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k-tuple domination for some chessboard graphs

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Abstract

In this paper we consider *k*-tuple domination on the $n \times n$ bishop's, rook's, and queen's graphs. For the $n \times n$ bishop's graph, we solve the *k*-tuple domination number when $k=n-1$ and when $k=n-3$. For the $n \times n$ rook's graph we find the *k*-tuple domination number for all $n > 1$ and all j, with $0 \leq n/2-1 \leq j$ and $k=2n-2-2j$. Finally, for the $n \times n$ queen's graph, a lower bound for the *k*-tuple domination number is found for $k=3n-3$.

1. Introduction

There are many types of chess themed problems that are of interest in mathematics and computer science. Hundreds of papers have been written in this subfield of mathematics. From the famous *n*-queen's independence problem, to domination problems and knight's tours and variations of these and other themes on different sized and shaped boards; there have been many different graph theoretic parameters explored for these graphs. For two good surveys on the types of problems that are of interest, see both [9,12].

A chessboard graph is formed by taking the squares on a $n \times n$ board as vertices. Two vertices are adjacent on these graphs if and only if they're separated by exactly one move of our piece type on an empty $n \times n$ board. Here our piece types are bishop, rook, or queen. These each form the bishop's, rook's, and queen's graph, denoted by B_n, R_n, and Q_n, respectively.

In this paper we will be looking at a variation of graph domination known as *k*-tuple domination. For a graph $G=(V,E)$ a set S is a *k*-tuple dominating set if and only if for every vertex in $v \epsilon V$, v has at least k open neighbors in S, or $v \epsilon S$ and v has at least $k-1$ open neighbors in S. Thus a set of pieces, all of one type, forms a *k*-tuple dominating set if and only if given any square on our $n \times n$ board, either the square is attacked at least k times or the square is occupied and attacked at least $k-1$ times. The minimum number of pieces of a certain type needed to provide a *k*-tuple dominating set is the *k*-tuple domination number, denoted by $\gamma_{\times k}(B_n)$, $\gamma_{\times k}(R_n)$, and $\gamma_{\times k}(Q_n)$ for the *k*-tuple domination numbers on the bishop's, rook's, and queen's graph, respectively. For more information on the *k*-tuple domination number in general, see [4,5,6,7,8,10,11]. For past work that considers *k*-tuple domination on chessboard graphs, see [1,2,3].

In **Section 2** of this paper we further explore work on k-tuple domination for the $n \times n$ bishop's graph. In particular, it found that for all $n>1$, $\gamma_{\times n-1}(B_n)=n^2-n$. In **Section** 2 we also find that for odd $n>3$, $\gamma_{\times n-3}(B_n)=n^2-4n-3$ and for even $n \geq 4$, $\gamma_{\times n-3}(B_n)=n^2-4n+4$. In **Section 3** of this paper we look at k-tuple domination for the $n \times n$ rook's graph. In this section it is shown that for all $n>1$ and all j, with $0 \leq j \leq n/2-1$, $\gamma_{\times 2n-2-2j}(R_n)=n^2-nj-j-1$. Finally, in **Section 4**, we arrive at a lower bound for the k-tuple domination number on the $n \times n$ queen's graph for when $k=3n-3$. The lower bound is $\gamma_{\times 3n-3}(Q_n) \geq n^2-\lfloor (n-1)/2 \rfloor$. Concluding this section is a list of $\gamma_{\times 3n-3}(Q_n)$ for some small board sizes.

1 *k*-tuple domination on the bishop's graph

Theorem 1 For all $n>1$, $\gamma_{\times n-1}(B_n) \geq n^2-n$.

Proof: It is easy to see the theorem holds for $n=2$. Thus, let $n>2$ by assumption.

Consider the border squares of our $n \times n$ board. Note that the all the vertices associated with the border squares all have degree of $n-1$. Thus, if a border square is occupied it can have, at most, one unoccupied square adjacent to the border. Likewise, if unoccupied, a border square can't have any adjacent, unoccupied squares.

Next, consider our unoccupied squares. To begin, consider any unoccupied square not on the border. This means there are exactly 4 border squares that are diagonally adjacent to it. It follows that these 4 squares can't be adjacent to anymore unoccupied squares.

Now, consider any non-corner, border square which is unoccupied. It follows that this unoccupied, border square is adjacent to two other border squares. These two border squares also have an additional common neighbor along the border. It follows that this second, common neighbor cannot be adjacent to any other unoccupied square - or else the two border squares would be adjacent to, at most, $n-3$ bishops. Thus, so long as we're considering non-corner squares, any unoccupied square eliminates 4 of our border squares from being adjacent to any additional unoccupied squares.

Finally, note if we have an unoccupied, corner square, this eliminates exactly 2 of our 4 corner squares from adjacency to any additional unoccupied square. This gives us an upper bound of $4(n-2)/4 +4/2=n$ unoccupied squares since there are exactly $4(n-1)$ border squares. Thus, it follows that $\gamma_{\times n-1}(B_n) \geq n^2-n$. □

Theorem 2 For all $n>1$, $\gamma_{\times n-1}(B_n)= n^2-n$.

Proof: Since we showed the lower bound in **Theorem 1**, it suffices to show that $\gamma_{\times n-1}(B_n) \leq n^2-n$. Consider the following formation of bishops in **Figure 1**, shown below, which generalizes to any $n \times n$ board for $n>1$.

	B	B	B	B	B	B
	B	B	B	B	B	B
	B	B	B	B	B	B
	B	B	B	B	B	B
	B	B	B	B	B	B
	B	B	B	B	B	B
	B	B	B	B	B	B

Figure 1 A formation of bishops on a 7×7 board which provides a minimum $\gamma_{\times 6}(B_7)$-set of 42 bishops.

To see the generalization of the formation to any $n\times n$ board, first note that we have the n squares in the left-most column unoccupied. All other squares are occupied.

It is straightforward to see that any of our occupied, border squares are adjacent to, at most, one unoccupied square. For the unoccupied squares, it is easy to see these squares are adjacent to no unoccupied square. Since these border squares have at least $n-1$ adjacent squares then these squares are dominated at least $n-1$ times.

Next, consider the interior squares. Note that all the vertices associated with these squares have degree of at least $n+1$. Thus, since these squares are adjacent to, at most, two unoccupied squares, then it is clear these squares are dominated at least $n-1$ times. Since all squares are dominated at least $n-1$ times, it follows that $\gamma_{\times n-1}(B_n) \leq n^2-n$. □

Theorem 3 For all odd $n>3$, $\gamma_{\times n-3}(B_n)\leq n^2-4n+3$.

Proof: Given odd $n>3$, consider a formation of bishops as follows. Every square on our board is occupied, except for the border squares and the center square. **Figure 2**, below, will illustrate.

	B	B	B	B	B	
	B	B	B	B	B	
	B	B		B	B	
	B	B	B	B	B	
	B	B	B	B	B	

Figure 2 A minimum $\gamma_{\times 4}(B_7)$-set of 24 bishops.

It is easy to see that the border squares are all adjacent to exactly 2 unoccupied squares and are themselves unoccupied. Thus, since all border squares have corresponding vertices with degree $n-1$, they're adjacent to exactly $n-3$ bishops in the set.

Likewise, it is easy to see that since the vertices corresponding to occupied squares have degree of at least $n+1$, and these squares are adjacent to at most 5 unoccupied squares, then it is clear these squares are attacked at least $n-4$ times with the squares being occupied.

Finally, consider the center square. The vertex associated with this square has degree $2n-2$. Since this square is adjacent to exactly 4 unoccupied squares, and $2n-6 \geq n-3$ for $n>4$, then we have this square attacked at least $n-3$ times.

For a count on the number of bishops in our set, note we have $4(n-1)$ border squares. Thus, we have $n^2-4(n-1)-2=n^2-4n+3$ bishops in our set. □

Theorem 4 For all $n>3$, $\gamma_{\times n-3}(B_n)= n^2-4n+3$.

Proof: First, it should be noted that by **Theorem 3**, that $\gamma_{\times n-3}(B_n)\leq n^2-4n+3$. Thus, it suffices to show that $\gamma_{\times n-3}(B_n) \geq n^2-4n+3$.

To see that $\gamma_{\times n-3}(B_n) \geq n^2-4n+3$, let a be the number of unoccupied squares in our set that are assigned to non-corner, border squares, b be the number of unoccupied squares assigned to corner squares, and c be the number of unoccupied squares in the interior of the board, but not along the main diagonal. Also, let d be the number of unoccupied squares along the main diagonals, except the corner squares or center square. Finally, let e be 0 if the center square is occupied and 1 if the center square is unoccupied.

Note first that we must have any unoccupied squares along the border adjacent to, at most, 2 other unoccupied squares. Likewise, any occupied, border square can be adjacent to, at most, 3 unoccupied squares. For the sake of this particular proof from here, let us define adjacency to include the closed neighborhood of a vertex and not merely the open neighborhood. Thus, a vertex (square) is adjacent to itself for our purposes.

Next, note it follows that all our non-corner, border squares are adjacent to exactly 3 of these same squares. Likewise, the unoccupied squares in the interior of the board, but not along the main diagonal, are adjacent to exactly 4 of our non-corner, border squares. Also, our unoccupied squares that are on the main diagonal, but not in the corners or center square are adjacent to exactly 2 of our non-corner, border squares. Note that the non-corner, border squares can be adjacent to, at most, 3 unoccupied squares – since their corresponding vertices all have degree $n-1$. Thus, since we have $4(n-2)$ non-corner, border squares, and each of these can be adjacent to, at most, 3 unoccupied squares, then we have **1)** $3a+4c+2d \leq 3\times 4(n-2)=12(n-2)$.

Now, consider the corner squares. Note that the center square is adjacent to all 4 corner squares and any other square along the main diagonal is adjacent to 2 corner squares. It is the case that $2b+2d+4e \leq 12$, since we have 4 corner squares that must be adjacent to, at most, 3 unoccupied squares. Thus, **2)** $b+d+2e \leq 6$. Note also that $b\leq 4$ and $e\leq 1$.

Note we're trying to maximize $a+b+c+d+e$. Thus, clearly $c=0$, since the slope of this variable is greater than the slope for variable a in **1)**. Consider then the cases for which $d=0,1,2,3,4,5,6$. We find that when $d=6$, $a+b+c+d+e \leq 4n-6$. Similarly, when $d=5$,

$a+b+c+d+e \leq 4n-5$. Also, when $d=3$ *or* $d=4$, $a+b+c+d+e \leq 4n-4$. Finally, when $d=0$, $d=1$, or $d=2$, $a+b+c+d+e \leq 4n-3$. Thus, the number of unoccupied squares is, at most, $4n-3$. This yields $\gamma_{\times n-3}(B_n) \geq n^2-4n+3$ for odd $n>3$. □

Theorem 5 For even $n>2$, $\gamma_{\times n-3}(B_n) \leq n^2-4n+4$.

Proof: Consider a formation of bishops for even $n>2$ for which every border square is unoccupied and every interior square is occupied. **Figure 3** will illustrate.

	B	B	B	B	B	B	
	B	B	B	B	B	B	
	B	B	B	B	B	B	
	B	B	B	B	B	B	
	B	B	B	B	B	B	
	B	B	B	B	B	B	

Figure 3 A minimum γ×5(B8)-set of 36 bishops

To see the set is a $\gamma_{\times n-3}(B_n)$-set for even $n>2$, first consider the interior squares. Each of these squares has corresponding vertices of degree $n+1$. It follows that since these squares are adjacent to, at most 4 unoccupied squares, then it is clear these squares are attacked $n-3$ times.

Next, consider the border squares. These squares have corresponding vertices of degree $n-1$. These squares are adjacent to exactly 2 other unoccupied squares. Thus, it is clear these squares are attacked $n-3$ times. For a quick count on the number of bishops, note there are $4(n-1)$ border squares. Thus, we have $n^2-4(n-1)=n^2-4n+4$ bishops. □

Theorem 6 For even $n>2$, $\gamma_{\times n-3}(B_n) = n^2-4n+4$.

Proof: Since it has been shown in **Theorem 5** that for even $n>2$, $\gamma_{\times n-3}(B_n) \leq n^2-4n+4$, then it suffices to show that for even $n>2$, $\gamma_{\times n-3}(B_n) \geq n^2-4n+4$.

Then let us in a similar fashion as **Theorem 4** label variables for the unoccupied squares. Thus, let a be the number of unoccupied squares in our set that are assigned to non-corner, border squares, let b be the number of unoccupied squares assigned to corner squares, and let c be the number of unoccupied squares in the interior of the board, but not along the main diagonal. Also, let d be the number of unoccupied squares along the main diagonals, except the corner squares or center square. Note for even $n>2$, there is no central square to consider. Let us also for the sake of this proof define a square to be adjacent to itself, as in **Theorem 4**.

We then arrive at **1)** $3a+4c+2d \leq 3\times4(n-2)=12(n-2)$ and **2)** $b+d\leq 6$, with $b\leq4$. Thus, we consider the 7 possible values for which $0\leq d \leq6$, and the upper bounds these cases provide for $a+b+c+d$. We find that when $d=5$ *or* $d=6$, $a+b+c+d \leq 4n-6$. Also, when $d=0$, $d=3$ *or* $d=4$, $a+b+c+d \leq 4n-4$. Likewise, when $d=1$ or $d=2$, $a+b+c+d \leq 4n-3$. Thus, the number of unoccupied squares is, at most, $4n$-3. Thus, $\gamma_{\times n\text{-}3}(B_n) \geq n^2$-$4n$+$3$ for even $n>4$. However, note that $\gamma_{\times n\text{-}3}(B_n)$ must be even since B_n is the union of two isomorphic graphs. Thus, $\gamma_{\times n\text{-}3}(B_n) \geq n^2$-$4n$+$4$ for even $n>2$. □

3 k-tuple domination on the rook's graph

Theorem 7 For all $n>1$, and all j with $0 \leq j \leq n/2\text{-}1$, $\gamma_{\times 2n\text{-}2\text{-}2j}(R_n) \leq n^2\text{-}\ nj\ \text{-}\ j\ \text{-}\ 1$.

Proof: We will first show that $\gamma_{\times 2n\text{-}2\text{-}2j}(R_n) \leq n^2\text{-}\ nj\ \text{-}\ j - 1$ when $n>1$ and $0 \leq j \leq n/2\text{-}1$ by referring to the formation shown below in **Figure 4**.

R	R	R			R	R	R
R	R	R		R	R	R	
R	R	R	R	R	R		
R	R	R	R	R			R
R	R	R	R			R	R
			R	R	R	R	R
			R	R	R	R	R
			R	R	R	R	R

Figure 4 A set of 45 rooks forming a minimum γ×10(R8)-set.

To form this k-tuple dominating set for general n, begin by assigning unoccupied squares in the lower-left, $(j+1)\times(j+1)$ subboard. In the $(j+1)\times(n\text{-}j\text{-}1)$ and the $(n\text{-}j\text{-}1)\times(j+1)$ subboards in the upper-left and lower-right of the board, respectively, assign all these squares as occupied. This leaves us with the upper-right, $(n\text{-}j\text{-}1)\times(n\text{-}j\text{-}1)$ subboard.

In this subboard place $n\text{-}2j\text{-}1$ sets of $n\text{-}j\text{-}1$ rooks. Each of these sets individually will form an independent set of rooks. More specifically, for $n\text{-}2j\text{-}1=1$, assign rooks to all squares along the main, positive sloping diagonal of our upper-right subboard. Then, for $n\text{-}2j\text{-}1=2$, assign rooks to squares directly above the previous occupied squares in the upper-right subboard and in the lower-right square of our $(n\text{-}j\text{-}1)\times(n\text{-}j\text{-}1)$ upper-right subboard. Such a process continues by assigning rooks to each of the diagonals' squares immediately above the previous step's diagonals, and in the $(n\text{-}j\text{-}1)\times(n\text{-}j\text{-}1)$ subboard by inductive step placement for higher values of $n\text{-}j\text{-}1$. These diagonals exist for so long as $0 \leq j \leq n/2\text{-}1$. All the remaining squares of our $(n\text{-}j\text{-}1)\times(n\text{-}j\text{-}1)$ subboard are unoccupied.

To see this is a $(2n\text{-}2j\text{-}2)$-tuple dominating set of rooks let us first examine the lower-left $(j+1)\times(j+1)$ subboard. These squares are adjacent to $n\text{-}j\text{-}1$ rooks in the upper-left subboard and $n\text{-}j\text{-}1$ rooks in the lower-right subboard. Thus, these squares are adjacent to exactly $2(n\text{-}j\text{-}1)=2n\text{-}2j\text{-}2$ rooks.

Next, consider the squares in the upper-left $(n\text{-}j\text{-}1)\times(j+1)$ subboard and the lower-right $(j+1)\times(n\text{-}j\text{-}1)$ subboard. These squares are all occupied and adjacent to exactly $j+n\text{-}j\text{-}2$ rooks in their own subboards. They're also adjacent to $n\text{-}2j\text{-}1$ rooks in the upper-right subboard. This gives us a total of 2n-2j-3 rooks that attack these squares. Thus, since these squares are all occupied, there's no reason to consider them any further.

Next, let us consider the occupied squares in the upper-right $(n-j-1)\times(n-j-1)$ subboard. They're adjacent to exactly $2(j+1)$ rooks from the other two subboards with rooks in them. Also, they're adjacent to exactly $2(n-2j-2)$ rooks from their own subboard. This gives us adjacency to exactly $2(j+1)+2(n-2j-2)=2n-2j-2$ rooks.

Finally, let us consider the squares in the upper-right subboad that aren't occupied. They're adjacent to exactly $2(j+1)$ rooks in the other subboards and $2(n-2j-1)$ rooks in their own subboard. This yields a total of $2(j+1)+2(n-2j-1)=2n-2j$ rooks that are adjacent to these squares.

Note for the total count of rooks we have $2(n-j-1)(j+1)$ rooks in the upper-left and lower right subboards. We also have $(n-2j-1)(n-j-1)$ rooks in the upper-right subboard. Summing these gives us exactly $2(n-j-1)(j+1)+ (n-2j-1)(n-j-1)=n^2-nj-j-1$ rooks in our set. □

Theorem 8 For all $n>1$, and all j with $0 \leq j \leq n/2-1$, $\gamma_{\times 2n-2-2j}(R_n)= n^2- nj - j - 1$.

Proof: First note that by **Theorem 7**, for all $n>1$, and all j with $0 \leq j \leq n/2-1$, $\gamma_{\times 2n-2-2j}(R_n) \leq n^2- nj - j - 1$. Thus all that needs to be shown is that for all $n>1$, and all j with $0 \leq j \leq n/2-1$, $\gamma_{\times 2n-2-2j}(R_n) \geq n^2- nj - j - 1$.

Suppose we're given a k-tuple dominating set of rooks on our $n\times n$ rook's graph, where $k=2n-2-2j$ for $0\leq j \leq n/2-1$ and $n>1$. Also, suppose we have all rows with at most j unoccupied squares in them. It follows that we can have, at most, nj unoccupied squares on our board. Thus, $\gamma_{\times 2n-2-2j}(R_n)\geq n^2- nj > n^2-nj- j - 1$. It follows this cannot be a minimum k-tuple dominating set of rooks.

Next, suppose there exists a row with $j+c$ unoccupied squares in it, with $c\geq 2$. It follows that there can be no more than $j+1-c$ unoccupied squares in each column, but not in the considered row - since we must dominate the squares in this row at least $k=2n-2-2j$ times. Thus, there are at most $j+c+n(j+1-c)$ unoccupied squares on our board. It follows that there are at least $n^2-j-c-nj-n+nc$ rooks in our set. However, since $nc>n+c-1$ for $n>1$ and $c\geq 2$, then it follows this cannot be a minimum k-tuple dominating set or rooks.

Finally, suppose there exists a row with exactly $j+1$ unoccupied squares in it, with $0 \leq j \leq n/2-1$. Note then that any column can have, at most, j additional unoccupied squares in the remaining n-1 column squares not in the considered row - since we need our set to be dominated at least $k=2n-2-2j$ times. It follows that we have, at most, $nj+j+1$ unoccupied squares on our board. Thus, we have at least $n^2-nj-j-1$ rooks in our set. □

4 k-tuple domination on the queen's graph

Theorem 9 For all $n>2$, $\gamma_{\times 3n-3}(Q_n) \geq n^2 - \lfloor (n-1)/2 \rfloor$.

Proof: It is easy to see that for $n=3$ and $n=4$ the theorem holds.

Next, to proceed for the other cases where $n>4$, note there must exist at least two empty squares on our board to contradict **Theorem 9**. First, note that we cannot have two unoccupied squares in any line since the line intersects the border squares somewhere - and these border squares have corresponding vertices of degree $3n$-3.

Consider now if we have two unoccupied squares with one of them on the border of the board, say without loss of generality in the far-left column. It follows that any other unoccupied square besides the known border square will be adjacent to a square in the far-left column. Thus, we have two unoccupied squares adjacent to a border square, or two unoccupied squares in the same row. In either case, since our border squares have corresponding vertices of degree $3n$-3, there is a square along the border not dominated $3n$-3 times. It follows that if we're to have two or more unoccupied squares none of these squares can be on the border.

Finally, note that any interior square is adjacent to exactly 8 squares along the border. Note that we cannot have two unoccupied square adjacent to the same border square, since these squares have corresponding vertices of degree $3n$-3. Thus, it follows that since we have $4(n-1)$ border squares, each of which can be adjacent to only one unoccupied square, then we have at most $4(n-1)/8=(n-1)/2$ unoccupied squares. Thus, for all $n>2$, $\gamma_{\times 3n-3}(Q_n) \geq n^2 - \lfloor (n-1)/2 \rfloor$. □

A quick, exhaustive hand search for $\gamma_{\times 3n-3}(Q_n)$ values was done. It should be noted that for $n=3$ and $n=4$ the value for $\gamma_{\times 3n-3}(Q_n)$ matches that of the lower bound given in **Theorem 9**. However, for $n=5$, $n=6$, $n=7$, and $n=8$ the lower bound provided in **Theorem 9** doesn't provide the appropriate $\gamma_{\times 3n-3}(Q_n)$ values. The table below shows $\gamma_{\times 3n-3}(Q_n)$ values up to, and including, the standard chessboard size. These values should be easy to verify with technology.

n	$\gamma_{\times 3n-3}(Q_n)$
2	3
3	8
4	15
5	24
6	35
7	48
8	62

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