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THE EQUIVALENCE OF TWO SEMI-FINITE FORMS OF THE QUINTUPLE PRODUCT IDENTITY

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Abstract

Two semi-finite forms of the quintuple product identity are proved to be equivalent by means of Abel's method on summation by parts.

Keywords: quintuple product identity; semi-finite form; Abel's method on summation by parts; q-series.

1. INTRODUCTION

The celebrated quintuple product identity states that [1, p. 82]:

$$(1.1)\qquad \sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}}(1+zq^k)z^{3k} = \frac{(q, q/z^2, z^2; q)_\infty}{(z, q/z; q)_\infty}, \quad z \neq 0.$$

Here and throughout this note, we define the products of q-shifted factorials as usual by

$$(a;q)_\infty = \prod_{l=0}^{\infty}(1-aq^l) \qquad \text{and} \qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

for $n \in \mathrm{Z}$ and $|q| < 1$, with the following abbreviated multiple parameter notation

$$(a, b, \cdots, c; q)_k = (a;q)_k(b;q)_k \cdots (c;q)_k,\ k \in \mathrm{Z} \cup \{\infty\}.$$

For the historical note and various proofs of this important identity (1.1), the reader can consult the paper [5]. A powerful generalization of (1.1) was presented with applications by Liu [6].

The author and Zhang [7, 8] gave three semi-finite form of the quintuple product identity, two of which are stated in the following two theorems, respecitvely.

Theorem 1.1. ([7, 8]) There holds

$$(1.2)\qquad \sum_{k=0}^{\infty}\frac{(z^2;q)_k}{(q;q)_k}q^{k^2}(1+zq^k)z^k=(-z;q)_\infty(z^2q;q^2)_\infty,\quad z\in\mathbb{C}.$$

Theorem 1.2. ([8]) There holds

$$(1.3)\qquad \sum_{k=0}^{\infty}\frac{(z^2q;q)_k}{(q;q)_k}q^{k^2}(1-z^2q^{2k+1})z^k=(-zq;q)_\infty(z^2q;q^2)_\infty,\quad z\in\mathbb{C}.$$

For the details of deducing the quintuple product identity (1.1) from Theorem 1.1 and 1.2, we refer the reader to [7] and [8], respectively.

In this short note, we will prove that Theorem 1.2 is equivalent to Theorem 1.1 by means of Abel's method on summation by parts.

The modified Abel's lemma on summation by parts is very effective in evalu- ating finite and infinite summations. See, [2–4], to name a few. Here we only employ this manner for the case of unilateral and nonterminating series, which can be stated in the following lemma.

For an arbitrary complex sequence $\{A_k\}$, let

$$\nabla A_k := A_k - A_{k-1} \qquad \text{and} \qquad \Delta A_k := A_k - A_{k+1}.$$

Lemma 1.3. Let $\{A_k\}$ and $\{B_k\}$ be two complex sequences. Then we have

$$\sum_{k=0}^{\infty}B_k\nabla A_k=[AB]_+-A_{-1}B_0+\sum_{k=0}^{\infty}A_k\Delta B_k,$$

provided that the series on both sides are convergent and there exists the limit

$[AB]_+ := \lim_{k\to\infty} A_k B_{k+1}$.

2. THE EQUIVALENCE of THEOREM 1.1 AND 1.2

Let

$$f(z):=\sum_{k=0}^{\infty}\frac{(z^2q;q)_k}{(q;q)_k}q^{k^2}(1-z^2q^{2k+1})z^k$$

And

$$g(z) := \sum_{k=0}^{\infty} \frac{(z^2;q)_k}{(q;q)_k} q^{k^2}(1+zq^k)z^k.$$

Define

$$A_k := \frac{(q^2z^2;q)_k}{(q;q)_k}(-1)^k q^{\frac{k^2+k}{2}} \qquad \text{and} \qquad B_k := (-1)^k q^{\frac{k^2+k}{2}} z^k.$$

Then, we have

$$A_{-1}B_0 = [AB]_+ = 0$$

With the differences

$$\nabla A_k = \frac{(1-z^2q^{2k+1})(z^2q;q)_k}{(1-z^2q)(q;q)_k}(-1)^k q^{\frac{k^2-k}{2}}$$

And

$$\vartriangle B_k = (-1)^k q^{\frac{k^2+k}{2}}(1+zq^{k+1})z^k.$$

Using Lemma 1.3, we get

$$\begin{aligned} f(z) &= (1-z^2q)\sum_{k=0}^{\infty} B_k \nabla A_k = (1-z^2q)\sum_{k=0}^{\infty} A_k \vartriangle B_k \\ &= (1-z^2q)\sum_{k=0}^{\infty} \frac{(q^2z^2;q)_k}{(q;q)_k} q^{k^2+k}(1+zq^{k+1})z^k \\ &= (1-z^2q)g(qz). \qquad (2.1) \end{aligned}$$

2.1. From Theorem 1.2 to Theorem 1.1. Combining (2.1) and Theorem 1.2, we get

$$g(qz) = \frac{f(z)}{1-z^2q} = \frac{(-zq;q)_\infty(z^2q;q^2)_\infty}{1-z^2q} = (-zq;q)_\infty(z^2q^3;q^2)_\infty.$$

Then, replacing z with zq−1 gives the identity (1.2), which completes the proof of Theorem 1.1.

2.2. From Theorem 1.1 to Theorem 1.2. Using (2.1) and Theorem 1.1, we have

$f(z) = (1 - z^2q)g(qz) = (1 - z^2q)(-zq; q)_\infty(z^2q^3; q^2)_\infty = (-zq; q)_\infty(z^2q; q^2)_\infty$, which is the identity (1.3). This ends the proof of Theorem 1.2.

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