t,\omega)| = \left|\int_0^t [\, f(s, x_1(s,\omega), \omega) - f(s, x_2(s,\omega), \omega)]\, ds\right| \tag{9}$$

Using the Lipschitz condition (H2):

$$\le \int_0^t L \cdot |x_1(s,\omega) - x_2(s,\omega)|\, ds \le L \cdot t \cdot \| x_1 - x_2 \|_\infty \tag{10}$$

Taking supremum over $t \in [0,T]$, we get:

$$\| \mathcal{A}x_1 - \mathcal{A}x_2 \|_\infty \le L \cdot T \cdot \| x_1 - x_2 \|_\infty \tag{11}$$

If $L \cdot T < 1$, then $\mathcal{A}$ is a contraction on $\mathcal{X}$.

Step 4: Apply Banach Fixed Point Theorem

Since $\mathcal{X}$ is a complete metric space and $\mathcal{A}$ is a contraction (when $L \cdot T < 1$), Banach's fixed point theorem guarantees the existence of a unique fixed point $x^* \in \mathcal{X}$ such that:

$$x^*(t,\omega) = x_0(\omega) + \int_0^t f\,(s, x^*(s,\omega), \omega)\, ds \tag{12}$$

Hence, x^* is the unique solution to the integral equation.

Step 5: Attractivity

Define the iterative sequence:

$$x^{(0)}(t,\omega) = x_0(\omega) \tag{13}$$

$$x^{(n+1)}(t,\omega) = x_0(\omega) + \int_0^t f\,(s, x^{(n)}(s,\omega), \omega)\, ds \tag{14}$$

Then we have:

$$\| x^{(n+1)} - x^* \|_\infty \leq L \cdot T \cdot \| x^{(n)} - x^* \|_\infty \Rightarrow \| x^{(n)} - x^* \|_\infty \leq (L \cdot T)^n \cdot \| x^{(0)} - x^* \|_\infty \rightarrow 0$$

as $n \rightarrow \infty$. Thus, the solution x^* is attractive. ▫

6. Verified Example

Let us consider the following first-order random differential equation:

$$\frac{dx(t,\omega)}{dt} = -\lambda x(t,\omega) + \sin(t) \cdot \xi(\omega), \quad x(0,\omega) = \xi(\omega) \tag{15}$$

where $\lambda > 0$ is a real constant and $\xi: \Omega \rightarrow \mathbb{R}$ is a bounded measurable random variable such that $\xi(\omega) \in [-1,1]$. This models a randomly forced damped system, where randomness arises from $\xi(\omega)$.

Step-by-Step Hypothesis Verification

(H1) Measurability and Continuity:

- The function $f(t,x,\omega) = -\lambda x + \sin(t) \cdot \xi(\omega)$ is continuous in (t,x) for each fixed ω, since $x \mapsto -\lambda x$ and $t \mapsto \sin(t)$ are continuous.
- For fixed (t,x), f is measurable in ω since $\xi(\omega)$ is measurable.

Thus, (H1) is satisfied.

(H2) Lipschitz Condition:

$$|f(t,x_1,\omega) - f(t,x_2,\omega)| = |-\lambda x_1 + \lambda x_2| = \lambda |x_1 - x_2| \tag{16}$$

Hence, Lipschitz constant $L = \lambda$. *So, (H2) is satisfied.*

(H3) Measurable and Bounded Initial Value:

$$x_0(\omega) = \xi(\omega) \tag{17}$$

Since $\xi(\omega)$ is given as a measurable and bounded random variable, *(H3) is satisfied.*

(H4) Integrability of $f(t, 0, \omega)$:

$$f(t, 0, \omega) = \sin(t) \cdot \xi(\omega) \tag{18}$$

Since $\sin(t)$ is continuous on $[0, T]$ and $\xi(\omega)$ is bounded, $f(t, 0, \omega)$ is integrable over $[0, T]$. *(H4) is satisfied.*

Contraction Check: Choose $\lambda = 0.5$ and $T = 1$. Then,

$$L \cdot T = 0.5 \cdot 1 = 0.5 < 1 \tag{19}$$

Hence, $\mathcal{A}$ is a contraction on $C([0, T], \mathbb{R})$.

Conclusion: All assumptions (H1)–(H4) are satisfied, and the contraction condition $L \cdot T < 1$ holds. Therefore, by Theorem 5.1, the unique random solution exists and is attractive.

7. Conclusion

In this paper, we studied the existence and attractivity of solutions to a class of first-order random differential equations using the Random Banach Fixed Point Theorem. We transformed the random differential equation into an equivalent random integral equation and constructed a corresponding random operator that satisfied the contraction property. By verifying standard hypotheses, including Carathéodory-type measurability and continuity, Lipschitz condition, and integrability, we successfully applied the fixed point theorem to guarantee a unique and attractive random solution.

A verified example was provided to illustrate the main result and demonstrate how each hypothesis is satisfied. The approach shown here contributes to the analytical foundation for random differential equations and has potential applications in stochastic dynamic systems. This framework can be extended in future work to study impulsive, fractional, and higher-order random differential systems, and can aid in the analysis of real-world systems driven by random behavior.

References

1. V. Lakshmikantham and S. Leela, *Nonlinear Differential Equations in Abstract Spaces*, Pergamon Press, 1981.
2. D. O'Regan and M. Meehan, *Existence Theory for Nonlinear Integral and Integro-differential Equations*, Kluwer Academic Publishers, 1998.
3. B. C. Dhage, "Generalized Contractions and Applications to Nonlinear Functional Equations", *Yokohama Mathematical Journal*, **48** (2000), 1–10.
4. J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Springer, 2000.
5. M. Branciari, "A Fixed Point Theorem of Banach–Caccioppoli Type on a Class of Generalized Metric Spaces", *International Journal of Mathematics and Mathematical Sciences*, **2002**(11), 645–650.
6. A.T. Bharucha-Reid, *Random Integral Equations*, Academic Press, 1972.
7. K. R. Parthasarathy, *Probability Measures on Metric Spaces*, AMS Chelsea Publishing, 2005.
8. B. Pathak and R. Sharma, "Existence of Random Solutions for Second-Order Differential Equations", *Random Operators and Stochastic Equations*, **23**(1), 2015, 31–42.
9. R. Gupta and V. Kumar, "Random Fixed Point Theorems in Metric Spaces and Applications", *Mathematics*, **9**(18), 2021, Article 2240. DOI: 10.3390/math9182240.
10. R. S. Yadav and M. Singh, "Existence Results for Random Differential Equations via Random Fixed Point Theorems", *International Journal of Nonlinear Analysis and Applications*, **13**(2), 2022, 277–289.
11. Kumar and S. Radenović, "Fixed Point Results for Hybrid Mappings in Random Metric Spaces with Applications", *Symmetry*, **14**(3), 2022, Article 478. DOI: 10.3390/sym14030478.
12. M. Abbas and M. Alghamdi, "Existence of Random Fixed Points in Generalized Metric Spaces Using Simulation Functions", *AIMS Mathematics*, **6**(6), 2021, 6522–6543. DOI: 10.3934/math.2021381.
13. S. Choudhary and P. Kumar, "Fixed Points of Weakly Contractive Random Operators and Applications to Stochastic DEs", *Journal of Fixed Point Theory and Applications*, **25**(1), 2023, Article 12. DOI: 10.1007/s11784-023-01020-1.
14. R. P. Agarwal and D. O'Regan, "Infinite Interval Problems for Differential Equations in Banach Spaces", *Nonlinear Analysis*, **75**(4), 2012, 2024–2033.
15. M. Kostić, *Abstract Volterra Integro-Differential Equations: Regularity and Solvability in Banach Spaces*, SpringerBriefs in Mathematics, Springer, 2022.