

Game number of double generalized Petersen graphs

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Abstract

The game chromatic number χ_g of double generalized Petersen graph $DP(n, m)$ is investigated. The definite values for the game chromatic number of $DP(n, 1)$, $DP(n, 2)$, $DP(n, 3)$ and $DP(n, m)$ are determined. These results extend the previous work of Shaheen, Ramy, Ziad Kanaya and Khaled Alshehadeh, who explored the game chromatic number of generalized Petersen graphs and Jahangir graphs.

Keywords: coloring of graphs; game chromatic number; double generalized Petersen graph.

1. Introduction

We consider the following well-known graph coloring game, played on a simple graph G with a color set C of cardinality k . Two players, Alice and Bob, alternately color an uncolored vertex of G with a color from C such that no adjacent vertices receive the same color (such a coloring of a graph G is known as proper coloring). Alice has the first turn and the game ends when no move is possible any more. If all the vertices are properly colored, Alice wins, otherwise Bob wins. The game chromatic number of G , denoted by $\chi_g(G)$, is the least cardinality k of the set C for which Alice has a winning strategy. In other words, necessary and sufficient conditions for k to be game chromatic number of a graph G are:

- i) Bob has winning strategy for $k - 1$ colors or less and
- ii) Alice has winning strategy for k colors.

This parameter is well defined, since it is easy to see that Alice always wins if the number of colors is larger than the maximum degree of G . Clearly, $\chi_g(G)$ is at least as large as the ordinary chromatic number $\chi(G)$, but it can be considerably more.

The obvious bounds for the game chromatic number are:

$$\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1 \quad (1)$$

where $\chi(G)$ is the chromatic number and $\Delta(G)$ is the maximum degree of the graph G .

A lot of attempts have been made to determine the game chromatic number for several classes of graphs. This work was initiated by Faigle et al. [6]. It was proved by Kierstead and Trotter [9] that the maximum of game chromatic number of a forest is 4, also that 33 is an upper bound for

game chromatic number of planar graphs. Bodlaender [5], found that the game chromatic number of Cartesian product is bounded above by constant in the family of planar graph. Later, Bartnicki et al. [4] determine the exact values of $\chi_g(GQH)$ when G and H belong to certain classes of graphs, and show that, in general, the game chromatic number $\chi_g(GQH)$ is not bounded from above by a function of game chromatic numbers of graphs G and H . After that, Zhu [12] established a bound for game coloring number and acyclic chromatic number for Cartesian product of two graphs H and S . In [10], Sia determined the exact values for the Cartesian product of different families of graph like $SmQP_n$, $SmQC_n$, $P2QW_n$. In [3], S. A. Bokhary and M. S. Akhtar found the the game chromatic number of some convex polytope graphs. Further game chromatic number of splitting graphs of path and cycle is determined by M. S. Akhtar et al. In [11] Shaheen, Ramy, Ziad Kanaya and Khaled Alshehada found the game chromatic number of generalized Petersen graphs and Jahangir graphs.

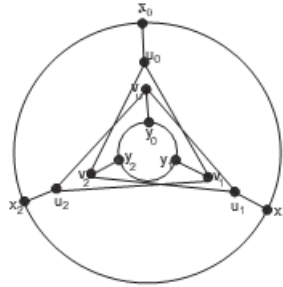
Here, we introduce some definitions and notational conventions. Suppose that Alice and Bob play the coloring game with k colors. We say that there is a threat to an uncolored vertex v if there are $k - 1$ colors in the neighborhood of v , and it is possible to color a vertex adjacent to v with the last color, so that all k colors would then appear in the neighborhood of v . The threat to the vertex v is said to be blocked or dealt if v is subsequently assigned a color, or it is no longer possible for v to have all k colors in its neighborhood. If two vertices x and y are under threat at a time then Alice can not block the threats on both these vertices and Bob wins the game for a set of k colors because on one hand if Alice blocks the threat on x then Bob plays k th color in the unique uncolored neighborhood of y , on the other hand if Alice blocks the threat on y then Bob plays k th color in the unique uncolored neighborhood of x .

In this article, we have extended the study on game chromatic number of families of double generalized Petersen graphs $DP(n, 1)$, $DP(n, 2)$, $DP(n, 3)$ and $DP(n, m)$.

2. Definite values of $\chi_g(DP(n, m))$, $m \in \{1, 2, 3\}$

The double generalized Petersen graph is the generalization of generalized Petersen graph. It is also three regular graph like generalized Petersen graph. In the double generalized Petersen graph, there are two cycle, one is called outer cycle and other is called inner cycle. Each vertex of the both cycles is attached to a pendent vertex, the pendent vertices that are attached to outer cycle are called outer pendent vertices and the pendent vertices that are attached to inner cycle are called inner pendent vertices. The double generalized Petersen graph $DP(n, m)$ is obtained by attaching the vertices of outer pendent vertices to inner pendent vertices lying at distance m . The length of the outer and inner cycle is n , thus the number of vertices are $4n$ and the number of edges in the $DP(n, m)$ are $6n$. $DP(n, m)$, $n \geq 3$ and $m \in \mathbb{Z}_n - \{0\}$, $2 \leq 2m < n$, has vertex set $\{x_i, y_i, u_i, v_i | i \in \mathbb{Z}_n\}$, edge set $\{x_i x_{i+1}, y_i y_{i+1}, u_i v_{i+m}, v_i u_{i+m}, x_i u_i, y_i v_i | i \in \mathbb{Z}_n\}$.

Lemma 1. $\chi_g(DP(3, 1)) = 4$.



Proof.

Fig. 1. DP (3, 1)

The double generalized Petersen graph DP (3, 1) with labeled vertices is shown in fig. 1.

It is accessible that $\chi_g(\text{DP}(3, 1)) > 2$.

Bob's winning strategy for 3 colors:

Let the set of useable colors be $\{1, 2, 3\}$, then there are two cases for Alice to play her first turn.

Case 1: If Alice plays color 1 in vertex x_i (or y_i), where $0 \leq i \leq 2$, then Bob responds by playing color 1 in the vertex y_i (or x_i). Then Alice has the following three choices to play her second turn.

Subcase 1.1: If Alice plays color 1 in the vertex u_{i-1} (or u_{i+1}) in her second turn then Bob plays color 2 in the vertex v_{i+1} in his second turn. In this way two vertices u_i and y_{i+1} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 1.2: If Alice plays color 1 in the vertex v_{i+1} (or v_{i-1}) in her second turn then Bob plays color 2 in the vertex u_{i-1} in his second turn. In this way two vertices x_{i-1} and v_i are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 1.3: In her second turn, if Alice plays some proper color in any other vertex of DP (3, 1) not described above then Bob plays according to the strategies of subcases 1.1 or 1.2 and wins the game.

Case 2: If Alice plays color 1 in vertex u_i (or v_i), where $0 \leq i \leq 2$, then Bob responds by playing color 1 in the vertex v_i (or u_i). Then Alice has the following three choices to play her second turn.

Subcase 2.1: If Alice plays color 1 in the vertex x_{i-1} (or x_{i+1}) in her second turn then Bob plays color 2 in the vertex y_{i+1} in his second turn. In this way two vertices y_i and v_{i+1} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 2.2: If Alice plays color 1 in the vertex y_{i+1} (or y_{i-1}) in her second turn then Bob plays color 2 in the vertex x_{i-1} in his second turn. In this way two vertices x_i and u_{i-1} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 2.3: In her second turn, if Alice plays some proper color in any other vertex of DP (3, 1) not described above then Bob plays according to the strategies of subcases 2.1 or 2.2 and wins the game.

From above discussion it implies that the game chromatic number of $DP(3, 1)$ is at least 4. Thus $\chi_g(DP(3, 1)) \geq 4$. Since maximum degree of $DP(3, 1)$ is 3, from equation 1 we have $\chi_g(DP(3, 1)) \leq 4$. Hence $\chi_g(DP(3, 1)) = 4$.

Theorem 1. $\chi_g(DP(n, 1)) = 4$ for all $n > 3$.

The double generalized Petersen graph $DP(n, 1)$ for $n = 5$ with labeled vertices is shown in fig. 2.

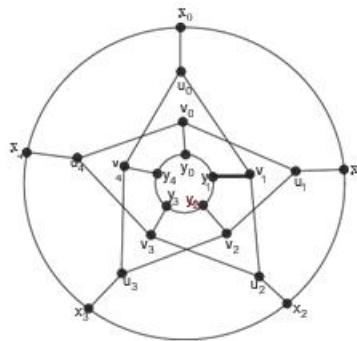
It is accessible that $\chi_g(DP(n, 1)) > 2$.

Bob's winning strategy for 3 colors:

Let the set of useable colors be $\{1, 2, 3\}$, then there are two cases for Alice to play her first turn.

Case 1: If Alice plays color 1 in vertex x_i (or y_i), where $0 \leq i \leq n - 1$, then Bob responds by playing color 1 in the vertex y_i (or x_i). Then Alice has the following three choices to play her second turn.

Subcase 1.1: If Alice plays color 1 in the vertex u_{i-1} in her second turn then Bob plays color 2 in the vertex v_{i+1} in his second turn.



Proof.

Fig. 2. $DP(5, 1)$

In this way two vertices u_i and y_{i+1} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 1.2: If Alice plays color 1 in the vertex v_{i+1} in her second turn then Bob plays color 2 in the vertex u_{i-1} in his second turn. In this way two vertices x_{i-1} and v_i are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 1.3: In her second turn, if Alice plays some proper color in any other vertex of $DP(n, 1)$ not described above then Bob plays according to the strategies of subcases 1.1 or 1.2 and wins the game.

Case 2: If Alice plays color 1 in vertex u_i (or v_i), where $0 \leq i \leq n - 1$, then Bob responds by playing color 1 in the vertex v_i (or u_i). Then Alice has the following three choices to play her second turn.

Subcase 2.1: If Alice plays color 1 in the vertex x_{i-1} in her second turn then Bob plays color 2 in the vertex y_{i+1} in his second turn. In this way two vertices y_i and v_{i+1} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

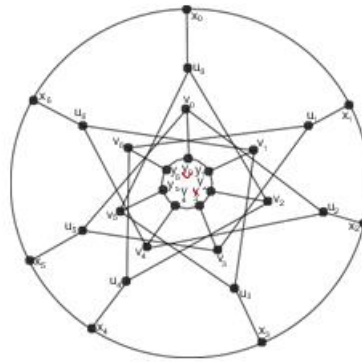
Subcase 2.2: If Alice plays color 1 in the vertex y_{i+1} in her second turn then Bob plays color 2 in the vertex x_{i-1} in his second turn. In this way two vertices x_i and u_{i-1} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 2.3: In her second turn, if Alice plays some proper color in any other vertex of $DP(n, 1)$ not described above then Bob plays according to the strategies of subcases 2.1 or 2.2 and wins the game.

From above discussion it implies that the game chromatic number of $DP(n, 1)$ is at least 4. Thus $\chi_g(DP(n, 1)) \geq 4$. Since maximum degree of $DP(n, 1)$ is 3, from equation 1 we have $\chi_g(DP(n, 1)) \leq 4$.

Hence $\chi_g(DP(n, 1)) = 4$ for $n > 3$.

Theorem 2. $\chi_g(DP(n, 2)) = 4$ for all $n > 4$



Proof.

Fig. 3. $DP(7, 2)$

The double generalized Petersen graph $DP(n, 2)$ for $n = 7$ with labeled vertices is shown in fig. 3.

It is accessible that $\chi_g(DP(n, 2)) > 2$.

Bob's winning strategy for 3 colors:

Let the set of useable colors be $\{1, 2, 3\}$, then there are two cases for Alice to play her first turn.

Case 1: If Alice plays color 1 in vertex x_i (or y_{i+1}), where $0 \leq i \leq n - 1$, then Bob responds by playing color 1 in the vertex y_{i+1} (or x_i). Then Alice has the following three choices to play her second turn.

Subcase 1.1: If Alice plays color 1 in the vertex u_{i-1} in her second turn then Bob plays color 2 in the vertex v_{i+2} in his second turn. In this way two vertices u_i and y_{i+2} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 1.2: If Alice plays color 1 in the vertex v_{i+2} in her second turn then Bob plays color 2 in the vertex u_{i-1} in his second turn. In this way two vertices x_{i-1} and v_{i+1} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 1.3: In her second turn, if Alice plays some proper color in any other vertex of $DP(n, 2)$ not described above then Bob plays according to the strategies of subcases 1.1 or 1.2 and wins the game.

Case 2: If Alice plays color 1 in vertex u_i (or v_{i+1}), where $0 \leq i \leq n - 1$, then Bob responds by playing color 1 in the vertex v_{i+1} (or u_i). Then Alice has the following three choices to play her second turn.

Subcase 2.1: If Alice plays color 1 in the vertex x_{i-1} in her second turn then Bob plays color 2 in the vertex y_{i+2} in his second turn. In this way two vertices y_{i+1} and v_{i+2} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

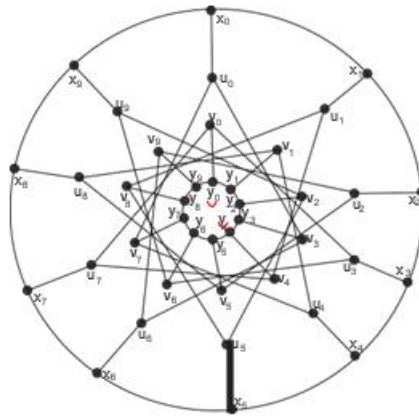
Subcase 2.2: If Alice plays color 1 in the vertex y_{i+2} in her second turn then Bob plays color 2 in the vertex x_{i-1} in his second turn. In this way two vertices x_i and u_{i-1} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 2.3: In her second turn, if Alice plays some proper color in any other vertex of $DP(n, 2)$ not described above then Bob plays according to the strategies of subcases 2.1 or 2.2 and wins the game.

From above discussion it implies that the game chromatic number of $DP(n, 2)$ is at least 4. Thus $\chi_g(DP(n, 2)) \geq 4$. Since maximum degree of $DP(n, 2)$ is 3, from equation 1 we have $\chi_g(DP(n, 2)) \leq 4$.

Hence $\chi_g(DP(n, 2)) = 4$ for $n > 4$.

Theorem 3. $\chi_g(DP(n, 3)) = 4$ for all $n > 6$



Proof.

Fig. 4. $DP(10, 3)$

The double generalized Petersen graph $DP(n, 3)$ for $n = 10$ with labeled vertices is shown in fig. 4.

It is accessible that $\chi_g(DP(n, 3)) > 2$.

Bob's winning strategy for 3 colors:

Let the set of useable colors be $\{1, 2, 3\}$, then there are two cases for Alice to play her first turn.

Case 1: If Alice plays color 1 in vertex x_i (or y_{i+2}), where $0 \leq i \leq n - 1$, then Bob responds by playing color 1 in the vertex y_{i+2} (or x_i). Then Alice has the following three choices to play her second turn.

Subcase 1.1: If Alice plays color 1 in the vertex u_{i-1} in her second turn then Bob plays color 2 in the vertex v_{i+3} in his second turn. In this way two vertices u_i and y_{i+3} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 1.2: If Alice plays color 1 in the vertex v_{i+3} in her second turn then Bob plays color 2 in the vertex u_{i-1} in his second turn. In this way two vertices x_{i-1} and v_{i+2} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 1.3: In her second turn, if Alice plays some proper color in any other vertex of $DP(n, 2)$ not described above then Bob plays according to the strategies of subcases 1.1 or 1.2 and wins the game.

Case 2: If Alice plays color 1 in vertex u_i (or v_{i+2}), where $0 \leq i \leq n-1$, then Bob responds by playing color 1 in the vertex v_{i+2} (or u_i). Then Alice has the following three choices to play her second turn.

Subcase 2.1: If Alice plays color 1 in the vertex x_{i-1} in her second turn then Bob plays color 2 in the vertex y_{i+3} in his second turn. In this way two vertices y_{i+2} and v_{i+3} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 2.2: If Alice plays color 1 in the vertex y_{i+3} in her second turn then Bob plays color 2 in the vertex x_{i-1} in his second turn. In this way two vertices x_i and u_{i-1} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 2.3: In her second turn, if Alice plays some proper color in any other vertex of $DP(n, 2)$ not described above then Bob plays according to the strategies of subcases 2.1 or 2.2 and wins the game.

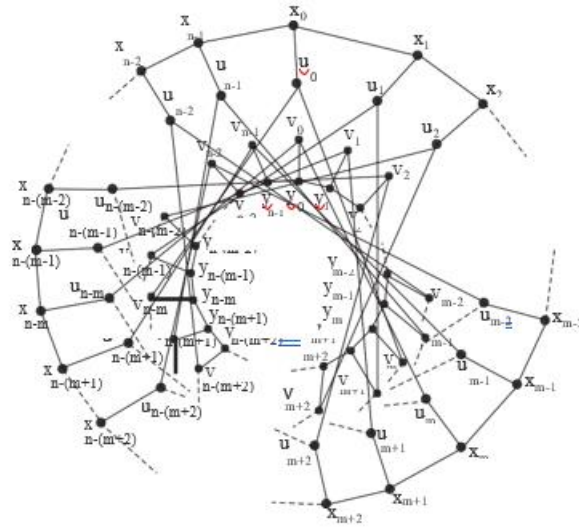
From above discussion it implies that the game chromatic number of $DP(n, 3)$ is at least 4. Thus $\chi_g(DP(n, 3)) \geq 4$. Since maximum degree of $DP(n, 3)$ is 3, from equation 1 we have $\chi_g(DP(n, 3)) \leq 4$.

$$\text{Hence } \chi_g(DP(n, 3)) = 4 \text{ for } n > 6.$$

3. Generalization of result for $DP(n, m)$, $n > 2m$

Now the game chromatic number is generalized for double generalized Petersen graph $DP(n, m)$ for $n > 2m$.

Theorem 4. $\chi_g(DP(n, m)) = 4$ for all $n > 2m$ and $m \geq 4$.



Proof.

Fig. 5. DP (n, m), $n > 2m$

The double generalized Petersen graph DP (n, m) with labeled vertices is shown in fig. 5.

It is accessible that $\chi_g(\text{DP}(n, m)) > 2$.

Bob's winning strategy for 3 colors:

Let the set of useable colors be $\{1, 2, 3\}$, then there are two cases for Alice to play her first turn.

Case 1: If Alice plays color 1 in vertex x_i (or $y_{i+(m-1)}$), where $0 \leq i \leq n-1$, then Bob responds by playing color 1 in the vertex $y_{i+(m-1)}$ (or x_i). Then Alice has the following three choices to play her second turn.

Subcase 1.1: If Alice plays color 1 in the vertex u_{i-1} in her second turn then Bob plays color 2 in the vertex v_{i+m} in his second turn. In this way two vertices u_i and y_{i+m} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 1.2: If Alice plays color 1 in the vertex v_{i+m} in her second turn then Bob plays color 2 in the vertex u_{i-1} in his second turn. In this way two vertices x_{i-1} and $v_{i+(m-1)}$ are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 1.3: In her second turn, if Alice plays some proper color in any other vertex of DP (n, m) not described above then Bob plays according to the strategies of subcases 1.1 or 1.2 and wins the game.

Case 2: If Alice plays color 1 in vertex u_i (or $v_{i+(m-1)}$), where $0 \leq i \leq n-1$, then Bob responds by playing color 1 in the vertex $v_{i+(m-1)}$ (or u_i). Then Alice has the following three choices to play her second turn.

Subcase 2.1: If Alice plays color 1 in the vertex x_{i-1} in her second turn then Bob plays color 2 in the vertex y_{i+m} in his second turn. In this way two vertices $y_{i+(m-1)}$ and v_{i+m} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 2.2: If Alice plays color 1 in the vertex y_{i+m} in her second turn then Bob plays color 2 in the vertex x_{i-1} in his second turn. In this way two vertices x_i and u_{i-1} are under threat at a time. Alice can not block the threats on both these vertices and Bob wins the game.

Subcase 2.3: In her second turn, if Alice plays some proper color in any other vertex of $DP(n, 2)$ not described above then Bob plays according to the strategies of subcases 2.1 or 2.2 and wins the game.

From above discussion it implies that the game chromatic number of $DP(n, m)$ is at least 4. Thus $\chi_g(DP(n, m)) \geq 4$. Since maximum degree of $DP(n, m)$ is 3, from equation 1 we have $\chi_g(DP(n, m)) \leq 4$.

Hence $\chi_g(DP(n, m)) = 4$ for $n > 2m$.

The following statement is an immediate consequence of the previous results.

Corollary 1. $\chi_g(DP(n, m)) = 4$ for all $n > 2m$ and $m \in \mathbb{N}$.

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