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(1, 4) AND (2, 4) SYSTEMS BASED ON SEQUENTIAL ORDER STATISTICS FOR EQUIVARIANT PARAMETERS ESTIMATION

Glory Prasanth. K[1], A. Venmani[2]
[1,2]Department of Mathematics and Statistics, SRMIST, Kattankulathur, Chennai
gk9617@srmist.edu.in

Abstract

Reliability theory is concerned with the study of structures /systems having components. The structure has a collection of components designed to perform a certain specific function. Systems are of various types depending on the relationship between the states of the system. One of the system we come across in reliability theory is (k,n) system. In this paper an extension of Sequential Order Statistics from (1, 3) and (2, 3) Systems is performed. The sequential (2,4) system and (1,4) system with absolutely continuous lifelength distributions were introduced. The distribution and probability density function of sequential order statistics from (2, 4) system and (1, 4) system and mean time before failure of these systems are evaluated. We also obtained minimum risk equivariant estimator for the location scale parameter taking into account the sequential (2, 4) and (1, 4) systems. Also MREE of location and scale parameter from each of the system are evaluated.

Keywords: Sequential (1, 4) and (2, 4) systems, Order Statistics, Sequential Order, Statistics, Mean time before failure, Minimum risk equivariant Estimator.

1. Introduction

Product reliability seems to be more essential than ever before at the moment. As more items enter the market, customers now have the option of expecting excellent quality and extended life from the things they buy. In such a tough and competitive industry, one approach for manufacturers to attract customers is to offer guarantees on product lives. A manufacturer must understand product failure-time distribution in order to create a cost effective warranty. Before releasing a product to the public, life testing and reliability tests are performed to gather this knowledge .More realistic examples of k-out-of –n systems include an aeroplane with four engines which will not collapse if atleast two of them fail or a satellite with adequate power to deliver communications if minimum four out of ten batteries are operating. The data obtained from life testing tests is also utilised for other objectives, such as calculating proper dose administration and expiry dates, as well as determining effective warranties. Order statistics are used in many fields of statistical theory and practise.

Suppose n items such as radio tubes, wire fuses, or light bulbs are placed on a life-test, the weakest fail first, followed by the second weakest and so on until all have failed. Thus, if the life-time X of a randomly chosen item has pdf $f(x)$, the life-test generates in turn ordered observations $X_*^{(1)}, X_*^{(2)}, X_*^{(3)}, \ldots, X_*^{(n)}$ from this distribution. The practical importance of such experiments is evident. They afford an ideal application of order statistics, since by nature of the experiment the observations arrive in ascending order of magnitude.

If out of n components atleast $k(1 \leq k \leq n)$ components operate then it is (k,n) system. Order statistics is the variables arising when n variables are identically independent distribution and the arrangement is in increasing order of magnitude. Order Statistics has a wide range of application in statistical science. One of the most flexible models is Sequential order statistics, which explains the sequential (k,n) systems in which any failure of component affects the remaining ones so that their inherent rate of failure is altered parametrically with regards to preceding number of failures.

Related works

Balakrishnan and Sandhu (1996) [10], pointed out that the censoring scheme has important characteristics of consuming experimenter's time as well as cost and proposed a general progressively Type-II right censored concept. Roberts (1962 a, b)[36], Cohen (1963)[15], Balakrishnan and Cohen(1991)[8] discussed samples that are progressively censored. For distributional related results on progressive censored samples Aggarwala (1996) [1] was referred. By assuming the above scheme they derived two-parameter exponential distributions. David (1981)[18], Arnold, Balakrishnan and Nagaraja (1992)[5] provide a detailed discussion on ordinary order statistics from an arbitrary continuous distribution. Scheffe and Turkey (1945)[38] derived conditional distributional results for ordinary order statistics. Sukhatme (1937)[39] obtained an independence results for spacing from a standard exponential distribution based on ordinary order statistics. Malmquist (1950)[31] derived a result for the ratios from a standard uniform distribution based on ordinary order statistics. . Sequential order statistics was introduced by Kamps as times of failure for (k,n) system in which every failure modifies the time of failure of the remaining active components in 1995a [23]. Chandrasekhar (2005) [13] used variously constructed independent sequential (k,n) systems to derive location-scale exponential distributions' minimum risk equivariant estimator(s).

Edwin Prabakaran and Chandrasekar (1994)[20] discussed equivariant estimation using simultaneous equivariant approach. Lawless (1982)[28] also provides results on statistical inference for (k,n) systems. Leo Alexander and Chandrasekar (1999) [27] derived MREE based on Type II right censored order statistics. Aggarwala and Balakrishnan (1998) [2] established some properties of progressively Type-II right censored order statistics from arbitrary continuous distributions. All these developments on progressively Type-II right censored order statistics and related results have been integrated in Balakrishnan and Aggarwala (2000) [7]. An exact inference on conditional distribution was developed by Viveros and Balakrishnan (1994)[42]. In this development and discussion on generalized order statistics, Kamps (1995a,1995b) has proved some general properties of progressive Type II right censored order statistics. Chanderasekar, B and et al (2002)[12], on the basis of Type-II Progresssively censored samples derived Equivariant estimation for parameters of exponential distribution. Kamps (1995 a,b) and Cramer and Kamps (1996)[17] developed some results related to the Sequential order statistics. Percentiles for location scale families of distributions were used to obtain Minimum risk equivariant estimators in Dutta and Ghosh (1988) [19]. Equivariant Estimation of Parameters

Based on Sequential Order Statistics from (1, 3) and (2, 3) Systems, was derived in Chandrasekhar (2007) [4].

Preliminaries

A k out of n system consists of n components that start working simultaneously. It is operating while atleast k components are functioning and it breaks down if $n-k+1$ or more components fail. Consider a (k,n) system and let $X_1, X_2, \ldots, X_n$ be the failure times of the components. If $X_1, X_2, \ldots, X_n$ are considered to be identical and independent distribution, then the system failure time $(n-k+1)th$ order statistics is associated with $X_1, X_2, \ldots, X_n$. It is usually denoted by $X_{(n-k+1)}$ .In a (k,n) system, it is usually assumed that the component failure times $X_1, X_2, \ldots, X_n$ are iid random variables. It is assumed that any component failure rate does not influence the remaining active components. But in practical situations, this is not possible. If any one of the components in the system breaks down, the whole system gets affected. Additional burden is placed on the remaining active components, and so the stress is more on the remaining active components. The consequences may be decrease in efficiency or increase in the failure rate or both.

In order to account for the fluctuational changes in the life lengths distribution of the active components, an alternative flexible model was designed. When a component fails in this model, the remaining active components take the burden, and so their distribution changes. This flexible model, especially constructed for this purpose is known as the sequential (k,n) system. The resulting order statistics of the sequential (k,n) system are known as sequential order statistics (SOS).Here we assume that failure of each component leads to different failure rate than before for the remaining active components. A sequential (k,n) system is a (k,n) system in which a component failure changes the lifelengths distribution of the remaining components. The life length of a sequential $(n-r+1)$ system is modelled by r^{th} SOS denoted by $X_*^{(r)}, 1 \leq r \leq n$.

2. Methods

2.1 Distributional Results

(Kamps, 1995a) Let $F_1, F_2, \ldots, F_n$ be absolutely continuous distribution functions with respective density function $f_1, f_2, \ldots, f_n$. The joint density function of the first r,

$(1 \leq r \leq n)$ SOS $X_*^{(1)}, X_*^{(2)}, \ldots, X_*^{(r)}$ based on these distributions is given by

$$f^*(x_1, x_2, \ldots, x_n) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \left[\left\{ \frac{1-F_i(x_i)}{1-F_i(x_{i-1})} \right\}^{n-i} \frac{f_i(x_i)}{1-F_i(x_{i-1})} \right],$$

$$-\infty = x_0 < x_1 < \cdots < x_n < \infty$$

Model 1

From absolutely continuous sequential (2,4) system with life length distributions F_1 , F_2 , F_3 having the respective density functions f_1 , f_2 , f_3 is given by

$$F_1(x) = 1 - e^{-\frac{1}{\tau}(x-\xi)}\left[1 + \frac{x-\xi}{\tau}\right], \quad x > \xi, \ \xi \in R, \ \tau > 0$$

and

$$F_2(x) = F_3(x) = 1 - e^{-\frac{1}{\tau}(x-\xi)}, \ x > \xi, \ \xi \in R, \ \tau > 0 \ .$$

The probability density function of (2, 4) system is

$$f_1(x) = \frac{1}{\tau^2}(x-\xi)e^{-\frac{1}{\tau}(x-\xi)}, \quad x > \xi, \ \xi \in R, \ \tau > 0$$

$$f_2(x) = f_3(x) = \frac{1}{\tau}e^{-\frac{1}{\tau}(x-\xi)}, \quad x > \xi, \ \xi \in R, \ \tau > 0$$

Thus the joint probability density function of $X_*^{(1)}, X_*^{(2)}$ and $X_*^{(3)}$ is

$$f^*(x_1, x_2, x_3) = 24\frac{1}{\tau^4}(x_1 - \xi)\left[1 + \frac{x_1 - \xi}{\tau}\right]^3 e^{-\frac{1}{\tau}(x_1 + x_2 + 2x_3 - 4\xi)},$$

$$\xi < x_1 < x_2 < x_3 < \infty, \xi \epsilon R, \tau > 0 \qquad (2.1.1)$$

Model 2

From absolutely continuous sequential (1,4) system with life length distributions F_1 , F_2 , F_3 , F_4 having the respective density functions f_1 , f_2 , f_3, f_4 is given by

$$F_1(x) = F_2(x) = 1 - e^{-\frac{1}{\tau}(x-\xi)}\left[1 + \frac{x-\xi}{\tau}\right], \quad x > \xi, \ \xi \in R, \ \tau > 0$$

$$F_3(x) = F_4(x) = 1 - e^{-\frac{1}{\tau}(x-\xi)}, \ x > \xi, \ \xi \in R, \ \tau > 0 \ .$$

The probability density function of (1, 4) system is

$$f_1(x) = f_2(x) = \frac{1}{\tau^2}(x-\xi)e^{-\frac{1}{\tau}(x-\xi)}, \quad x > \xi, \ \xi \in R, \ \tau > 0$$

$$f_3(x) = f_4(x) = \frac{1}{\tau} e^{-\frac{1}{\tau}(x-\xi)}, \quad x > \xi, \ \ \xi \in R, \ \ \tau > 0$$

Thus the joint probability density function of $X_*^{(1)}, X_*^{(2)}, X_*^{(3)}$and $X_*^{(4)}$ is

$$f^*(x_1, x_2, x_3, x_4) = \frac{24}{\tau^6}(x_1 - \xi)(x_2 - \xi)\left[1 + \frac{x_2 - \xi}{\tau}\right]^2 e^{-\frac{1}{\tau}(x_1+x_2+x_3+x_4-4\xi)},$$

$$\xi < x_1 < x_2 < x_3 < x_4 < \infty, \xi \, \epsilon \, R, \tau > 0 \qquad (2.1.2)$$

2.2 Mean time before failure

The failure time of the system is $X_*^{(3)}$ and its pdf is given by

$f_3^*(x_3) = \int_{\xi}^{x_3} \int_{x_1}^{x_3} f^*(x_1, x_2, x_3)\, dx_2 dx_1, \quad \xi < x_1 < x_2 < x_3 < \infty,$

Considering (2.1.1),

$$f_3^*(x_3) = \frac{12}{\tau^5}(x_3 - \xi)^4 e^{-\frac{4}{\tau}(x_3-\xi)} + \frac{108}{\tau^4}(x_3 - \xi)^3 e^{-\frac{4}{\tau}(x_3-\xi)} + \frac{450}{\tau^3}(x_3 - \xi)^2 e^{-\frac{4}{\tau}(x_3-\xi)}$$

$$+ \frac{1038}{\tau^2}(x_3 - \xi) e^{-\frac{4}{\tau}(x_3-\xi)} + \frac{1107}{\tau^3} e^{-\frac{4}{\tau}(x_3-\xi)} - \frac{1176}{\tau^2} e^{-\frac{3}{\tau}(x_3-\xi)} + \frac{69}{\tau} e^{-\frac{2}{\tau}(x_3-\xi)}$$

MTBF=$\int_{\xi}^{\infty} x_3\, f_3^*(x_3)\, \mathrm{d}x_3$

$$= \frac{9}{32}\left(\frac{5\tau}{4} + \xi\right) + \frac{81}{32}(\tau + \xi) + \frac{225}{16}\left(\frac{3\tau}{4} + \xi\right) + \frac{519}{8}\left(\frac{\tau}{2} + \xi\right)$$

$$+ \frac{1107}{4}\left(\frac{\tau}{4} + \xi\right) - 392\left(\frac{\tau}{3} + \xi\right) + \frac{69}{2}\left(\frac{\tau}{2} + \xi\right)$$

$$= \tau \frac{629}{384} + \xi$$

The failure time of the system is $X_*^{(4)}$ and its pdf is given by

$f_4^*(x_4) = \int_{\xi}^{x_4}\int_{x_1}^{x_4}\int_{x_2}^{x_4} f^*(x_1,x_2,x_3,x_4)\, dx_3 dx_2 dx_1,\ \xi < x_1 < x_2 < x_3 < x_4 < \infty$

Considering (2.1.2),

$$f_4^*(x_4) = -\frac{4}{\tau^5}(x_4-\xi)^4 e^{-\frac{4}{\tau}(x_4-\xi)} - \frac{76}{3\tau^4}(x_4-\xi)^3 e^{-\frac{4}{\tau}(x_4-\xi)} - \frac{202}{3\tau^3}(x_4-\xi)^2 e^{-\frac{4}{\tau}(x_4-\xi)}$$
$$-\frac{818}{9\tau^2}(x_4-\xi) e^{-\frac{4}{\tau}(x_4-\xi)} - \frac{1439}{27\tau} e^{-\frac{4}{\tau}(x_4-\xi)} + \frac{305}{27\tau} e^{-\frac{2}{\tau}(x_4-\xi)} - \frac{195}{\tau} e^{-\frac{2}{\tau}(x_4-\xi)}$$
$$+\frac{12}{\tau^4}(x_4-\xi)^3 e^{-\frac{3}{\tau}(x_4-\xi)} + \frac{78}{\tau^3}(x_4-\xi)^2 e^{-\frac{3}{\tau}(x_4-\xi)} + \frac{210}{\tau^2}(x_4-\xi) e^{-\frac{3}{\tau}(x_4-\xi)}$$
$$+\frac{237}{\tau} e^{-\frac{3}{\tau}(x_4-\xi)}$$

$$\text{MTBF} = \int_{\xi}^{\infty} x_4\ f_4^*(x_4)\, dx_4$$

$$= -\frac{3}{32}\left(\frac{5\tau}{4}+\xi\right) - \frac{19}{32}(\tau+\xi) - \frac{101}{48}\left(\frac{3\tau}{4}+\xi\right) - \frac{409}{72}\left(\frac{\tau}{2}+\xi\right)$$
$$-\frac{1439}{108}\left(\frac{\tau}{4}+\xi\right) - \frac{195}{2}\left(\frac{\tau}{2}+\xi\right) + \frac{305}{27}(\tau+\xi) + \frac{8}{9}\left(\frac{4\tau}{3}+\xi\right) + \frac{52}{9}(\tau+\xi)$$
$$+\frac{70}{3}\left(\frac{2\tau}{3}+\xi\right) + 79\left(\frac{\tau}{3}+\xi\right)$$
$$= \tau\,\frac{10153}{3456} + \xi$$

2.3 Minimum Risk Equivariant Estimator for location-scale parameter

From (2.1.1), it is seen that the distribution of $\left(\frac{X_*^{(1)}-\xi}{\tau}, \frac{X_*^{(2)}-\xi}{\tau}, \frac{X_*^{(3)}-\xi}{\tau}\right)$ does not depend on ξ as well as τ, the distribution of $\left(X_*^{(1)}, X_*^{(2)}, X_*^{(3)}\right)$ belongs to a location-scale parameter(ξ,τ).

Our interest is to estimate $\eta = \alpha\xi + \beta\tau, \alpha, \beta \in R$.

The linear functions of ξ and τ are the percentiles of the distribution.

Dutta and Ghosh (1988) [19]

If δ_0 is equivariant, then the positive valued function g satisfies

$g(a + b\,\underline{x}) = bg(\underline{x}) \quad \forall\, a \in R\,, b > 0,$

w^* minimizes $E_{0,1}\{\rho(\delta_0(\underline{x}) - g(\underline{x})w(\underline{z}))|\underline{z}\}$ with respect to w, ρ is an loss function which is invariant

$\underline{z} = (z_1\,, z_2, \dots, z_{n-1})$ and $z_i = \frac{x_i - x_n}{g(\underline{x})}\,, i = 1,2,3, \dots, n-1$

If there is squared error loss function then w^* minimizes $E_{0,1}\{\delta_0 - gw - \beta\}^2$, w.r.t w. Choose, $\delta_0(\underline{x}) = \alpha x_1 + \beta(x_3 - x_1)$ and $g(\underline{x}) = x_3 - x_1$.

When $\xi = 0$ and $\tau = 1$,

$$f^*(x_1, x_2, x_3) = 24x_1(1 + x_1)^3 e^{-(x_1 + x_2 + 2x_3)}\,, 0 < x_1 < x_2 < x_3 < \infty$$

Making the transformation $u = x_1$, $v = x_3 - x_1$, $z = \frac{x_2 - x_1}{x_3 - x_1}$, we get

$$f_1^*(u, v, z) = 24uv(1 + u)^3 e^{-(4u + 2v + vz)}\,, 0 < u < \infty\,, 0 < v < \infty, 0 < z < 1$$

$$f_2^*(v, z) = 6ve^{-v(2+z)}\,, 0 < v < \infty, 0 < z < 1$$

$$f_3^*(z) = \frac{6}{(2+z)^2}\,, 0 < z < 1$$

$$f_4^*(v|\mathrm{z}) = v(2 + z)^2\, e^{-v(2+z)}, 0 < v < \infty, 0 < z < 1$$

$$f_5^*(u, v|\mathrm{z}) = 4uv(1 + u)^3(2 + z)^2 e^{-(4u + 2v + vz)}\,,$$

$$0 < u < \infty\,, 0 < v < \infty, 0 < z < 1$$

The Expectation corresponding is

$$E_{0,1}(V|Z) = \frac{2}{(2 + z)}$$

$$E_{0,1}(UV|Z) = \frac{103}{64}\frac{1}{(2 + z)}$$

$$E_{0,1}(V^2|Z) = \frac{6}{(2 + z)^2}$$

$$w^* = \frac{\alpha E_{0,1}\{x_1\,(x_3 - x_1)|z\} + \beta E_{0,1}\{(x_3 - x_1)^2|z\} - \beta E_{0,1}\{(x_3 - x_1)|z\}}{E_{0,1}\{(x_3 - x_1)^2|z\}}$$

$$w^* = \alpha\frac{103}{384}(2+z) + \beta - \frac{\beta}{3}(2+z)$$

Estimate the expectations and substitute, we get

MREE of $\alpha\xi + \beta\tau$

$$\delta^*(X) = \alpha X_*^{(1)} + \left(\frac{\beta}{3} - \frac{103}{384}\alpha\right)\left(2X_*^{(3)} + X_*^{(2)} - 3X_*^{(1)}\right)$$

From (2.1.2), it is seen that the distribution of $\left(\frac{X_*^{(1)}-\xi}{\tau}, \frac{X_*^{(2)}-\xi}{\tau}, \frac{X_*^{(3)}-\xi}{\tau}, \frac{X_*^{(4)}-\xi}{\tau}\right)$ does not depend on ξ and τ, the distribution of $\left(X_*^{(1)}, X_*^{(2)}, X_*^{(3)}, X_*^{(4)}\right)$ belongs to a location-scale parameter(ξ, τ).

Choose, $\delta_0(\underline{x}) = \alpha x_1 + \beta(x_3 - x_1)$ and $g(\underline{x}) = x_3 - x_1$.

When $\xi = 0$ and $\tau = 1$,

$$f^*(x_1, x_2, x_3, x_4) = 24x_1x_2(1+x_2)^2 e^{-(x_1+x_2+x_3+x_4)},$$

$$0 < x_1 < x_2 < x_3 < x_4 < \infty$$

Making the transformation $u = x_1$, $v = x_3 - x_1$, $z_1 = \frac{x_2 - x_1}{x_3 - x_1}, z_2 = \frac{x_4 - x_1}{x_3 - x_1}$, we get

$$f_1^*(u, v, z_1, z_2) = 24uv^2(u + vz_1)(1 + u + vz_1)^2 e^{-(4u+vz_1+vz_2+v)},$$

$$0 < u < \infty, 0 < v < \infty, 0 < z_1 < 1, 1 < z_2 < \infty$$

$$f_2^*(v, z_1, z_2) = \frac{3}{16}v^2 e^{-v(z_1+z_2+1)}\{13 + 33vz_1 + 28v^2z_1^2 + 8v^3z_1^3\},$$

$$0 < v < \infty,\ 0 < z_1 < 1,\ 1 < z_2 < \infty$$

$$f_3^*(z_1, z_2) = \frac{3}{8(z_1+z_2+1)^6}\left\{\begin{matrix} 13(z_1+z_2+1)^3 + 99z_1(z_1+z_2+1)^2 + \\ 336{z_1}^2(z_1+z_2+1) + 480{z_1}^3 \end{matrix}\right\}$$

$$f_4^*(v|z_1, z_2) = \frac{v^2(z_1+z_2+1)^6 e^{-v(z_1+z_2+1)}\{13 + 33vz_1 + 28v^2z_1^2 + 8v^3z_1^3\}}{2\{13(z_1+z_2+1)^3 + 99z_1(z_1+z_2+1)^2 + 336{z_1}^2(z_1+z_2+1) + 480{z_1}^3\}}$$

$$f_5^*(u, v|z_1, z_2) = \frac{6uv^2(u+vz_1)(1+u+vz_1)^2(z_1+z_2+1)^6 e^{-(4u+vz_1+vz_2+v)}}{\{13(z_1+z_2+1)^3 + 99z_1(z_1+z_2+1)^2 + 336{z_1}^2(z_1+z_2+1) + 480{z_1}^3\}}$$

The Expectation corresponding is

$$E_{0,1}\left(V\middle|\underline{Z}\right) = \frac{\{39(z_1+z_2+1)^3+396z_1(z_1+z_2+1)^2+1680{z_1}^2(z_1+z_2+1)+2880{z_1}^3\}}{(z_1+z_2+1)\{13(z_1+z_2+1)^3+99z_1(z_1+z_2+1)^2+336{z_1}^2(z_1+z_2+1)+480{z_1}^3\}}$$

$$E_{0,1}\left(V^2\middle|\underline{Z}\right) = \frac{4\{39(z_1+z_2+1)^3+495z_1(z_1+z_2+1)^2+2520{z_1}^2(z_1+z_2+1)+5040{z_1}^3\}}{(z_1+z_2+1)^2\{13(z_1+z_2+1)^3+99z_1(z_1+z_2+1)^2+336{z_1}^2(z_1+z_2+1)+480{z_1}^3\}}$$

$$E_{0,1}\left(UV\middle|\underline{Z}\right) = \frac{\{153(z_1+z_2+1)^3+1200z_1(z_1+z_2+1)^2+4080{z_1}^2(z_1+z_2+1)+5760{z_1}^3\}}{4(z_1+z_2+1)\{13(z_1+z_2+1)^3+99z_1(z_1+z_2+1)^2+336{z_1}^2(z_1+z_2+1)+480{z_1}^3\}}$$

The MREE of $\alpha\xi+\beta\tau$

$$\begin{aligned}
\delta^*\left(\underline{X}\right) = {} & \alpha X_*^{(1)} \\
& + \beta\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)\Big\{13\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)^3 \\
& + 132\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)^2\left(X_*^{(2)}-X_*^{(1)}\right) \\
& + 560\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)\left(X_*^{(2)}-X_*^{(1)}\right)^2+960\left(X_*^{(2)}-X_*^{(1)}\right)^3\Big\} \\
& /4\Big\{13\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)^3+165\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)^2\left(X_*^{(2)}-X_*^{(1)}\right) \\
& + 840\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)\left(X_*^{(2)}-X_*^{(1)}\right)^2+1680\left(X_*^{(2)}-X_*^{(1)}\right)^3\Big\} \\
& - \alpha\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)\Big\{51\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)^3 \\
& + 400\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)^2\left(X_*^{(2)}-X_*^{(1)}\right) \\
& + 1360\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)\left(X_*^{(2)}-X_*^{(1)}\right)^2+1920\left(X_*^{(2)}-X_*^{(1)}\right)^3\Big\} \\
& /16\ \Big\{13\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)^3 \\
& + 165\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)^2\left(X_*^{(2)}-X_*^{(1)}\right) \\
& + 840\left(X_*^{(2)}+X_*^{(3)}+X_*^{(4)}-3X_*^{(1)}\right)\left(X_*^{(2)}-X_*^{(1)}\right)^2+1680\left(X_*^{(2)}-X_*^{(1)}\right)^3\Big\}
\end{aligned}$$

2.4 Minimum Risk Equivariant Estimator for Scale parameter

The MREE of τ for (2,4) system

$$\delta^*(X) = \frac{1}{3}\left(2X_*^{(3)} + X_*^{(2)} - 3X_*^{(1)}\right)$$

The MREE of τ for (1,4) system

$$\begin{aligned}\delta^*(\underline{X}) = \left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)&\Big\{13\left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)^3 \\ &+ 132\left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)^2\left(X_*^{(2)} - X_*^{(1)}\right) \\ &+ 560\left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)\left(X_*^{(2)} - X_*^{(1)}\right)^2 + 960\left(X_*^{(2)} - X_*^{(1)}\right)^3\Big\} \\ &/4\Big\{13\left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)^3 \\ &+ 165\left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)^2\left(X_*^{(2)} - X_*^{(1)}\right) \\ &+ 840\left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)\left(X_*^{(2)} - X_*^{(1)}\right)^2 + 1680\left(X_*^{(2)} - X_*^{(1)}\right)^3\Big\}\end{aligned}$$

2.5 Minimum Risk Equivariant Estimator for Location parameter

The MREE of ξ for (2,4) system is

$$\delta^*(X) = \frac{1}{384}\left(693X_*^{(1)} - 103X_*^{(2)} - 206X_*^{(3)}\right)$$

The MREE of ξ for (1,4) system

$$\delta^*(\underline{X}) = X_*^{(1)} - \left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)\left\{51\left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)^3 \right.$$
$$+ 400\left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)^2\left(X_*^{(2)} - X_*^{(1)}\right)$$
$$\left. + 1360\left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)\left(X_*^{(2)} - X_*^{(1)}\right)^2 + 1920\left(X_*^{(2)} - X_*^{(1)}\right)^3\right\}$$
$$/16\left\{13\left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)^3 \right.$$
$$+ 165\left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)^2\left(X_*^{(2)} - X_*^{(1)}\right)$$
$$\left. + 840\left(X_*^{(2)} + X_*^{(3)} + X_*^{(4)} - 3X_*^{(1)}\right)\left(X_*^{(2)} - X_*^{(1)}\right)^2 + 1680\left(X_*^{(2)} - X_*^{(1)}\right)^3\right\}$$

3. Results

Sequential (k,n) system was discussed. Sequential (2, 4) system and (1, 4) were introduced. The distribution of the sequential order statistics from each of the systems and Mean time before failure (MTBF) of these systems were evaluated. The Equivariant estimation of the location-scale parameter in sequential (2,4) system and (1,4) system. The Minimum risk equivariant estimation for location and scale parameter were also discussed.

4. Conclusion

This paper provides mean time before failure and MREE under invariant loss function for estimating location-Scale parameter of exponential distribution. This problem can be studied for Bayes estimator under general convex and invariant loss function

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