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A Fixed Point Theorem in T- Metric Space

Süheyla Elmas
suheylaelmas@atauni.edu.tr

Abstract

In this article, the concept of T-metric spaces will be introduced.I have present some fixed point theorems for two maps on full T-metric spaces. I have show that two unovalent mapping can be obtained in T-metric spaces using the similar method in a common fixed point theorem.

Keywords: Metric space, fixed point iterations, convergence speed and fixed point.

Introduction

Banach constriction theory is an important fixed point theorem. This theorem is highly generalized.

Fixed point problems for contractive mappings in metric spaces with a partial order have been studied by many authors (see [2]-[4]).

In the present paper, we introduce the notion of S-metric spaces and give some properties of them. Implicit relations on S-metric spaces have been used in many articles (see [5]-[9]).

I will prove fixed point theorems for two mappings in complete T metric spaces. Also, I give an illustrative example for the monovalent case.

Let's start by giving specific definitions in the literature.

Definition 1.1.

Let X be a nonempty set. A function $F : X^3 \to [0,\infty)$ is said to be an F-metric on X, if for each $x,y,z,t \in X$,

$$\text{M1. } T(x,y,z) \geq 0,$$
$$\text{M2. } T(x,y,z) = 0 \text{ if and only if } x = y = z,$$
$$\text{M3. } T(x,y,z) \leq T(x,x,t) + T(y,y,t) + T(z,z,t).$$

The pair (X,T) is called an T-metric space. [6]

Definition 1.2.

Let (X,T) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball BS(x,r) and closed ball BS[x,r] with center x and radius r as follows, respectively:

The pair (X,T) is called an T-metric space. [6]

Definition 1.2.

Let (X,T) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x,r)$ and closed ball $B_S[x,r]$ with center x and radius r as follows, respectively:

$$B_S[x,r] = \{y \in X : T(y,y,x) \leq r\}. \text{ [3]}$$

Definition 1.3.

Let (X,T) and (X',T') be two T-metric spaces. A function $f : (X,T) \rightarrow (X',T')$ is said to be continuous at a point $a \in X$ if for every sequence $\{x_n\}$ in X with $T(x_n,x_n,a) \rightarrow 0$, $T'(f(x_n),f(x_n),f(a)) \rightarrow 0$. I say that f is continuous on X if f is continuous at every point $a \in X$.

Definition 1.4.

Let (X,T) be an F-metric space and $A \subset X$. [11]

1. The set A is said to be an open subset of X, if for every $x \in A$ there exists $r > 0$ such that $B_S(x,r) \subset A$.

2. The set A is said to be T-bounded if there exists $r > 0$ such that $T(x,x,y) < r$ for all $x,y \in A$.

3. A sequence $\{x_n\}$ in X converges to x if $T(x_n,x_n,x) \rightarrow 0$ as $n \rightarrow \infty$, that is for every $\varepsilon > 0$ there exists

$n_0 \in N$ such that for $n \geq n_0$, $T(x_n,x_n,x) < \varepsilon$. In this case, we denote by $\lim_{n\to\infty} x_n \rightarrow x$ and we say that x is the limit of $\{x_n\} \subset X$.

4. A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$T(x_n,x_n,x_m) < \varepsilon \text{ for each } n,m \geq n_0.$$

5. The T-metric space (X,T) is said to be complete if every Cauchy sequence is convergent.

Let r be the set of all $A \subset X$ with $x \in A$ and there exists $r > 0$ such that $B_S(x,r) \subset A$. Then r is a topology on X [8]

Lemma 1.1.

Let (X,T) be an T- Metric Space and suppose that $\{x_n\}$ and $\{y_n\}$ are T-convergent to x,y, respectively. Then I have

$$\lim_{n\to\infty} \sup T(x_n, z, y_n) \leq T(z,z,x) + T(x,x,y)$$

$$\lim_{n\to\infty} \sup T(x_n, z, y_n) \leq T(z,z,x)$$

In particular, if y = x, then I have .

Proof : Let . $\lim_{n\to\infty} y_n \rightarrow y$ *and* $\lim_{n\to\infty} x_n \rightarrow x$

Then for each $\delta > 0$ there exist $n_1,n_2 \in N$ such that for all $n \geq n_1$

$$T(x_n, x_n, x) < \delta/2$$

and for all $n \geq n_2$

$$. \quad T(y_n, y_n, y) < \delta/_4$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ by condition of T-metric, I have

$$T(x_n, z, y_n) \leq T(x_n, x_n, x) + T(z, z, x) + T(y_n, y_n, x)$$

$$\leq T(x, x, y) + T(x_n, x_n, x) + T(z, z, x) + 2\,T(y_n, y_n, y)$$

In the above inequality, I get the first desired result for the upper limit n → ∞.. The second conclusion seems clear.

Theorem 1.1 Let (X,T) be an T-metric space. Then the convergent sequence $\{x_n\}$ in X is Cauchy

Theorem 1.2 Let (T,X) be anT -metric space. Then, I have $x, y \in T$ and $T(x,x,y) = T(y,y,x)$

2. Main Results

Theorem 2.1. *Let (X,T) be a complete T-metric space and F,G : X → X be mappings satisfying the following conditions:*

1. $F(X) \subseteq G(X)$ and either F(X) or G(X) is a closed subset of X,
2. The pair (F,G) is weakly compatible,
3. $T(Fx, Fy, Fz) \leq \psi(\max\{T(Gx, Gy, Gz), a_1 T(Gz, Fx, Fz), a_2 T(Gz, Fy, Fz)\})$ for all $x,y,z \in X$ and $0 < a_1, a_2 < 1$, where $\psi \in \Phi$.

Then the maps F and G have a unique common fixed point. If G is continuous at the fixed point p, then F is also continuous at p.

Note: Φ is reflec the class of all functions $\psi : R^+ \to R^+$ such that ψ is nondecreasing, continuous and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. It is clear that $\psi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$ and hence,

I have $\psi(t) < t$ for all $t > 0$.

Proof : Let $x_0 \in X$. Define the sequence $y_n = Fx_n = Gx_{n+1}$, $n = 0,1,2,\cdots$ and let
$L_{n+1} = T(y_n, y_n, y_{n+1})$.

Then we have L

$$\begin{aligned} L_{n+1} &= T(y_{n-1}, y_{n-1}, y_n) \\ &= T(Ax_n, Ax_n, Ax_{n+1}) \end{aligned}$$

$$\leq \psi(\max\{T(Gx_n, Gx_n, Gx_{n+1}),\ a_1T(Bx_{n+1}, Ax_n, Ax_{n+1}), a_2T(Gx_{n+1}, Fx_n, Fx_{n+1})\})$$

$$\leq \psi(\max\{L_n,\ a_1L_{n+1}, a_2L_{n+1}\}).$$

There for $L_{n+1} \leq \psi(L_n)$, $n = 1,2,3,\cdots$.

Depending on this I have,

$$T(y_n, y_n, y_{n+1}) \leq \psi T(y_{n-1}, y_{n-1}, y_n)$$

$$\leq \psi^2 T(y_{n-2}, y_{n-2}, y_{n-1})$$

$$\leq \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\leq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\leq \psi^n T(y_0, y_0, y_{+1})$$

Therefore, according to the condition of T- metric (Theorem 2.1.3), for every m > n,

I have $T(y_n, y_n, y_m) \leq 2\,T(y_n, y_n, y_{n+1}) + T(,y_{n+1}, y_{n+1}, y_{n+2})$

$$\leq \sum_{i=n}^{m-3} 2[T(y_i, y_i, y_{i+1}) + T(y_{m-2}, y_{m-2}, y_{m-3})]$$

$$\leq 2[\psi^n(T(y_0, y_0, y_1) + \psi^{n+1}\,T(y_0, y_0, y_1) + \cdots .. + \psi^{m-2}T(y_0, y_0, y_1)].$$

$$= 2\sum_{i=n}^{m-3} \psi^i[T(y_0, y_0, y_1)]$$

Therefore ; $\sum_{i=1}^{\infty} \psi^i(s) < \infty$ for all $s > 0$, $T(y_n, y_n, y_m) \to 0$ as $n \to \infty$.

So that each $\delta > 0$, there is $n_0 \in N$ such that for each $m,n \geq n_0$ and $T(y_n, y_n, y_m) < \delta$.

This means that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $q \in X$ such that

$$\lim_{n\to\infty} y_n = q \quad \text{and} \quad q = \lim_{n\to\infty} y_n = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} G(x_{n+1})$$

Let G(X) be a closed subset of X. Then there exists z ∈ X such that G(z) = q. I prove that F(z) = q.

Since,

$$T(Fz, Fz, Fx_n) \leq \psi\,[\max\{T(Gz, Gz, Gx_n),\ a_1T(Gx_n, Fz, Fx_n),\ a_2T(Gx_n, Fz, Fx_n)\}]$$
$$= \psi\,[\max\{T(q, q, y_{n-1}),\ a_1T(y_{n-1}, Fz, y_n),\ a_2T(y_{n-1}, Fz, y_n)\}]$$

Setting the limit as n → ∞ in the above inequality, I obtain

$$\mathrm{T}(\mathrm{Fz}, \mathrm{Fz}, \mathrm{p}) \leq \psi\left[\max\{0,\ a_1 \lim_{n\to\infty} sup\, T(y_{n-1}, Fz, y_n),\ a_2 \lim_{n\to\infty} sup\, T(y_{n-1}, Fz,)y_n\}\right]$$

$$\leq \psi\,[\max\{0,\ a_1T(Fz, Fz, q),\ a_2T(Fz, Fz, q)\}]$$

$$\leq \max\left\{a_{1,2}\right\}T(Fz, Fz, q)$$

This shows that $1 \leq \max\left\{a_{1,2}\right\}$, it's a contradiction, it's a mistake.

Therefore, from $\psi(t) < t$ for all $t > 0$, I have Fz = Gz = q.

From the poor compatibility of the couple (F; G), I have F(Gz) = G(Fz) and therefore

Fz = Gz .

Let's assume that $\mathrm{Fz} \neq z$.

Then

$$\mathrm{T}(\mathrm{Fq}, \mathrm{Fq}, \mathrm{Fxn}) \leq \psi\left[\max\{T(Gq, Gq, G_{x_n}),\ a_1T(G_{x_n}, Fq, F_{x_n}),\ a_2T(G_{x_n}, Fq, F_{x_n})\}\right]$$

$$= \psi\,[\max\{T(Gq, Gq, y_{n-1}),\ a_1T(y_{n-1}, Fq, y_n),\ a_2T(y_{n-1}, Fq, y_n)\}]$$

Taking the upper limit as n → ∞ in the above inequality, I obtain.

$$\mathrm{T}(F_q, F_q, \mathrm{q}) \leq \psi\left[\max\{a_1 \lim_{n\to\infty} sup\, T\big(y_{n-1}, F_q, y_n\big),\ a_2 \lim_{n\to\infty} sup\, T\big(y_{n-1}, F_q, y_n\big), \mathrm{T}(F_q, F_q, \mathrm{q})\}\right]$$

$$\leq \psi\left[\max\{a_1\mathrm{T}\big(F_q, F_q, \mathrm{q}\big),\ a_2\mathrm{T}\big(F_q, F_q, \mathrm{q}\big), \mathrm{T}(F_q, F_q, \mathrm{q})\}\right]$$

$$\leq \max\{a_1, a_2\}\,\mathrm{T}\big(F_q, F_q, \mathrm{q}\big)$$

Since $\psi(t) < t$ for all $t > 0$, I have Gp = Fp = p. Thus p is a common fixed point of F and G.

Suppose q' is another common fixed point of F and G. Then, I have

$$\mathrm{T}(q, q, q') = \mathrm{T}(F_q, F_q, q')$$

$$\leq \psi\,[\max\{a_1\mathrm{T}(q', q, q'), a_2\mathrm{T}(q', q, q'), \mathrm{T}(q, q, q')\}]$$

If $\mathrm{T}(q, q, q') \leq \psi\{\mathrm{T}(q, q, q')\}$ then $\mathrm{T}(q, q, q') \leq \psi\{\mathrm{T}(q, q, q')\} < \mathrm{T}(q, q, q')$

which one is a contraction.Hence, I have q = q'.

$$\text{If } \mathrm{T}(q, q, q') < a\mathrm{T}(q', q, q') < a\mathrm{T}(q', q, q'),$$

Then

$$\mathrm{T}(q, q, q') < a\mathrm{T}(q', q, q')$$

$$\leq a(\mathrm{T}(q, q, q') + 2\mathrm{T}(q', q', q')) = a\,\mathrm{T}(q, q, q')$$

Where $a = \max\{a_1, a_2\}$. This is also a contraction. . Hence, I have q = q'. Thus, q is the unique common fixed point of F and G.

Later, I will prove the continuity of mapping in T-metric spaces.
Let $\{a_n\}$ be any sequence in X such that $\{a_n\}$ is convergent to q.

Then I have

$$\mathrm{T}(F_q, F_q, F_{a_n}) \leq \psi\left[\max\{T(Gq, Gq, F_{a_n}), a_1T(G_{a_n}, Fq, F_{a_n}), a_2T(G_{a_n}, Fq, F_{a_n})\}\right]$$

Taking the upper limit as $n \to \infty$ in the above inequality, from the continuity of *G* at a point q I get

$$\lim_{n\to\infty} sup\, T(q, q, F_{a_n}) = \lim_{n\to\infty} sup\; \mathrm{T}(F_q, F_q, F_{a_n})$$

$$\leq$$

$$\psi\left[\max\left(a_1 \lim_{n\to\infty} sup\, T(G_{a_n}, Fq, F_{a_n}), a_2 \lim_{n\to\infty} sup\, T(G_{a_n}, Fq, F_{a_n}), \lim_{n\to\infty} sup\; \mathrm{T}(F_q, F_q, F_{a_n})\right)\right]$$

$$\leq \psi\left[\max\left(a_1 \lim_{n\to\infty} sup\, T(q, q, F_{a_n}), a_2 \lim_{n\to\infty} sup\, T(q, q, F_{a_n}), 0\right)\right]$$

$$\leq \max\{a_1, a_2\}\, T(q, q, F_{a_n})$$

after this

$$\mathrm{a}_1 \lim_{n\to\infty} sup\, T\big(G_{a_n}, Fq, F_{a_n}\big)$$
$$\le \mathrm{a}_1 \Big\{\lim_{n\to\infty} sup\, T\big(G_{a_n}, G_{a_n}, G_q\big) + \lim_{n\to\infty} sup\ \mathrm{T}\big(F_q, F_q\ , G_q\ \big) + \lim_{n\to\infty} sup\, T\big(F_{a_n}, F_{a_n}, G_q\big)\Big\}$$

And

$$\mathrm{a}_2 \lim_{n\to\infty} sup\, T\big(G_{a_n}, Fq, F_{a_n}\big)$$
$$\le \mathrm{a}_2 \Big\{\lim_{n\to\infty} sup\, T\big(G_{a_n}, G_{a_n}, G_q\big) + \lim_{n\to\infty} sup\ \mathrm{T}\big(F_q, F_q\ , G_q\ \big) + \lim_{n\to\infty} sup\, T\big(F_{a_n}, F_{a_n}, G_q\big)\Big\}$$

I have

$$\lim_{n\to\infty} sup\, T\big(q, q, F_{a_n}\big) \le \max\{\mathrm{a}_1, \mathrm{a}_2\} \lim_{n\to\infty} sup\, T\big(q, q, F_{a_n}\big)$$

This means that

$$\lim_{n\to\infty} sup\, T\big(q, q, F_{a_n}\big) = 0 .$$

Then, I deduce that F is continuous at q.

Corollary : Let (X; T) be a complete T-metric space and A : X →X be a mapping satisfying the following

inequality.

$$\mathrm{T}\big(F_{x_1}, F_{x_2}\ , F_{x_3}\ \big) \le \psi \Big[\max\{\mathrm{T}\big(G_{x_1}, G_{x_2}\ , G_{x_3}\ \big),\ \mathrm{a}_1 T(G_{x_3}, x_3),\ \mathrm{a}_2 T(x_3, F_{x_2}\ , F_{x_3})\}\Big]$$

for all $x_1, x_2, x_3 \in X$, where ψ ∈ Φ. Then the mapping F has a unique common fixed point q ∈X .

And, the mapping F is continuous at q.

References

[1] S.Elmas, Fixed Point Iteration and Newton's Method,Journal of Advanced Mathematics And Mathematics Education. 1 (1), 1-10

[2] R.P. Agarwal, M. A. El-Gebeily, D. O'regan, Generalized contractions in partially ordered metric spaces, Appl. Anal., 87(2008), 109-116

[3]S Elmas ,S.Hızarcı, The Golden Ratio, the Golden Number and a review on Fixed Point,Ciencia e tecnica Vitivinicola 31 (12)

[4] Lj. Ciri´c, D. Mihet, R. Saadati,´ Monotone generalized contractions in partiality ordered probabilistic metric spaces, Topology Appl., 17(2009), 2838-2844.

[5] S. Sedghi, N. Shobe, H. Zhou, A common fixed point theorem in D∗-metric spaces. Fixed Point Theory Appl., 2007(2007), Article ID 7906, 13 pages

[6] S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorems in S-metric spaces, Mat. Vanik, 64(2012), 258 - 266.

[7] J.K. Kim, S. Sedghi, N. Shobkolaei, Common Fixed Point Theorems for the R-weakly Commuting Mappings in S-metric Spaces. J.Comput. Anal. Appl., 19(2015), 751-759.

[8] S. Sedghi, NV. Dung, Fixed point theorems on S-metric spaces. Mat.Vensnik 66(2014), 113-124.

[9] S. Sedghi, I. Altun, N. Shobe, M.A. Salahshour, Some Properties of S-metric Spaces and Fixed Point

Results, Kyungpook Math. J., 54(2014), 113-122.

[10] S. Sedghi, N. Shobe, T. Dosenovic, fixed point results in S-metric spaces, Nonlinear Functional Anal. and Appl., 20(2015), 55-67.

[11] S.Elmas, S.Hızarcı, A.Kaplan, A Fixed Point Theorem for Surfaces, ARPN Journal of Science and Technology.