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Group Distance Magic Labeling of Graphs and their Direct Product

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Abstract

A graph G is said to have the group distance magic labeling if there exists an abelian group H and one-one map A from the vertex set of G to the group elements such that $_{x\in N(u)} A(x) = \mu$ for all $u \in V$, where $N(u)$ is the open neighborhood of u and $\mu \in H$ is the magic constant, more specifically such graph is called H-distance magic graph. In this paper, we prove anti-prism graphs are $Z2n$, $Z2 \times Zn$, $Z3 \times Z6m$, $Z4 \times Z6m$, and $Z6 \times Z6m$-distance magic graphs. This paper also concludes the group distance magic labeling of direct product of the anti-prism graphs.

1. Introduction

Graph labeling is an assignment the labels by elements from certain set to the vertices or edges, or both subject to certain conditions. For any graph G of order n, the distance magic labeling (also called Sigma Labeling) is defined as a bijection $\lambda : V(G) \rightarrow \{1, 2, 3, ..., n\}$ such that for every $x \in V$

$$w(x) = \sum_{y\in N_G(x)} \lambda(y) = k,$$

where $N_G(x)$, the neighborhood of vertex x, is the set of vertices adjacent to x, $w(x)$ is the weight of each vertex of the graph G and k is the positive integer called magic constant [1, 7]. Motivated from the idea of distance magic labeling, Froncek introduced a group distance magic labeling (GDML) in 2013 [6].

For a given graph G of order n and an abelian group H of order n, the group distance magic labeling is a one-one map $A : V(G) \to H$ such that for every $x \in V$,

$$w(x) = \sum_{y \in N_G(x)} \lambda(y) = \mu,$$

where $\mu \in H$. Generally, we can say that elements of an abelian group are used to assign the labels to the vertices of the graph G. It is the proved fact that every distance magic graph is also group distance magic graph with respect to modulo group Z_n, where n is the order of the graph, but the problem of finding group distance magic labeling still retains its interest for other abelian groups other than Z_n. Another interesting aspect of the problem is the converse of this fact is not true in general.

A cycle cannot have a GDML for any group, since if there is a GDML then the magic constant μ should be $n-1$ which is impossible. However, Froncek [6] can prove the GDML for Cartesian product and direct product of cycles for different conditions on order of graph, that is $C_n \times C_m$ ($n \leq m$) admits GDML iff nm is even or n, m both even. In 2015, Anholcer et al. proved a GDML of direct product of graphs [2]. They proved GDML for $C_n \times C_m$ for $Z_m \times Z_n$ if $m, n \equiv 0(mod 4)$. They also proved, the direct product of a r-regular graph G of order n with C_4 is GDML. They proved GDML for $C_n \times C_m$ for group $Z_t \times A$ where A is abelian group of order $\frac{mn}{}$ if $m, n \equiv 0 \pmod 4$. The direct product of C_m with C_n is not GDML for any abelian group Γ and $m, n\, f\equiv 0(mod 4)$. The direct product of a r_1-regular graph G_1 with a r_2- regular graph G_2 is $\Gamma_1 \times \Gamma_2$-distance magic whenever G_1 is Γ_1-distance magic and G_2 is Γ_2-distance magic. They also proved GDML for $G \times H$ where G is a balanced magic graph and H is an r-regular graph for $r \geq 1$.

In 2013, Cichacz [3] proved a GDML for lexicographic product of regular graphs with cycles, composition of regular graphs with complete bipartite graphs. She gave the formula $\mu = \frac{n+1}{}$ for regular graph G. According to her, the lexicographic product of graph G of order n with $^2C_4$ is GDML for abelian group Γ of order $4n$ such that $\Gamma \cong Z_2 \times Z_2 \times A$ for some abelian group A of order n [4, 5]. The lexicographic product of complete bipartite graph $K_{m,n}$ (m is an even and n is an odd) with C_4 is GDML for abelian group Γ of order $4(m + n)$. She proved GDML in $G \times C_4$ where G is Eulerian graph of odd order n and abelian group Γ of order $4n$.

If we consider for r-regular graph, any 2-regular graphs cannot have a GDML as we mentioned above. By a simple calculation we can conclude that any r-regular graph with r odd, cannot have a GDML. In this paper, we target one family of 4-regular graph, which is the anti-prism family of graphs for finding the group distance magic labeling with respect to modulo group and the product of modulo groups. We present the Z_{2n}, $Z_2 \times Z_n$, $Z_3 \times Z_{6m}$, $Z_4 \times Z_{6m}$, and $Z_6 \times Z_{6m}$-

distance magic labeling for the anti-prism. We also provide the $Z_3 \times Z_{4n}$, Z_{4mn} and $Z_2 \times Z_{mn}$-distance magic labeling for the direct product of the anti-prism graphs.

2. Discussion and Main Results

In this section we present our main results providing the group distance magic labeling for anti-prism and their direct product corresponding to different abelian groups.

2.1. GDML of Anti-Prism Graph

We determine the GDML of Anti-prism graph of order $2n$ in theorems which have been given below. Before presenting our primary findings, the vertex set and edge set of Anti-prism graphs A_n as follows

$V(A_n) = \{x_i, y_i, 0 \leq i \leq n-1\}$

$E(A_n) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i, x_i y_{i+1}, 0 \leq i \leq n-2\} \cup \{x_0 x_{n-1}, y_0 y_{n-1}, y_0 x_{n-1}, x_{n-1} y_{n-1}\}$

Theorem 1 *Let $G \cong A_n$ where A_n is an anti-prism graph and the module $2n$ group is Z_{2n}, then G allows a Z_{2n}-DML.*

Proof. Let A_n be the anti-prism graph, we know that A_n is a 4-regular graph of order $2n$. The vertex and edge representations of A_n, that follow are used as

$V(A_n) = \{x_i, y_i, 0 \leq i \leq n-1\}$

$E(A_n) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i, x_i y_{i+1}, 0 \leq i \leq n-2\} \cup \{x_0 x_{n-1}, y_0 y_{n-1}, y_0 x_{n-1}, x_{n-1} y_{n-1}\}$

$A : V(G) \rightarrow Z_{2n}$ that is defined as

Case(i) If n is even then the labeling of each vertex of graph A_n is given as

$$\ell(x_i) = 2i \quad \text{for} \quad 0 \leq i \leq n-1$$
$$\ell(y_j) = 2(n-j)-1 \quad \text{for} \quad 0 \leq j \leq n-1$$

Case(ii) If n is odd then the labeling of each vertex of graph A_n is given as

$$\ell(x_i) = 2i+1 \quad \text{for} \quad 0 \leq i \leq n-1$$
$$\ell(y_j) = 2(n-j-1) \quad \text{for} \quad 0 \leq j \leq n-1$$

Under l, A_n is a magic graph with Z_{2n}-distance and a magic constant

$$\mu = 2n - 4$$

Since $Z_{2n} \cong Z_2 \times Z_n$ if $gcd(2, n) = 1$ which are used for GDML of graph A_n in theorem 1. Now we discuss module group $Z_2 \times Z_n$ if $gcd(2, n) f= 1$ for GDML of graph A_n in the following theorem

Theorem 2 *Let $G \cong A_n$ where A_n is anti-prism graph and the module group is $Z_2 \times Z_n$ such that $gcd(2, n)$ f= 1. Then G admits a $Z_2 \times Z_n$-DML.*

Proof. We use the vertex set and edge set of A_n given in theorem 1 and $A : V(G) \longrightarrow Z_2 \times Z_n$ must be defined as follows

$$A(x_i) = (0, i) \quad \text{for} \quad 0 \leq i \leq n-1$$
$$A(y_j) = (1, (n-1)-j) \quad \text{for} \quad 0 \leq j \leq n-1$$

Theorem 3 *Let $G \cong A_n$ where A_n is anti-prism graph such that $n = 9m$, $m \in Z^+$ and m f= $3k$,*

$k \in N$. Then G allows a $Z_3 \times Z_{6m}$-DML.

Proof. We use the vertex set and edge set of A_n given in theorem 1 and $A : V(G) \longrightarrow Z_3 \times Z_{6m}$ must be defined as follows

$$\ell(x_i) = \begin{cases} (0, 2i \mod 6m) & \text{for} \quad 9t \leq i \leq 2+9t, t \geq 0 \\ (1, 2i \mod 6m) & \text{for} \quad 3+9t \leq i \leq 5+9t, t \geq 0 \\ (2, 2i \mod 6m) & \text{for} \quad 6+9t \leq i \leq 8+9t, t \geq 0 \end{cases}$$

And

$$\ell(y_j)=\begin{cases}(1,(6m-(2j+1)) \mod 6m) & \text{for} \quad 2+9t\leq j\leq 4+9t, t\geq 0\\ (0,(6m-(2j+1)) \mod 6m) & \text{for} \quad 5+9t\leq j\leq 7+9t, t\geq 0\\ (2,(6m-(2j+1)) \mod 6m) & \text{for} \quad 8+9t\leq j\leq 10+9t, t\geq 0 \;\; or \;\; j=0,1\end{cases}$$

Under l, A_n is a magic graph with $Z_3 \times Z_{6m}$-distance and a magic constant

$$\mu = (0, 6m - 4).$$

Theorem 4 *Let* $G \cong A_n$ *where* A_n *is anti-prism graph such that* $n = 12m$ *and* $m = 2k+1$, $k \geq 0$. *Then* G *allows a* $Z_4 \times Z_{6m}$*-DML.*

Proof. We use the vertex set and edge set of A_n given in theorem 1 and $A : V(G) \to Z_4 \times Z_{6m}$ must be defined as follows

$$\ell(x_i)=\begin{cases}(0,2i \mod 6m) & \text{for} \quad 12t\leq i\leq 2+12t, t\geq 0\\ (1,2i \mod 6m) & \text{for} \quad 3+12t\leq i\leq 5+12t, t\geq 0\\ (2,2i \mod 6m) & \text{for} \quad 6+12t\leq i\leq 8+12t, t\geq 0\\ (3,2i \mod 6m) & \text{for} \quad 9+12t\leq i\leq 11+12t, t\geq 0\end{cases}$$

And

$$\ell(y_j)=\begin{cases}(2,(6m-(2j+1)) \mod 6m) & \text{for} \quad 2+12t\leq j\leq 4+12t, t\geq 0\\ (1,(6m-(2j+1)) \mod 6m) & \text{for} \quad 5+12t\leq j\leq 7+12t, t\geq 0\\ (0,(6m-(2j+1)) \mod 6m) & \text{for} \quad 8+12t\leq j\leq 10+12t, t\geq 0\\ (3,(6m-(2j+1)) \mod 6m) & \text{for} \quad 11+12t\leq j\leq 13+12t, t\geq 0 \;\; or \;\; j=0,1\end{cases}$$

Under A, A_n is a magic graph with $Z_4 \times Z_{6m}$-distance and a magic constant

$$\mu = (1, 6m - 4).$$

Theorem 5 *Let* $G \cong A_n$ *where* A_n *is anti-prism graph such that* $n = 18m$, $m \geq 1$ *and* $m = k$, $k \equiv 1 \mod 6$ *or* $5 \mod 6$. *Then* G *allows a* $Z_6 \times Z_{6m}$*-DML*
Proof. We use the vertex set and edge set of A_n given in theorem 1 and $A : V(G) \to Z_6 \times Z_{6m}$ must be defined as follows

$$\ell(x_i)=\begin{cases}(0,2i \mod 6m) & \text{for} \quad 18t\le i\le 2+18t, t\ge 0\\ (1,2i \mod 6m) & \text{for} \quad 3+18t\le i\le 5+18t, t\ge 0\\ (2,2i \mod 6m) & \text{for} \quad 6+18t\le i\le 8+18t, t\ge 0\\ (3,2i \mod 6m) & \text{for} \quad 9+18t\le i\le 11+18t, t\ge 0\\ (4,2i \mod 6m) & \text{for} \quad 12+18t\le i\le 14+18t, t\ge 0\\ (5,2i \mod 6m) & \text{for} \quad 15+18t\le i\le 17+18t, t\ge 0\end{cases}$$

And

$$\ell(y_j)=\begin{cases}(4,(6m-(2j+1)) \mod 6m) & \text{for} \quad 2+18t\le j\le 4+18t, t\ge 0\\ (3,(6m-(2j+1)) \mod 6m) & \text{for} \quad 5+18t\le j\le 7+18t, t\ge 0\\ (2,(6m-(2j+1)) \mod 6m) & \text{for} \quad 8+18t\le j\le 10+18t, t\ge 0\\ (1,(6m-(2j+1)) \mod 6m) & \text{for} \quad 11+18t\le j\le 13+18t, t\ge 0\\ (0,(6m-(2j+1)) \mod 6m) & \text{for} \quad 14+18t\le j\le 16+18t, t\ge 0\\ (5,(6m-(2j+1)) \mod 6m) & \text{for} \quad 17+18t\le j\le 19+18t, t\ge 0 \quad or \quad j=0,1\end{cases}$$

Under A, A_n is a magic graph with $Z_6 \times Z_{6m}$-distance and a magic constant

$$\mu = (3, 6m - 4).$$

2.2 Group Distance Magic Labeling of Direct Product of Anti-Prism Graphs
The vertex set V (G) × V (H) and edge set of graph G× H which is the direct product of graphs G and H as follow

$$E(G\times H)=\{(u,v)(u',v') \quad | \quad u,v\in V(G), u',v'\in V(H), uu'\in E(G), vv'\in E(H)\},$$

that is any two vertices (u, v) and (u', v') are adjacent in $G \times H$ if and only if u is adjacent to u' in G and v is adjacent to v' in H [5].

Lemma 1 *[2] If an r_1-regular graph G_1 is a Γ_1-distance magic and an r_2-regular graph G_2 is a Γ_2-distance magic, then the direct product $G_1 \times G_2$ is a $\Gamma_1 \times \Gamma_2$- distance magic graph.*

Based on the above Lemma, the existence of the GDML has already been proved and we can construct the GDML for the direct product graphs for specific groups but the problem is still open for finding the complete list of groups for which GDML exits for the direct product of graphs. In the following theorems, we present the group distance magic labeling for direct product of anti-prisms for several groups.

Theorem 6 *Let $G \cong A_3$ and $H \cong A_n$, where A_3 and A_n be anti-prism graphs such that $n = 3m$, $m \geq 1$. The module group of order $12n$ is $Z_3 \times Z_{4n}$. Then the graph $G \times H$ allows a $Z_3 \times Z_{4n}$-DML.*

Proof. The vertex and edge representations of A_3 and A_n that follow are used as

$V(A_3) = \{x_i, y_i | 0 \leq i \leq 2\}$

$E(A_3) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i, x_i y_{i+1} | 0 \leq i \leq 1\} \cup \{x_0 x_2, y_0 y_2, y_0 x_2, x_2 y_2\}$

$V(A_n) = \{x'_i, y'_i | 0 \leq i \leq n-1\}$

$E(A_n) = \{x'_i x'_{i+1}, y'_i y'_{i+1}, x'_i y'_i, x'_i y'_{i+1} | 0 \leq i \leq n-2\} \cup \{x'_0 x'_{n-1}, y'_0 y'_{n-1}, y'_0 x'_{n-1}, x'_{n-1} y'_{n-1}\}$

The following vertex represents of A3 × An, according to the notion of direct product

$$V(A_3 \times A_n) = \{(x_i, y_i), (x'_j, y'_j) | 0 \leq i \leq 2\ ,\ 0 \leq j \leq n-1\}$$

$l : V(A_3 \times A_n) \to Z_3 \times Z_{4n}$ must be defined as follows,

$$\ell(x_i, x'_j) = (i, 2j), \qquad \text{for} \quad 0 \leq i \leq 2, 0 \leq j \leq n-1$$

$$\ell(x_i, y'_j) = (i, 2(n + 2ni + j) \mod 12m), \qquad \text{for} \quad 0 \leq i \leq 2, 0 \leq j \leq n-1$$

$$\ell(y_i, x'_j) = (2 - i, (2(2n + 2ni - j) - 1) \mod 12m), \qquad \text{for} \quad 0 \leq i \leq 2, 0 \leq j \leq n-1$$

$$\ell(y_i, y'_j) = (2 - i, (2(n + 2ni - j) - 1) \mod 12m), \qquad \text{for} \quad 0 \leq i \leq 2, 0 \leq j \leq n-1$$

Then under A, $A_3 \times A_n$ is a magic graph with $Z_3 \times Z_{4n}$-distance and a magic constant

$$\mu = (0, 4n - 8)$$

Theorem 7 *Let $G \cong A_m$ and $H \cong A_n$, where A_m and A_n be anti-prism graphs such that $m \leq n$ and Z_{4mn} be the module group of order $4mn$. Then the graph $G \times H$ admits a Z_{4mn}-distance magic labeling for all $m, n \geq 3$.*

Proof. The vertex and edge representations of A_m and A_n that follow are used as

$V(A_m) = \{x_i, y_i | 0 \leq i \leq m-1\}$

$E(A_m) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i, x_i y_{i+1} | 0 \leq i \leq m-2\} \cup \{x_0 x_{m-1}, y_0 y_{m-1}, y_0 x_{m-1}, x_{m-1} y_{m-1}\}$

$V(A_n) = \{x'_i, y'_i | 0 \leq i \leq n-1\}$

$E(A_n) = \{x'_i x'_{i+1}, y'_i y'_{i+1}, x'_i y'_i, x'_i y'_{i+1} | 0 \leq i \leq n-2\} \cup \{x'_0 x'_{n-1}, y'_0 y'_{n-1}, y'_0 x'_{n-1}, x'_{n-1} y'_{n-1}\}$

The following vertex represents of $A_m \times A_n$, according to the notion of direct product

$$V(A_m \times A_n) = \{(x_i, y_i), (x'_j, y'_j) | 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\}$$

$l : V(A_m \times A_n) \rightarrow Z_{4mn}$ must be defined as follows,

$$\begin{aligned}
\ell(x_i, x'_j) &= 4ni + 2j, && \text{for} \quad 0 \leq i \leq m-1, 0 \leq j \leq n-1 \\
\ell(x_i, y'_j) &= 4ni + 2(j+n), && \text{for} \quad 0 \leq i \leq m-1, 0 \leq j \leq n-1 \\
\ell(y_i, x'_j) &= (4mn-1) - 2(2ni+j), && \text{for} \quad 0 \leq i \leq m-1, 0 \leq j \leq n-1 \\
\ell(y_i, y'_j) &= (4mn-1) - 2[n(2i+1)+j], && \text{for} \quad 0 \leq i \leq m-1, 0 \leq j \leq n-1
\end{aligned}$$

Then under l, $A_m \times A_n$ is a magic graph with Z_{4mn}-distance and a magic constant

$$\mu = \begin{cases} 8((m-2)n-1) & \text{for} \quad m = 3, 4 \\ 4((m-4)n-2) & \text{for} \quad m > 4 \end{cases}$$

Theorem 8 *Let $G \cong A_m$ and $H \cong A_n$, where A_m and A_n be anti-prism graphs such that $m \leq n$ and The module group of order 4mn is $Z_2 \times Z_{2mn}$. Then the graph $G \times H$ allows a $Z_2 \times Z_{2mn}$-DML for all $m, n \geq 3$.*

Proof. The vertex and edge representations of A_m and A_n that follow are used as

$V(A_m) = \{x_i, y_i | 0 \leq i \leq m-1\}$

$E(A_m) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i, x_i y_{i+1} | 0 \leq i \leq m-2\} \cup \{x_0 x_{m-1}, y_0 y_{m-1}, y_0 x_{m-1}, x_{m-1} y_{m-1}\}$

$V(A_n) = \{x'_i, y'_i | 0 \leq i \leq n-1\}$

$E(A_n) = \{x'_i x'_{i+1}, y'_i y'_{i+1}, x'_i y'_i, x'_i y'_{i+1} | 0 \leq i \leq n-2\} \cup \{x'_0 x'_{n-1}, y'_0 y'_{n-1}, y'_0 x'_{n-1}, x'_{n-1} y'_{n-1}\}$

The following vertex represents of $A_m \times A_n$, according to the notion of direct product

$$V(A_m \times A_n) = \{(x_i, y_i), (x'_j, y'_j) | 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\}$$

$l : V(A_m \times A_n) \rightarrow Z_2 \times Z_{2mn}$ must be defined as follows,

$$\ell(x_i, x'_j) = (0, 2ni + j), \quad \text{for} \quad 0 \le i \le m-1, 0 \le j \le n-1$$
$$\ell(x_i, y'_j) = (0, n(2i+1) + j), \quad \text{for} \quad 0 \le i \le m-1, 0 \le j \le n-1$$
$$\ell(y_i, x'_j) = (1, 2n(m-i) - 1 - j), \quad \text{for} \quad 0 \le i \le m-1, 0 \le j \le n-1$$
$$\ell(y_i, y'_j) = (1, n[2(m-i) - 1] - 1 - j), \quad \text{for} \quad 0 \le i \le m-1, 0 \le j \le n-1$$

Then under l, $A_m \times A_n$ is a magic graph with $Z_2 \times Z_{2mn}$-distance and a magic constant

$$\mu = \begin{cases} (0, 4((m-2)n-2)) & \text{for} \quad m = 3, 4 \\ (0, 2((m-4)n-4)) & \text{for} \quad m > 4 \end{cases}$$

3. Conclusion

Graph theory and groups are connected by Group Distance Magic Labeling (GDML). Due to this feature, we define the relationship using GDML between the group Z_{2n} and anti-prism graph of order $2n$. For the first time, we determine GDML of anti-prism graph by the groups $Z_2 \times Z_n$, $Z_3 \times Z_{6m}$, $Z_4 \times Z_{6m}$, $Z_6 \times Z_{6m}$ other than Z_{2n}. We also extended our work from GDML of anti-prism graph to the GDML of direct product of anti-prism graph by Z_{4mn}, $Z_3 \times Z_{4n}$ and $Z_2 \times Z_{2mn}$.

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