UtilitasMathematica
ISSN 0315-3681 Volume 119, 2022

ON THE (s, t)-PADOVAN AND (s, t)-LUCAS-PADOVAN QUATERNIONS AND THEIR MATRIX SEQUENCES

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Abstract

In this paer, we introduce the (s, t)-Lucas-Padovan quaternions which generalizes Padovan and Lucas- Padovan quaternions. We derive the Binet-like formula and generating function for the (s, t)-Lucas-Padovan quater- nions. Also, we obtain certain binomial sums regarding the (s, t)-Padovan and (s, t)-Lucas-Padovan quaternions. (s, t)-Padovan, (s, t)-Lucas-Padovan, quaternions, matrix sequences *r*

Keywords: (s, t)-Padovan Quaternions, (s, t)-Lucas-Padovan Quaternions, Quaternions Matrix Sequences.

1. Introduction

Many authors have studied about Padovan quaternions, [3, 4, 7, 8, 12, 13]. Quaternions are used in such fields as quantum physics and computer science, and in differential equations and group theory [5, 9, 11, 10]. As we know, many authors gave a generalization for (s, t)-Fibonacci, (s, t)-Padovan, (s, t)-Perrin, (s, t)-Pell and $(s.t)$-Pell-Lucas for Fibonacci, Padovan, Perrin, Pell and Pell-Lucas numbers, [1, 2, 3, 6, 16]. In [15, 16, 17], the authors have studied matrix sequences obtained from them. In [3, 4, 15, 6, 15, 16], the authors have studied (s, t) quaternions for (s, t)-Fibonacci, (s, t)-Padovan and (s, t)-Perrin. In particular, in [8], the author introduced a generalization of quaternion sequences for Fibonacci sequence which called Horadam quaternions.

A quaternions is described by:

$$q = a + bi + cj + dk,$$

where a, b, c are real numbers and i, j, k the orthogonal part at the base in R^3. We note that the quaternion multiplication is defined using the rules $i^2 = j^2 = k^2 = ijk = -1$. The conjugate and norm of a quaternions are defined by $q^* = a - bi - cj - dk$ and $N(q) = qq^* = a^2 + b^2 + c^2 + d^2$.

The Padovan sequence is the sequence of integers P_n defined by initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation

$$P_n = P_{n-2} + P_{n-3} \quad (1.1)$$

for all $n \geq 3$. The first few values of P_n are 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, . . .

The Padovan quaternions are defined by

$$QP_n = P_n + P_{n+1}i + P_{n+2}j + P_{n+3}k,$$

where P_n is the nth Padovan number and i, j and k are orthonormal bases.

Now, we will define a new sequence to be called *Lucas-Padovan* (abbr. *Ludovan*) sequence L_n in the same way that we define Lucas sequence for Fibonacci sequence. The Ludovan sequence is defined by the following rules; let $L_0 = 1$ and, for $n \geq 1$,

$$L_n = P_{n-1} + P_{n+1},$$

where P_n is the nth Padovan number. The first few values of Ludovan sequence L_n, for $n \geq 0$, are

1, 2, 3, 3, 5, 6, 8, 11, 14, 19, 25, 33, 44, 58, 77, 102, 135, . . .

The Ludovan quaternions are defined by, for $n \geq 0$,

$$QL_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k,$$

where Ln is the nth Ludovan number and i, j and k are orthonormal bases.

2. (*s, t*)-PADOVAN AND (*s, t*)-LUCAS-PADOVAN QUATERNIONS

In this section, we give definitions of the (s, t)-Padovan and (s, t)-Ludovan. We also investigate their properties. In [4], the authors defined the (s, t)-Padovan sequence as follows: For any real numbers s, t and $n \geq 2$, let $27t^2 - 4s^3 \neq 0$, $s > 0$ and $t \neq 0$. Then the (s, t)-Padovan sequence $\{P_n(s, t)\}$ is defined by

$$P_{n+3}(s, t) = sP_{n+1}(s, t) + tP_n(s, t),$$

where $P_0(s, t) = 0$, $P_1(s, t) = 1$, $P_2(s, t) = 0$.

Using the definition of the (s, t)-Padovan sequence, we define the (s, t)-Ludovan sequence.

Definition 2.1. *The* (s, t)*-Ludovan sequence* $\{L_n(s, t)\}$ *is defined by*

$\geq$

$$L_n(s, t) = P_{n-1}(s, t) + P_{n+1}(s, t),$$

where $L_0(s, t) = 1$, $L_1(s, t) = 0$.

In this paper, to simplify notation, we take $P_n(s, t) = p_n$ and $L_n(s, t) = l_n$. That is, $p_{n+3} = sp_{n+1} + tp_n$ and $l_n = p_{n-1} + p_{n+1}$. The first few terms of p_n are

$$\{p_n\}_{n\geq 0} : 0,\ 1,\ 0,\ s,\ t,\ s^2,\ 2st,\ s^3 + t^2,\ 3s^2t,\ s^4 + 3st^2,\ \ldots,$$

and, we define $p-2 = 1/t$ and $p-1 = 0$. Also, the first few terms of l_n are

$$\{l_n\}_{n\geq 0} : 1,\ 0,\ s+1,\ t,\ s+s^2,\ 2st+t,\ s^2+s^3+t^2,\ 3s^2t+2st,\ s^3+s^4+3st^2+t^2,\ \ldots,$$

and we define $l-2 = -s/t^2$ and $l-1 = 1/t$.

Lemma 2.1. *Let* $\{l_n\}$ *be the* (s, t)*-Ludovan sequence. Then, for any real numbers* s, t *and* $n \geq 0$ *we have* $l_{n+3} = sl_{n+1} + tl_n$.

Proof. From Definition 2.1, since $l_0 = 1$, $l_1 = 0$, $l_2 = s + 1$, and $l_3 = t$, we have $l_3 = sl_1 + tl_0$. By induction on n, since $l_n = p_{n-1} + p_{n+1}$ and $p_n = p_{n-2} + p_{n-3}$, we have

$$sl_{n+1} + tl_n = s(p_n + p_{n+2}) + t(p_{n-1} + p_{n+1})$$

$$= (sp_{n+2} + tp_{n+1}) + (sp_n + tp_{n-1})$$

$$= pn+4 + pn+2 = ln+3.$$

Therefore, the proof is completed. Q

Now, we define the (s, t)-Padovan quaternions and (s, t)-Ludovan quaternions.

Definition 2.2. *The* (s, t)*-Padovan quaternion sequence* Qp_n *and* (s, t)*-Ludovan quaternion sequence* Ql_n *are defined respectively by*

$$Qp_n = p_n + p_{n+1}i + p_{n+2}j + p_{n+3}k,$$

$$Ql_n = l_n + l_{n+1}i + l_{n+2}j + l_{n+3}k,$$

where p_n *and* l_n *are the nth* (s, t)*-Padovan number and* (s, t)*-Ludovan number, respectively.*

From Definition 2.2, we can get $Qp_{n+3} = sQp_{n+1} + tQp_n$ and the following result.

Lemma2.2. *Let* $\{Ql_n\}$ *be the* (s, t)*-Ludovan quaternion sequence. Then,* $n \geq 0$*, we have*

$Ql_{n+1} = Qp_{n-1} + Qp_{n+1}$.

$Ql_{n+3} = sQl_{n+1} + tQl_n$.

Proof. (i)From Definition 2.1, this result is easy to get. (ii)From Lemma 2.1, we have

$$Ql_{n+3} = l_{n+3} + l_{n+4}i + l_{n+5}j + l_{n+6}k$$

$$= (sl_{n+1} + tl_n) + (sl_{n+2} + tl_{n+1})i + (sl_{n+3} + tl_{n+2})j + (sl_{n+4} + tl_{n+3})k$$

$$= sQl_{n+1} + tQl_n.$$

Therefore, the proof is completed.

In [4], we have the following two theorems which among them, the first is about the Binet-like formula for the nth (s, t)-Padovan quaternion, and the second is about the generating function for the (s, t)-Padovan quaternion.

Theorem 2.1. [4] *The Binet-like formulas for the nth* (s, t)*-Padovan quaternion is,* $n \geq 0$,

$$Qp_n = a\hat{\alpha}\alpha^n + b\hat{\beta}\beta^n + c\hat{\gamma}\gamma^n,$$

where α, β *and* γ *are the roots of the characteristic equation*

$$x^3 - sx - t = 0 \ (2.1)$$

associted with (1.1)

$$a = \frac{(\beta-1)(\gamma-1)}{(\alpha-\beta)(\alpha-\gamma)}, \ b = \frac{(\alpha-1)(\gamma-1)}{(\beta-\alpha)(\beta-\gamma)}, \ c = \frac{(\alpha-1)(\beta-1)}{(\gamma-\alpha)(\gamma-\beta)},$$

and

$$\hat{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k, \ \hat{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k, \ \hat{\gamma} = 1 + \gamma i + \gamma^2 j + \gamma^3 k.$$

Theorem 2.2. [4] The generating function for the (s, t)-Padovan quaternion is

$$g_p(x) = \frac{i + sk + (1 + sj + tk)x + tjx^2}{1 - sx^2 - tx^3}.$$

From Lemma 2.2 and Theorem 2.1, we have the Binet-like formula for the (s, t)-Ludovan quaternion.

Theorem 2.3. (Binet-like formula) The Binet-like formula for the nth (s, t)-Ludovan quaternion is

$$Ql_n = a\hat{\alpha}\alpha^n(\frac{1}{\alpha} + \alpha) + b\hat{\beta}\beta^n(\frac{1}{\beta} + \beta) + c\hat{\gamma}\gamma^n(\frac{1}{\gamma} + \gamma).$$

From the Binet-like formula for the (s, t)-Ludovan quaternion, we have the following theorem.

Theorem 2.4. For positive integer m,

(i) $\sum_{n=0}^{m} \binom{m}{n} s^n t^{m-n} Ql_n = Ql_{3m}$.

(ii) $\sum_{u=0}^{m} \binom{m}{u} s^{m-u} t^u Ql_{n-u} = Ql_{n+2m}$.

Proof. From (2.1), we know that $\alpha^3 - s\alpha - t = 0$, $\beta^3 - s\beta - t = 0$ and $\gamma^3 - s\gamma - t = 0$.

(i)Since $s\alpha + t = \alpha^3$, $s\beta + t = \beta^3$ and $s\gamma + t = \gamma^3$, we have

$$\begin{aligned}
\sum_{n=0}^{m}\binom{m}{n}s^{n}t^{m-n}Ql_{n} &= \sum_{n=0}^{m}\binom{m}{n}s^{n}t^{m-n}\left(a\hat{\alpha}\alpha^{n}(\frac{1}{\alpha}+\alpha)+b\hat{\beta}\beta^{n}(\frac{1}{\beta}+\beta)+c\hat{\gamma}\gamma^{n}(\frac{1}{\gamma}+\gamma)\right)\\
&= \sum_{n=0}^{m}\binom{m}{n}\left(a\hat{\alpha}(s\alpha)^{n}t^{m-n}(\frac{1}{\alpha}+\alpha)+b\hat{\beta}(s\beta)^{n}t^{m-n}(\frac{1}{\beta}+\beta)+c\hat{\gamma}(s\gamma)^{n}t^{m-n}(\frac{1}{\gamma}+\gamma)\right)\\
&= a\hat{\alpha}(s\alpha+t)^{m}(\frac{1}{\alpha}+\alpha)+b\hat{\beta}(s\beta+t)^{m}(\frac{1}{\beta}+\beta)+c\hat{\gamma}(s\gamma+t)^{m}(\frac{1}{\gamma}+\gamma)\\
&= a\hat{\alpha}\alpha^{3m}(\frac{1}{\alpha}+\alpha)+b\hat{\beta}\beta^{3m}(\frac{1}{\beta}+\beta)+c\hat{\gamma}\gamma^{3m}(\frac{1}{\gamma}+\gamma)\\
&= Ql_{3m}.
\end{aligned}$$

In the same way as proof of (i), we can get the result $\sum_{u=0}^{m}\binom{m}{u}s^{m-u}t^{u}Ql_{n-u}=Ql_{n+2m}$.
Therefore, the proof is completed

Using the generating function, we can obtain binary coefficient representations for the nth (s, t)-Padovan quaternion.

Theorem 2.5. *For the nth* (*s*, *t*)*-Padovan quaternion* Qp_n,

$$\begin{aligned}
Qp_{n} = \sum_{m=0}^{n}\binom{m}{3m-n+1}s^{3m-n+1}t^{n-2m-1}+i\sum_{m=0}^{n}\binom{m}{3m-n}s^{3m-n}t^{n-2m}\\
+j\sum_{m=0}^{n}\binom{m+1}{3m-n+2}s^{3m-n+2}t^{n-2m-1}+k\sum_{m=0}^{n}\binom{m+1}{3m-n+1}s^{3m-n+1}t^{n-2m}.
\end{aligned}$$

Proof. In this proof, let's put (i + sk) as A, (1 + sj + tk) as B and (tj) as C to proceed concisely with the proof. Then we have the generating function of the Padovan quaternions as follows:

$$\begin{aligned}
g_{p}(x) = \sum_{u\geq 0}Qp_{n}x^{n} &= (A+Bx+Cx^{2})\frac{1}{1-sx^{2}-tx^{3}}\\
&= (A+Bx+Cx^{2})\sum_{u\geq 0}(sx^{2}+tx^{3})^{u}\\
&= A\sum_{u\geq 0}(s+tx)^{u}x^{2u}+B\sum_{u\geq 0}(s+tx)^{u}x^{2u+1}+C\sum_{u\geq 0}(s+tx)^{u}x^{2u+2}\\
&= A\sum_{u\geq 0}\sum_{v=0}^{u}\binom{u}{v}s^{v}t^{u-v}x^{3u-v}+B\sum_{u\geq 0}\sum_{v=0}^{u}\binom{u}{v}s^{v}t^{u-v}x^{3u-v+1}\\
&\quad +C\sum_{u\geq 0}\sum_{v=0}^{u}\binom{u}{v}s^{u}t^{u-v}x^{3u-v+2}. \qquad (2.2)
\end{aligned}$$

In the first formula of (2.2), if we set n = 3u − v then we can obtain that

$$A\sum_{u\geq 0}\sum_{v=0}^{u}\binom{u}{v}s^{v}t^{u-v}x^{3u-v}=A\sum_{n\geq 0}\sum_{m=0}^{n}\binom{m}{3m-n}s^{3m-n}t^{n-2m}x^{n}.$$

Using the same process, we can obtain the following equations

$$\begin{aligned}
g_p(x) = \sum_{u\geq 0} Qp_n x^n &= A\sum_{n\geq 0}\sum_{m=0}^{n}\binom{m}{3m-n} s^{3m-n}t^{n-2m}x^n + B\sum_{n\geq 0}\sum_{m=0}^{n}\binom{m}{3m-n+1} s^{3m-n+1}t^{n-2m-1}x^n \\
&\quad + C\sum_{n\geq 0}\sum_{m=0}^{n}\binom{m}{3m-n+2} s^{3m-n+2}t^{n-2m-2}x^n. \\
&= (i+sk)\sum_{n\geq 0}\sum_{m=0}^{n}\binom{m}{3m-n} s^{3m-n}t^{n-2m}x^n \\
&\quad + (1+sj+tk)\sum_{n\geq 0}\sum_{m=0}^{n}\binom{m}{3m-n+1} s^{3m-n+1}t^{n-2m-1}x^n \\
&\quad + (tj)\sum_{n\geq 0}\sum_{m=0}^{n}\binom{m}{3m-n+2} s^{3m-n+2}t^{n-2m-2}x^n. \\
&= \sum_{n\geq 0}\left\{\sum_{m=0}^{n}\binom{m}{3m-n+1} s^{3m-n+1}t^{n-2m-1} + \sum_{m=0}^{n}\binom{m}{3m-n} s^{3m-n}t^{n-2m} i\right. \\
&\quad \left. + \sum_{m=0}^{n}\binom{m+1}{3m-n+2} s^{3m-n+2}t^{n-2m-1} j + \sum_{m=0}^{n}\binom{m+1}{3m-n+1} s^{3m-n+1}t^{n-2m} k\right\} x^n.
\end{aligned}$$

Therefore, the proof is completed.

Now, we consider the generating function for the (s, t)-Ludovan quaternion.

Theorem 2.6. The generating function for the (s, t)-Ludovan quaternion is

$$g_l(x) = \frac{(1+(s+1)j+tk)+((s+1)i+tj+(s+s^2)k)x+(1+ti+(st+t)k)x^2}{1-sx^2-tx^3}.$$

Proof. Assume that the function

$$g_l(x) = \sum_{n\geq 0} Ql_n x^n = Ql_0 + Ql_1 x + Ql_2 x^2 + \ldots + Ql_n x^n + \ldots$$

be generating function of the (s, t)-Ludovan quaternions. Multiply both of side of the equality by term sx2 such as

$$sx^2 g_l(x) = sQl_0 x^2 + sQl_1 x^3 + sQl_2 x^4 + \ldots + sQl_n x^{n+2} + \ldots$$

and multiply by term tx3 such as

$$tx^3 g_l(x) = tQl_0 x^3 + tQl_1 x^4 + tQl_2 x^5 + \ldots + tQl_n x^{n+3} + \ldots$$

Then, we write

$(1 - sx^2 - tx^3)g_l(x) = Ql_0 + Ql_1 x + (Ql_2 - sQl_0)x^2 + (Ql_3 - sQl_1 - tQl_0)x^3 + \ldots + (Ql_n - sQl_{n-2} - tQl_{n-3})x^n + \ldots$

From (ii) of Lemma 2.2, we obtain that

$(1 - sx^2 - tx^3)g_l(x) = (1 + (s + 1)j + tk) + ((s + 1)i + tj + (s + s^2)k)x + (1 + ti + (st + t)k)x^2.$

Therefore, the proof is completed.

Using the generating function, we can obtain binary coefficient representations for the *n*th (*s*, *t*)-Ludovan quater- nion.

Theorem 2.7. *For the nth* (*s*, *t*)*-Ludovan quaternion* Ql_n,

$$\begin{aligned} Ql_n = \sum_{m=0}^{n} \left(\binom{m}{3m-n+2} s^2 + \binom{m}{3m-n} t^2 \right) s^{3m-n} t^{n-2m-2} \\ + i \sum_{m=0}^{n} \left(\binom{m}{3m-n+1} + \binom{m+1}{3m-n+2} s \right) s^{3m-n+1} t^{n-2m-1} \\ + j \sum_{m=0}^{n} \left(\binom{m}{3m-n} + \binom{m+1}{3m-n+1} s \right) s^{3m-n} t^{n-2m} \\ + k \sum_{m=0}^{n} \left(\binom{m}{3m-n} t^2 + \binom{m+1}{3m-n+2} (s^2+s^3) \right) s^{3m-n} t^{n-2m-1}. \end{aligned}$$

Proof. In this proof, let’s put (1 + (s + 1)j + tk) as A, ((s + 1)i + tj + (s + s2)k) as B and (1 + ti + (st + t)k) as C to proceed concisely with the proof. Then, in the same way as proof of Theorem 2.5, we have the generating function of the (s, t)-Ludovan quaternions as follows:

$$\begin{aligned} g_l(x) = \sum_{u \geq 0} Ql_n x^n = (A + Bx + Cx^2) \frac{1}{1 - sx^2 - tx^3} \\ = A \sum_{n \geq 0} \sum_{m=0}^{n} \binom{m}{3m-n} s^{3m-n} t^{n-2m} x^n + B \sum_{n \geq 0} \sum_{m=0}^{n} \binom{m}{3m-n+1} s^{3m-n+1} t^{n-2m-1} x^n \\ + C \sum_{n \geq 0} \sum_{m=0}^{n} \binom{m}{3m-n+2} s^{3m-n+2} t^{n-2m-2} x^n. \end{aligned}$$

Now using $A = (1 + (s + 1)j + tk)$, $B = ((s + 1)i + tj + (s + s^2)k)$, $C = (1 + ti + (st + t)k)$ and the properties of binomial coefficients, we can get the conclution we want.

Therefore, the proof is completed

Since $Qp_n = p_n + p_{n+1}i + p_{n+2}j + p_{n+3}k$ and $Ql_n = l_n + l_{n+1}i + l_{n+2}j + l_{n+3}k$, we can get many identities for the binomial coefficients.

Corollary 2.1. *For the nth* (*s*, *t*)*-Padovan number* p_n *and the nth* (*s*, *t*)*-Ludovan number* l_n*, we have the following identities.*

(i) $p_n = \sum_{m=0}^{n} \binom{m}{3m-n+1} s^{3m-n+1} t^{n-2m-1}$.

(ii) $l_n = \sum_{m=0}^{n} \binom{m}{3m-n+2} s^{3m-n+2} t^{n-2m-2} + \sum_{m=0}^{n} \binom{m}{3m-n} s^{3m-n} t^{n-2m}$.

(iii) $\sum_{m=0}^{n+1} \binom{m}{3m-n-1} s^{3m-n-1} t^{n-2m+1} = \sum_{m=0}^{n} \binom{m+1}{3m-n+2} s^{3m-n+2} t^{n-2m-1}$.

(iv) $\sum_{m=0}^{n} \binom{m}{3m-n} s^{3m-n+1} t^{n-2m} + \sum_{m=0}^{n} \binom{m}{3m-n+1} s^{3m-n+1} t^{n-2m} = \sum_{m=0}^{n} \binom{m+1}{3m-n+1} s^{3m-n+1} t^{n-2m}$.

(v) $\sum_{m=0}^{n+1} \binom{m}{3m-n+1} s^{3m-n-1} t^{n-2m+1} + \sum_{m=0}^{n+1} \binom{m+1}{3m-n} s^{3m-n} t^{n-2m+1} = \sum_{m=0}^{n} \binom{m}{3m-n} s^{3m-n} t^{n-2m+1}$
$+ \sum_{m=0}^{n} \binom{m+1}{3m-n+2} s^{3m-n+2} t^{n-2m-1} + \sum_{m=0}^{n} \binom{m+1}{3m-n+2} s^{3m-n+3} t^{n-2m-1}$.

(vi) $\sum_{m=0}^{n} \binom{m}{3m-n+2} s^{3m-n+2} t^{n-2m-1} + \sum_{m=0}^{n} \binom{m}{3m-n} s^{3m-n} t^{n-2m+1} + \sum_{m=0}^{n} \binom{m}{3m-n+1} s^{3m-n+2} t^{n-2m-1}$
$+ \sum_{m=0}^{n} \binom{m+1}{3m-n+2} s^{3m-n+3} t^{n-2m-1}$
$= \sum_{m=0}^{n} \binom{m}{3m-n} s^{3m-n} t^{n-2m+1} + \sum_{m=0}^{n} \binom{m+1}{3m-n+2} s^{3m-n+2} t^{n-2m-1} + \sum_{m=0}^{n} \binom{m+1}{3m-n+2} s^{3m-n+3} t^{n-2m-1}$.

Proof. Since $Qp_n = p_n + p_{n+1}i + p_{n+2}j + p_{n+3}k$ and $Ql_n = l_n + l_{n+1}i + l_{n+2}j + l_{n+3}k$, from Theorem 2.5 and Theorem 2.7, we cna get (i) and (ii). Since $p_{n+3} = p_{(n+1)+2}$ and $l_{n+3} = l_{(n+1)+2}$, we can get (iii) and (v). Since $p_{n+3} = sp_{n+1} + tp_n$ and $l_{n+3} = sl_{n+1} + tl_n$, we can get (iv) and (vi).

Therefore, the proof is complete.

From Corollary 2.1, if we let $s = t = 1$, then, for the nth Padovan number P_n and Ludovan number L_n, we have the following identities for binomial coefficients.

(i) $P_n = \sum_{m=0}^{n} \binom{m}{3m-n+1}$.

(ii) $L_n = \sum_{m=0}^{n} \binom{m}{3m-n+2} + \sum_{m=0}^{n} \binom{m}{3m-n}$.

(iii) $\sum_{m=0}^{n+1} \binom{m}{3m-n-1} = \sum_{m=0}^{n} \binom{m+1}{3m-n+2}$.

(iv) $\sum_{m=0}^{n} \binom{m}{3m-n} + \sum_{m=0}^{n} \binom{m}{3m-n+1} = \sum_{m=0}^{n} \binom{m+1}{3m-n+1}$.

(v) $\sum_{m=0}^{n+1} \binom{m}{3m-n+1} + \sum_{m=0}^{n+1} \binom{m+1}{3m-n} = \sum_{m=0}^{n} \binom{m}{3m-n} + \sum_{m=0}^{n} \binom{m+1}{3m-n+2} + \sum_{m=0}^{n} \binom{m+1}{3m-n+2}$.

(vi) $\sum_{m=0}^{n} \binom{m}{3m-n+2} + \sum_{m=0}^{n} \binom{m}{3m-n} + \sum_{m=0}^{n} \binom{m}{3m-n+1} + \sum_{m=0}^{n} \binom{m+1}{3m-n+2}$
$= \sum_{m=0}^{n} \binom{m}{3m-n} + \sum_{m=0}^{n} \binom{m+1}{3m-n+2} + \sum_{m=0}^{n} \binom{m+1}{3m-n+2}$.

3. (*s*, *t*)-PADOVAN AND (*s*, *t*)-LUDOVAN QUATERNIONS MATRIX SEQUENCES

In [14], the authors gave some matrix representations associated with the Horadam quaternions. In [13], the author gave the matrix representation of Padovan quaternions. And, in [15], the authors introduced the (s, t)- Padovan quaternions matrix sequence using the expression in [13].

In this section, we give definition of the (s, t)-Ludovan quaternions matrix sequence.

In [15], the authors gave the definition for the (s, t)-Padovan quaternions matrix sequence, $P_n(s, t)$ as follows: For $n \geq 0$, $s > 0$, $t \neq 0$ and $27t^2 - 4s^3 \neq 0$,

$P_{n+3}(s, t) = sP_{n+1}(s, t) + tP_n(s, t)$, *(3.1)*
Where

$$\mathcal{P}_0(s,t) = \begin{bmatrix} Qp_2 & Qp_1 & Qp_0 \\ Qp_1 & Qp_0 & Qp_{-1} \\ Qp_0 & Qp_{-1} & Qp_{-2} \end{bmatrix}, \; \mathcal{P}_1(s,t) = \begin{bmatrix} 0 & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathcal{P}_0(s,t), \; \mathcal{P}_2(s,t) = \begin{bmatrix} s & t & 0 \\ 0 & s & t \\ 1 & 0 & 0 \end{bmatrix} \mathcal{P}_0(s,t).$$

Then, by induction on n, we can get, for $n \geq 0$

$$\mathcal{P}_n(s,t) = \begin{bmatrix} 0 & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \mathcal{P}_0(s,t) = \begin{bmatrix} Qp_{n+2} & Qp_{n+1} & Qp_n \\ Qp_{n+1} & Qp_n & Qp_{n-1} \\ Qp_n & Qp_{n-1} & Qp_{n-2} \end{bmatrix}.$$

In this paper, we let

$$\mathcal{R} = \begin{bmatrix} 0 & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then, by induction on n,

$$\mathcal{R}^n = \begin{bmatrix} 0 & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} p_{n+1} & p_{n+2} & tp_n \\ p_n & p_{n+1} & tp_{n-1} \\ p_{n-1} & p_n & tp_{n-2} \end{bmatrix} \tag{3.2}$$

So as it follows:

$$P_n(s, t) = R\, P_0(s, t).$$

Now, let us define the (s, t)-Ludovan quaternions matrix sequence by using (s, t)-Padovan quaternions matrix sequence.

Definition 3.1. For n ≥ 1, the (s, t)-Ludovan quaternions matrix sequence Ln(s, t) is defined by

$$L_n(s, t) = P_{n-1}(s, t) + P_{n+1}(s, t),$$

and L0(s, t) is defined

$$L_0(s, t) = VP_0(s, t),$$

where $P_n(s, t)$ is the nth (s, t)-Padovan quaternions matrix sequence and

$$\mathcal{V} = \begin{bmatrix} 0 & s+1 & t \\ 1 & 0 & 1 \\ \frac{1}{t} & 1 & -\frac{s}{t} \end{bmatrix}.$$

From definition 3.1, we have

$$\begin{aligned} \mathcal{L}_n(s,t) &= \mathcal{P}_{n-1}(s,t) + \mathcal{P}_{n+1}(s,t) \\ &= \begin{bmatrix} Qp_{n+1} & Qp_n & Qp_{n-1} \\ Qp_n & Qp_{n-1} & Qp_{n-2} \\ Qp_{n-1} & Qp_{n-2} & Qp_{n-3} \end{bmatrix} + \begin{bmatrix} Qp_{n+3} & Qp_{n+2} & Qp_{n+1} \\ Qp_{n+2} & Qp_{n+1} & Qp_n \\ Qp_{n+1} & Qp_n & Qp_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} Ql_{n+2} & Ql_{n+1} & Ql_n \\ Ql_{n+1} & Ql_n & Ql_{n-1} \\ Ql_n & Ql_{n-1} & Ql_{n-2} \end{bmatrix}. \end{aligned}$$

In this paper, to simplify notation, we take $P_n(s, t) = P_n$ and $L_n(s, t) = L_n$. That is,

$$\begin{aligned} \mathcal{L}_0 &= \mathcal{V}\mathcal{P}_0 \\ &= \begin{bmatrix} 0 & s+1 & t \\ 1 & 0 & 1 \\ \frac{1}{t} & 1 & -\frac{s}{t} \end{bmatrix} \begin{bmatrix} si+tj+s^2k & 1+sj+tk & i+sk \\ 1+sj+tk & i+sk & j \\ i+sk & j & \frac{1}{t}+k \end{bmatrix} \\ &= \begin{bmatrix} Ql_2 & Ql_1 & Ql_0 \\ Ql_1 & Ql_0 & Ql_{-1} \\ Ql_0 & Ql_{-1} & Ql_{-2} \end{bmatrix}. \end{aligned}$$

and $P_n = R^n P_0$. It is also easy to prove that, for $n \geq 0$,

$$\mathcal{R}^n\mathcal{V}=\begin{bmatrix} p_{n+1} & p_{n+2} & tp_n \\ p_n & p_{n+1} & tp_{n-1} \\ p_{n-1} & p_n & tp_{n-2}\end{bmatrix}\begin{bmatrix} 0 & s+1 & t \\ 1 & 0 & 1 \\ \frac{1}{t} & 1 & -\frac{s}{t}\end{bmatrix}=\begin{bmatrix} l_{n+1} & l_{n+2} & tl_n \\ l_n & l_{n+1} & tl_{n-1} \\ l_{n-1} & l_n & tl_{n-2}\end{bmatrix}=\mathcal{V}\mathcal{R}^n. \tag{3.3}$$

Lemma 3.1. Let $\{L_n\}$ be the (s, t)-Ludovan quaternions matrix sequence. Then, for $n \geq 0$,

$$L_{n+3} = sL_{n+1} + tL_n.$$

Proof. It can be proved in the same way as the proof of Lemma 2.1 by using (3.1). Q

Theorem 3.1. *For the nth* (*s, t*)*-Ludovan number* l_n*, the nth* (*s, t*)*-Ludovan quaternions matrix* L_n *and* $m, n \geq 0$*,*

(i) $L_n = R^n L_0$.

(ii) $L_n = VP_n$.

(iii) $L_{m+n} = R^m L_n = R^n L_m$.

Proof. (i)Since $l_n = p_{n-1} + p_{n+1}$, we have, from (3.2),

$$L_n = P_{n-1} + P_{n+1}$$

$$= (R^{n-1} + R^{n+1})P_0$$

$$=\left(\begin{bmatrix} p_n & p_{n+1} & tp_{n-1} \\ p_{n-1} & p_n & tp_{n-2} \\ p_{n-2} & p_{n-1} & tp_{n-3}\end{bmatrix}+\begin{bmatrix} p_{n+2} & p_{n+3} & tp_{n+1} \\ p_{n+1} & p_{n+2} & tp_n \\ p_n & p_{n+1} & tp_{n-1}\end{bmatrix}\right)\mathcal{P}_0$$

$$=\begin{bmatrix} l_{n+1} & l_{n+2} & tl_n \\ l_n & l_{n+1} & tl_{n-1} \\ l_{n-1} & l_n & tl_{n-2}\end{bmatrix}\mathcal{P}_0$$

$$=\mathcal{R}^n\mathcal{V}\mathcal{P}_0.$$

Since $L_0 = VP_0$, we have $L_n = R^n L_0$.

(ii) Since (3.3) is established, we have $L_n = R^n L_0 = R^n VP_0 = VR^n P_0 = VP_n$.

(iii) Since $P_{m+n} = R^{m+n}P_0$, $P_{m+n} = R^m P_n = R^n P_m$. From (ii), since (3.3) is established, we have

$$L_{m+n} = VP_{m+n}$$

$$= VR^{m+n}P_0$$

$$= R^m VR^n P_0$$

$$= R^m L_n = R^n L_m.$$

Therefore, the proof is completed.

Corollary 3.1. *For nonnegative integers m, n,*

$$Qlm+n = pm+1Qln + pm+2Qln-1 + tpmQln-2$$

$$= pn+1Qlm + pn+2Qlm-1 + tpnQlm-2.$$

Proof. From (iii) fo Theorem 3.1, the proof is completed.

In [15], the authors gave the Binet-like formula and the generating function for the *n*th (*s, t*)-Padovan quaternions matrix sequence as follows: for $n \geq 0$,

$$\mathcal{P}_n = \bar{a}\alpha^n + \bar{b}\beta^n + \bar{c}\gamma^n, \tag{3.4}$$

where α, β and γ are roots of the equation (2.1),

$$\bar{a} = \frac{\mathcal{P}_2 + \mathcal{P}_1\alpha + \mathcal{P}_0\beta\gamma}{(\alpha-\beta)(\alpha-\gamma)},\ \bar{b} = \frac{\mathcal{P}_2 + \mathcal{P}_1\beta + \mathcal{P}_0\alpha\gamma}{(\beta-\alpha)(\beta-\gamma)},\ \bar{c} = \frac{\mathcal{P}_2 + \mathcal{P}_1\gamma + \mathcal{P}_0\alpha\beta}{(\gamma-\alpha)(\gamma-\beta)}.$$

$$g_{\mathcal{P}}(x) = \frac{\mathcal{P}_0 + \mathcal{P}_1 x + (\mathcal{P}_2 - s\mathcal{P}_0)x^2}{1 - sx^2 - tx^3}. \tag{3.5}$$

Using the equation (3.4), we can obtain the Binet-like formula for the (s, t)-Ludovan quaternions matrix sequence.

Theorem 3.2. The Binet-like formula for the nth (s, t)-Ludovan quaternions matrix sequence is

$$\mathcal{L}_n = \bar{a}\alpha^n\left(\frac{1}{\alpha} + \alpha\right) + \bar{b}\beta^n\left(\frac{1}{\beta} + \beta\right) + \bar{c}\gamma^n\left(\frac{1}{\gamma} + \gamma\right).$$

From the Binet-like fomulas for the (s, t)-Padovan quaternions matrix sequence and the (s, t)-Ludovan quaternions matrix sequence, we have the following thoerem.

Theorem 3.3. For pasitive integer m,

(i) $\sum_{n=0}^{m} \binom{m}{n} s^n t^{m-n} \mathcal{P}_n = \mathcal{P}_{3m}$.
(ii) $\sum_{u=0}^{m} \binom{m}{u} s^{m-u} t^u \mathcal{P}_{n-u} = \mathcal{P}_{n+2m}$.
(iii) $\sum_{n=0}^{m} \binom{m}{n} s^n t^{m-n} \mathcal{L}_n = \mathcal{L}_{3m}$.
(iv) $\sum_{u=0}^{m} \binom{m}{u} s^{m-u} t^u \mathcal{L}_{n-u} = \mathcal{L}_{n+2m}$.

Proof. In the same way as proof of Theorem 2.4, we can get the results.

Using the equation (3.5) for the generating function, we can obtain binary coefficient representations for the *n*th (*s, t*)-Padovan quaternions matrix sequence P_n.

Theorem 3.4. *For the nth* (*s, t*)*-Padovan quaternions matrix sequence* P_n*,*

$$\mathcal{P}_n = \sum_{m=0}^{n} s^{3m-n} t^{n-2m-2} \left\{ \binom{m}{3m-n} t^2 I + \binom{m}{3m-n+1} st\mathcal{R} + \binom{m}{3m-n+2} s^2(\mathcal{R}^2 - sI) \right\} \mathcal{P}_0,$$

where I is the identity matrix of order 3.

Proof. From (3.5), in the same way as proof of Theorem 2.5, we have

$$g_{\mathcal{P}}(x) = \sum_{n\geq 0}\sum_{m=0}^{n}\binom{m}{3m-n}s^{3m-n}t^{n-2m}x^n\mathcal{P}_0 + \sum_{n\geq 0}\sum_{m=0}^{n}\binom{m}{3m-n+1}s^{3m-n+1}t^{n-2m-1}x^n\mathcal{P}_1$$
$$+\sum_{n\geq 0}\sum_{m=0}^{n}\binom{m}{3m-n+2}s^{3m-n+2}t^{n-2m-2}x^n(\mathcal{P}_2 - s\mathcal{P}_0).$$

Since $P_1 = RP_0$ and $P_2 = R^2P_0$, we have

$$g_{\mathcal{P}}(x) = \sum_{n\geq 0}\mathcal{P}_n x^n$$
$$= \sum_{n\geq 0}\sum_{m=0}^{n}s^{3m-n}t^{n-2m-2}\left\{\binom{m}{3m-n}t^2I + \binom{m}{3m-n+1}st\mathcal{R} + \binom{m}{3m-n+2}s^2(\mathcal{R}^2 - sI)\right\}\mathcal{P}_0x^n.$$

Theorefore, the proof is completed.
Since $P_n = R^nP_0$ and

$$\mathcal{R} = \begin{bmatrix} 0 & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ \mathcal{R}^2 = \begin{bmatrix} s & t & 0 \\ 0 & s & t \\ 1 & 0 & 0 \end{bmatrix},$$

we have

$$\mathcal{P}_n = \begin{bmatrix} Qp_{n+2} & Qp_{n+1} & Qp_n \\ Qp_{n+1} & Qp_n & Qp_{n-1} \\ Qp_n & Qp_{n-1} & Qp_{n-2} \end{bmatrix}$$
$$= \begin{bmatrix} p_{n+1} & p_{n+2} & tp_n \\ p_n & p_{n+1} & tp_{n-1} \\ p_{n-1} & p_n & tp_{n-2} \end{bmatrix}\mathcal{P}_0$$
$$= \sum_{m=0}^{n}s^{3m-n}t^{n-2m-2}\left\{\binom{m}{3m-n}t^2I + \binom{m}{3m-n+1}st\mathcal{R} + \binom{m}{3m-n+2}s^2(\mathcal{R}^2 - sI)\right\}\mathcal{P}_0$$
$$= \sum_{m=0}^{n}s^{3m-n}t^{n-2m-2}\left\{\begin{bmatrix} \binom{m}{3m-n}t^2 & \binom{m+1}{3m-n+2}s^2t & \binom{m}{3m-n+1}st^2 \\ \binom{m}{3m-n+1}st & \binom{m}{3m-n}t^2 & \binom{m}{3m-n+2}s^2t \\ \binom{m}{3m-n+2}s^2 & \binom{m}{3m-n+1}st & \binom{m}{3m-n}t^2 - \binom{m}{3m-n+2}s^3 \end{bmatrix}\right\}\mathcal{P}_0. \tag{3.6}$$

Corollary 3.2. *For the nth* (*s*, *t*)*-Padovan quaternion Qp*$_n$*,*

$$Qp_n = \sum_{m=0}^{n}\left\{\binom{m}{3m-n+1}s + i\binom{m}{3m-n}t + j\binom{m+1}{3m-n+2}s^2 + k\binom{m+1}{3m-n+1}st\right\}s^{3m-n}t^{n-2m-1}$$
$$= \sum_{m=0}^{n+1}\left\{\binom{m}{3m-n+1}s^2 + i\binom{m}{3m-n}st + j\binom{m}{3m-n-1}t^2 + k\binom{m+1}{3m-n+1}s^2t\right\}s^{3m-n-1}t^{n-2m-1}.$$

Proof. Since Qp_{n-1} is the (2, 3)-entry of the matrix P_n

$$\mathcal{P}_0 = \begin{bmatrix} si+tj+s^2k & 1+sj+tk & i+sk \\ 1+sj+tk & i+sk & j \\ i+sk & j & \frac{1}{t}+k \end{bmatrix},$$

we can get, from (3.6),

$$\begin{aligned} Qp_{n-1} &= \sum_{m=0}^{n} s^{3m-n}t^{n-2m-2}\left\{\binom{m}{3m-n+1}st(i+sk)+\binom{m}{3m-n}t^2(j)+\binom{m}{3m-n+2}s^2t\left(\frac{1}{t}+k\right)\right\} \\ &= \sum_{m=0}^{n} s^{3m-n}t^{n-2m-2}\left\{\binom{m}{3m-n+2}s^2+i\binom{m}{3m-n+1}st+j\binom{m}{3m-n}t^2+k\binom{m+1}{3m-n+2}s^2t\right\}. \end{aligned}$$

Since $Qp_{(n+1)-1} = Qp_n$, from Theorem 2.5, we can get the conclution.

Now, we can obtain the generation function for the *n*th (*s*, *t*)-Ludovan quaternions matrix *L*
sequence $_n$ by the way we obtained the generation function for the (*s*, *t*)-Padovan quaternions matrix sequence as in the following theorem.

Theorem 3.5. *The generating function the nth* (*s*, *t*)*-Ludovan quaternions matrix sequence* L_n *is*

$$g_{\mathcal{L}}(x) = \frac{\mathcal{L}_0+\mathcal{L}_1x+(\mathcal{L}_2-s\mathcal{L}_0)x^2}{1-sx^2-tx^3}.$$

We can obtain the *n*th (*s*, *t*)-Ludovan quaternions matrix sequence L_n by the same way as proof of Theorem 3.4.

Theorem 3.6. *For the nth* (*s*, *t*)*-Ludovan quaternions matrix sequence* L_n,

$$\mathcal{L}_n = \sum_{m=0}^{n} s^{3m-n}t^{n-2m-2}\left\{\binom{m}{3m-n}t^2I+\binom{m}{3m-n+1}st\mathcal{R}+\binom{m}{3m-n+2}s^2(\mathcal{R}^2-sI)\right\}\mathcal{L}_0$$

$$= \sum_{m=0}^{n} s^{3m-n}t^{n-2m-2}\left\{\binom{m}{3m-n}t^2\mathcal{V}+\binom{m}{3m-n+1}st\mathcal{R}\mathcal{V}+\binom{m}{3m-n+2}s^2(\mathcal{R}^2\mathcal{V}-s\mathcal{V})\right\}\mathcal{P}_0,$$

where I is the identity matrix of order 3.

Since $L_n = R^nL_0$, $L_0 = VP_0$ and

$$\mathcal{R}\mathcal{V} = \begin{bmatrix} s+1 & t & 0 \\ 0 & s+1 & t \\ 1 & 0 & 1 \end{bmatrix}, \ \mathcal{R}^2\mathcal{V}-s\mathcal{V} = \begin{bmatrix} t & 0 & t \\ 1 & t & -s \\ -\frac{s}{t} & 1 & t+\frac{s^2}{t} \end{bmatrix},$$

We have

$$
\begin{aligned}
\mathcal{L}_n &= \begin{bmatrix} Ql_{n+2} & Ql_{n+1} & Ql_n \\ Ql_{n+1} & Ql_n & Ql_{n-1} \\ Ql_n & Ql_{n-1} & Ql_{n-2} \end{bmatrix} \\
&= \sum_{m=0}^{n} s^{3m-n} t^{n-2m-2} \left\{ \binom{m}{3m-n} t^2 I + \binom{m}{3m-n+1} st\mathcal{R} + \binom{m}{3m-n+2} s^2(\mathcal{R}^2 - sI) \right\} \mathcal{L}_0 \\
&= \sum_{m=0}^{n} s^{3m-n} t^{n-2m-2} M\mathcal{P}_0, \qquad (3.7)
\end{aligned}
$$

where the matrix M is

$$
\begin{bmatrix}
\binom{m}{3m-n+1} st + \binom{m+1}{3m-n+2} s^2 t & \binom{m}{3m-n} t^2 + \binom{m+1}{3m-n+1} st^2 & \binom{m}{3m-n} t^3 + \binom{m}{3m-n+2} s^2 t \\
\binom{m}{3m-n} t^2 + \binom{m}{3m-n+2} s^2 & \binom{m}{3m-n+1} st + \binom{m+1}{3m-n+2} s^2 t & \binom{m}{3m-n} t^2 + \binom{m}{3m-n+1} st^2 - \binom{m}{3m-n+2} s^3 \\
\binom{m}{3m-n} t + \binom{m}{3m-n+1} st - \binom{m}{3m-n+2} \frac{s^3}{t} & \binom{m}{3m-n+2} s^2 & \left(\binom{m}{3m-n+1} - \binom{m}{3m-n}\right) st + \binom{m}{3m-n+2} \left(s^2 t + \frac{s^4}{t}\right)
\end{bmatrix}.
$$

Corollary 3.3. *For the nth* (s, t)*-Ludovan quaternion* Ql_n

$$
\begin{aligned}
Ql_n = &\sum_{m=0}^{n} \left(\binom{m}{3m-n+2} s^2 + \binom{m}{3m-n} t^2 \right) s^{3m-n} t^{n-2m-2} \\
&+ i \sum_{m=0}^{n} \left(\binom{m}{3m-n+1} + \binom{m+1}{3m-n+2} s \right) s^{3m-n+1} t^{n-2m-1} \\
&+ j \sum_{m=0}^{n} \left(\binom{m}{3m-n} + \binom{m+1}{3m-n+1} s \right) s^{3m-n} t^{n-2m} \\
&+ k \sum_{m=0}^{n} \left(\binom{m}{3m-n} t^2 + \binom{m+1}{3m-n+2} (s^2 + s^3) \right) s^{3m-n} t^{n-2m-1} \\
= &\sum_{m=0}^{n+1} \left\{ \left(\binom{m}{3m-n+1} t + \binom{m}{3m-n} st + \binom{m}{3m-n+1} \frac{s^3}{t} \right) + i \left(\binom{m}{3m-n-1} t^2 + \binom{m}{3m-n+1} s^2 \right) \right. \\
&\left. + j \left(\binom{m}{3m-n} st + \binom{m+1}{3m-n+1} s^2 t \right) + k \left(\binom{m+1}{3m-n} st^2 + \binom{m}{3m-n-1} t^2 \right) \right\} s^{3m-n-1} t^{n-2m-1}.
\end{aligned}
$$

Proof. Since Ql_{n-1} is the $(2,3)$-entry of the matrix $\mathcal{L}_n$, we can get, from (3.7),

$$
\begin{aligned}
Qp_{n-1} = &\sum_{m=0}^{n} s^{3m-n} t^{n-2m-2} \left\{ \left(\binom{m}{3m-n} t^2 + \binom{m}{3m-n+2} s^2 \right) (i + sk) + \left(\binom{m}{3m-n+1} st + \binom{m+1}{3m-n+2} s^2 t \right) (j) \right. \\
&\left. + \left(\binom{m}{3m-n} t^2 + \binom{m}{3m-n+1} st^2 - \binom{m}{3m-n+2} s^3 \right) \left(\frac{1}{t} + k \right) \right\} \\
= &\sum_{m=0}^{n} s^{3m-n} t^{n-2m-2} \left\{ \binom{m}{3m-n} t + \binom{m}{3m-n+1} st - \binom{m}{3m-n+2} \frac{s^3}{t} + i \left(\binom{m}{3m-n} t^2 + \binom{m}{3m-n+2} s^2 \right) \right. \\
&\left. + j \left(\binom{m}{3m-n+1} st + \binom{m}{3m-n+2} s^2 t \right) + k \left(\binom{m+1}{3m-n+1} st^2 + \binom{m}{3m-n} t^2 \right) \right\}.
\end{aligned}
$$

Since $Ql_{(n+1)-1} = Ql_n$, from Theorem 2.7, we can get the conclution

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