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New Determinantal Inequalities for Positive Definite Matrices

Ronak Pashaei[1], Mohammad Sadegh Asgari[2*]
[1,2]Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.
pashai.ronak@gmail.com[1], msasgari00@gmail.com[2]

Abstract

R In this article, we explore some new determinant inequalities for matrices. A concavity approach is implemented to show many sub and super-additive inequalities for the determinant. This approach is a new direction in this type of inequality. In the end, many determinant inequalities are presented for accretive-dissipative matrices.

Keywords: Subadditive matrix inequalities, determinant inequalities, concave function, accretive- dissipative matrices.

1. Introduction

Matrix inequalities, which extend some of certain scalar inequalities, play an important role in almost all branches of mathematics and other areas of science. This paper investigates the determinant inequalities of a matrix.

In the sequel, Mn denotes the algebra of all $n \times n$ complex matrices, whose elements will be denoted by upper case letters, in the sequel. The determinant function is a complex-valued function defined on Mn and is denoted by det. The determinant has numerous applications and significance in the study of matrix analysis.

In this article, we will present some new inequalities for the determinant. In our analysis, we will use a concavity approach as a new approach to tickle determinants inequalities. Recall that a function $f : (a, b) \to R$ is said to be concave if

$$f((1-t)\alpha + t\beta) \geq (1-t)f(\alpha) + tf(\beta),$$

for all $\alpha, \beta \in (a, b)$ and $0 \leq t \leq 1$. It is interesting that this inequality can be reversed as

$$f((1-t)\alpha + t\beta) \leq (1-t)f(\alpha) + tf(\beta) + 2R\ .f\ .\ \alpha + \beta\ \Sigma - f(\alpha) + f(\beta)\ \Sigma\ , \qquad (1.1)$$

where $R = \max\{t, 1-t\}$, [7]. We also refer the reader to a detailed discussion of this inequality in [8,9,11].

The function $f(A) = \det n(A)$ is concave on the cone of positive semidefinite matrices. We will use this concavity to find new sub and super-additivity results for the determinant. In particular, we prove that when $A, B \in Mn$ are positive definite, then

$$\det^{\frac{1}{n}}(A+B) \le \det^{\frac{1}{n}} B\left(1+\frac{\operatorname{tr}(AB^{-1})}{n}\right).$$

Although this inequality can be shown using other techniques, we present a concavity proof. Further, we

find that this latter inequality has some impact in proving many other known results for the determinant. The significance of these results is not the results themselves only but the new technique used to prove them. Once this has been done, we abandon concavity and start studying the determinant of matrices having the form A + iB. Many results will be presented as an extension of known results.

2. Determinantal inequalities via concavity

In this part of the paper, we present related determinant inequalities using a concavity approach. In particular, we use the fact that the function f defined on the cone of positive definite matrices in Mn by f (A) = det $^{1/n}$ (A) is a concave function. That is, if $0 \le t \le 1$ and A, B ∈ Mn are positive definite, then [1, Corollary 11.3.21]

det n ((1 − t)A + tB) ≥ (1 − t)det n (A) + tdet n (B). (2.1)

Further, noting Young's inequality, we have

det n ((1 − t)A + tB) ≥ det n (A)det n (B),

which implies

det((1 − t)A + tB) ≥ det1−t(A)dett(B), (2.2)

showing that the function A ›→ log det A is concave, [2, Lemma II.3].

Now since f (A) = log det A is concave on positive definite matrices, then we have the inequality

$$f((1-t)A+tB) \le (1-t)f(A)+tf(B)+2R\left(f\left(\frac{A+B}{2}\right)-\frac{f(A)+f(B)}{2}\right), \qquad (2.3)$$

where $0 \le t \le 1$ and R = max{t, 1 − t}. We refer the reader to [7–10] for a detailed discussion of this and related inequalities, as we mentioned earlier in the introduction.

The following result shows a complement of the inequality (2.2):

Theorem 2.1. Let A, B ∈ Mn be positive definite and let R = max{t, 1 − t}, $0 \le t \le 1$. Then

$$\det((1-t)A+tB) \le \left(\prod_{i=1}^{n} S\left(\lambda_i\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)\right)^{2R} \det^{1-t}(A)\det^{t}(B),$$

where S(h) is the Specht's ratio defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}, h > 0, h \neq 1.$$

Proof. Since f (T) = log (det (T)) is a concave function on the convex set of positive definite matrices [2, Lemma II.3], (2.3) implies that

$$\det\left((1-t)\,A + tB\right) \leq \left(\frac{\det\left(\frac{A+B}{2}\right)}{\sqrt{\det\left(AB\right)}}\right)^{2R} \det^{1-t}\left(A\right) \cdot \det^{t}\left(B\right).$$

On the other hand,

$$\begin{aligned}
\frac{\det\left(\frac{A+B}{2}\right)}{\sqrt{\det\left(AB\right)}} &= \frac{\det^{\frac{1}{2}}\left(A\right)\det\left(\frac{I+A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2}\right)\det^{\frac{1}{2}}\left(A\right)}{\det^{\frac{1}{2}}\left(A\right)\det^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\det^{\frac{1}{2}}\left(A\right)} \\
&= \frac{\prod_{i=1}^{n} \frac{1+\lambda_i\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)}{2}}{\prod_{i=1}^{n} \lambda_i^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)} \\
&= \prod_{i=1}^{n} \frac{1+\lambda_i\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)}{2\lambda_i^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)}.
\end{aligned}$$

From [12], we know that

$$\frac{a+b}{2} \leq S\left(h\right)\sqrt{ab}; \quad m \leq a, b \leq M,\ h = \frac{M}{m}.$$

Thus,

$$\prod_{i=1}^{n} \frac{1+\lambda_i\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)}{2\lambda_i^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)} \leq \prod_{i=1}^{n} S\left(\lambda_i\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right).$$

Whence

$$\frac{\det\left(\frac{A+B}{2}\right)}{\sqrt{\det\left(AB\right)}} \leq \prod_{i=1}^{n} S\left(\lambda_i\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right).$$

Notice that replacing A by 1/1-t A and B by 1/t B with 0 < t < 1 in (2.1), we reach the super additivity inequality [1, Corollary 11.3.21], known as the Minkowski determinant inequality,

det 1/n (A + B) ≥ det 1/n (A) + det 1/n (B). (2.4)

The following result presents a reversed version of this inequality via concavity.

The following result presents a reversed version of (2.4) via concavity.

Theorem 2.2. Let A, B ∈ Mn be positive definite and let R = max{t, 1 − t}, 0 < t < 1. Then

$$\det^{\frac{1}{n}}(A+B) \le \det^{\frac{1}{n}}(A) + \det^{\frac{1}{n}}(B) + \frac{R}{t(1-t)}\Big(\det^{\frac{1}{n}}(tA+(1-t)B) - t\det^{\frac{1}{n}}(A) - (1-t)\det^{\frac{1}{n}}(B)\Big).$$

Proof. Since f(T) = det 1/n (T) is a concave function on the convex set of positive definite Hermitian matrices, we infer from (1.1) that

$$\det^{\frac{1}{n}}((1-t)A+tB) \le (1-t)\det^{\frac{1}{n}}(A) + t\det^{\frac{1}{n}}(B) + 2R\Big(\det^{\frac{1}{n}}(\frac{A+B}{2}) - \frac{\det^{\frac{1}{n}}(A)+\det^{\frac{1}{n}}(B)}{2}\Big). \qquad (2.5)$$

Replacing A and B by A/(1 − t) and t/B in (2.5), respectively, we obtain the desired result.

As a consequence of this, we have the following interesting observation.

Corollary 2.1. Let A, B ∈ Mn be positive definite. Then

$$\det^{\frac{1}{n}}(A+B) \le \det^{\frac{1}{n}}(B)\big(1+\frac{1}{n}tr(AB^{-1})\big),$$

and

$$\det^{\frac{1}{n}}(A+B) \le \det^{\frac{1}{n}}(A)\big(1+\frac{1}{n}tr(BA^{-1})\big).$$

Proof. From Theorem 2.2, we have

$$\det^{\frac{1}{n}}(A+B) \le \det^{\frac{1}{n}}(A) + \det^{\frac{1}{n}}(B) + \frac{R}{t(1-t)}\Big(\det^{\frac{1}{n}}(tA+(1-t)B) - t\det^{\frac{1}{n}}(A) - (1-t)\det^{\frac{1}{n}}(B)\Big),$$

for 0 < t < 1. In particular, this is still true when t → 0+ and t → 1−. Now, we evaluate these limits. Notice that when t → 0+, R = 1 − t, and hence (2) becomes (when t → 0+)

$$\begin{aligned}\det^{\frac{1}{n}}(A+B) &\le \det^{\frac{1}{n}}(A)+\det^{\frac{1}{n}}(B)\\ &\quad+\frac{1}{t}\left(\det^{\frac{1}{n}}(tA+(1-t)B)-t\det^{\frac{1}{n}}(A)-(1-t)\det^{\frac{1}{n}}(B)\right)\\ &=\det^{\frac{1}{n}}(B)+\frac{\det^{\frac{1}{n}}(tA+(1-t)B)-(1-t)\det^{\frac{1}{n}}(B)}{t}. \qquad (2.6)\end{aligned}$$

We evaluate

$$\begin{aligned}&\lim_{t\to 0^+}\frac{\det^{\frac{1}{n}}(tA+(1-t)B)-(1-t)\det^{\frac{1}{n}}(B)}{t}\\ &=\det^{\frac{1}{n}}(B)\lim_{t\to 0^+}\frac{\det^{\frac{1}{n}}\left(tB^{-\frac{1}{2}}AB^{-\frac{1}{2}}+(1-t)I\right)-1+t}{t}\\ &=\det^{\frac{1}{n}}(B)\lim_{t\to 0^+}\frac{\left(\prod_{i=1}^{n}\lambda_i\left(tB^{-\frac{1}{2}}AB^{-\frac{1}{2}}+(1-t)I\right)\right)^{\frac{1}{n}}-1+t}{t}\\ &=\det^{\frac{1}{n}}(B)\lim_{t\to 0^+}\frac{\left(\prod_{i=1}^{n}t\lambda_i(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})+1-t\right)^{\frac{1}{n}}}{t}\\ &=\det^{\frac{1}{n}}(B)\lim_{t\to 0^+}\left\{1+\frac{1}{n}\left(\prod_{i=1}^{n}t\lambda_i(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})+1-t\right)^{\frac{1}{n}-1}\right.\\ &\qquad\left.\times\sum_{j=1}^{n}\left[\left(\lambda_j(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})-1\right)\prod_{i\neq j}\left(t\lambda_i(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})+1-t\right)\right]\right\}\end{aligned}$$

Thus

$$\begin{aligned}&\lim_{t\to 0^+}\frac{\det^{\frac{1}{n}}(tA+(1-t)B)-(1-t)\det^{\frac{1}{n}}(B)}{t}\\ &=\det^{\frac{1}{n}}(B)\left\{1+\frac{1}{n}\sum_{j=1}^{n}\left(\lambda_j(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})-1\right)\right\}\\ &=\frac{1}{n}\det^{\frac{1}{n}}(B)\sum_{j=1}^{n}\lambda_j(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})=\frac{1}{n}\det^{\frac{1}{n}}(B)\mathrm{tr}(AB^{-1}).\end{aligned}$$

Consequently, (2.6) becomes

$$\det^{\frac{1}{n}}(A+B)\le\det^{\frac{1}{n}}(B)+\frac{1}{n}\det^{\frac{1}{n}}(B)\mathrm{tr}(AB^{-1}),$$

which implies the first desired inequality. The second inequality follows similarly, by taking $t\to 1-$.

We should notice here that due to (2.4), we have

$$\begin{aligned}\det^{\frac{1}{n}}(A+B) &= \det^{\frac{1}{n}}(B)\det^{\frac{1}{n}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}+I\right)\\ &\geq \det^{\frac{1}{n}}(B)\left\{\det^{\frac{1}{n}}\left(AB^{-1}\right)+1\right\}.\end{aligned}$$

This observation and Corollary 2.1 imply the following double-sided inequality.

Corollary 2.2. Let A, B ∈ Mn be positive definite. Then

$$\det^{\frac{1}{n}}(B)\left\{1+\det^{\frac{1}{n}}\left(AB^{-1}\right)\right\} \leq \det^{\frac{1}{n}}(A+B) \leq \det^{\frac{1}{n}}(B)\left(1+\frac{1}{n}tr(AB^{-1})\right).$$

The significance of Corollary2.2is the fact that $\det^{\frac{1}{n}}(AB^{-1}) \leq \frac{1}{n}\mathrm{tr}(AB^{-1})$, implying that

$$\det^{\frac{1}{n}}(B)\left\{1+\det^{\frac{1}{n}}\left(AB^{-1}\right)\right\} \leq \det^{\frac{1}{n}}(B)\left(1+\frac{1}{n}\mathrm{tr}(AB^{-1})\right).$$

Consequently, Corollary2.2provides an intermediate-term to the above inequality!

Further, recalling [1, Proposition II.3.20] stating that

$$\det^{\frac{1}{n}}(A) = \min\left\{\frac{\mathrm{tr}(AB)}{n} : B \text{ is positive definite and } \det(B)=1\right\},$$

and noting that $\det\left(\frac{B^{-1}}{\det^{\frac{1}{n}}(B^{-1})}\right)=1$, we have

$$\begin{aligned}\det^{\frac{1}{n}}(A+B) &\leq \frac{1}{n}\mathrm{tr}\left((A+B)\frac{B^{-1}}{\det^{\frac{1}{n}}(B^{-1})}\right)\\ &= \frac{\det^{\frac{1}{n}}(B)}{n}\mathrm{tr}(AB^{-1}+I)\\ &= \frac{\det^{\frac{1}{n}}(B)}{n}(\mathrm{tr}(AB^{-1})+n)\\ &= \det^{\frac{1}{n}}(B)\left(1+\frac{1}{n}\mathrm{tr}(AB^{-1})\right);\end{aligned}$$

which has been shown differently in Corollary2.1.

Another observation that follows from Corollary2.1is when B = ∈I, for ∈ > 0. By Corollary2.1, we have

$$\det^{\frac{1}{n}}(A+\epsilon I) \leq \det^{\frac{1}{n}}(\epsilon I)\left(1+\frac{1}{n}\mathrm{tr}(\epsilon^{-1}A)\right) = \epsilon\left(1+\frac{1}{n\epsilon}\mathrm{tr}(A)\right).$$

Letting s → 0+ implies the well known inequality for the positive definite matrix A that

$$\det^{\frac{1}{n}}A \leq \frac{1}{n}\mathrm{tr}(A).$$

The last observation we would like to pay the attention of the reader to is how we can use Corollary 2.1 to prove [1, Proposition II.3.20]. This discussion acknowledges the concavity discussion of the function $A \mapsto \det^{\frac{1}{n}}(A)$. Concavity of this function implies the superadditivity inequality (2.4) and Corollary2.1.

These two consequences imply

$$\begin{aligned} \det^{\frac{1}{n}}(B) + \frac{1}{n}\mathrm{tr}(\det^{\frac{1}{n}}(B)\cdot AB^{-1}) &= \det^{\frac{1}{n}}(B)\left(1+\frac{1}{n}\mathrm{tr}(AB^{-1})\right) \\ &\geq \det^{\frac{1}{n}}(A+B) \\ &\geq \det^{\frac{1}{n}}(A) + \det^{\frac{1}{n}}(B). \end{aligned}$$

This implies that if A ∈ Mn is positive definite, then for any other positive definite matrix B with determinant 1; one has

$$\det^{\frac{1}{n}}(A) \leq \frac{1}{n}\mathrm{tr}(AB).$$

Since this is true for any positive B with det(B) = 1, we may write

$$\det^{\frac{1}{n}}(A) \leq \min\left\{\frac{\mathrm{tr}(AB)}{n} : B \text{ is positive with } \det(B) = 1\right\}.$$

Since det $\left(\det^{\frac{1}{n}}(A)\cdot A^{-1}\right) = 1,$ replacing B in the above inequality by $\det^{\frac{1}{n}}(A)\cdot A^{-1}$ implies an identity, which proves the well known identity [1, Proposition II.3.20]

$$\det^{\frac{1}{n}}(A) = \min\left\{\frac{\mathrm{tr}(AB)}{n} : B \text{ is positive with } \det(B) = 1\right\}.$$

Therefore, the concavity approach we adopted in the above results provides an alternative way to prove some well-known determinant identities.

For the rest of this section, we present sub and super additive determinantal inequalities for matrices of the form A + iB, where A, B ∈ Mn are positive (i.e., accretive-dissipative matrices) and i2 = −1. Although the approach differs from the above approach, we find it convenient to add these results as they resemble the same theme of this paper. The following two lemmas will be needed.

Lemma 2.1. ([6]) Let A, B ∈ Mn be positive definite. Then

$$|\det(A+\mathrm{i}B)| \leq \det(A+B) \leq 2^{\frac{n}{2}}|\det(A+\mathrm{i}B)|. \qquad (2.7)$$

Lemma 2.2. ([5]) (Fan's determinant inequality) Let H, K ∈ Mn be Hermitian and let A = H + iK, If H is positive definite, then

$$|\det(H+\mathrm{i}K)|^{\frac{2}{n}} \geq \det{}^{\frac{2}{n}} H + \det{}^{\frac{2}{n}} K,$$

with equality if and only if all the eigenvalues of H−1K have the same absolute value.

The following lemma is shown in [13, p. 48], but here we present its extension to complex matrices.

Lemma 2.3. Let A, B ∈ Mn be positive definite. Then

$$|\det(A+\mathrm{i}B)|^2 = \begin{vmatrix} A & -B \\ B & A \end{vmatrix} = \det A \det\left(A + BA^{-1}B\right).$$

The first result in this direction is a simple proof of Fan's determinant inequality.

Theorem 2.3. Let A, B ∈ Mn be positive definite. Then

$$|\det(A+\mathrm{i}B)|^{\frac{2}{n}} \geq \det{}^{\frac{2}{n}} A + \det{}^{\frac{2}{n}} B.$$

Proof. We have

$$\begin{aligned} |\det(A+\mathrm{i}B)|^{\frac{2}{n}} &= \begin{vmatrix} A & -B \\ B & A \end{vmatrix}^{\frac{1}{n}} \\ &= \det{}^{\frac{1}{n}} A . \det{}^{\frac{1}{n}} \left(A + BA^{-1}B\right) \quad \text{(by Lemma 2.3)} \\ &\geq \det{}^{\frac{1}{n}} A . \left(\det{}^{\frac{1}{n}} A + \det{}^{\frac{1}{n}} \left(BA^{-1}B\right)\right) \quad \text{(by (2.4))} \\ &= \det{}^{\frac{2}{n}} A + \det{}^{\frac{2}{n}} B, \end{aligned}$$

which completes the proof.

The following proposition aims to present an upper bound for |det (A + iB)| .

Proposition 2.1. Let A, B ∈ Mn be positive definite. Then

$$|\det(A+\mathrm{i}B)| \leq \det\left(A + \frac{BA^{-1}B}{2}\right). \tag{2.8}$$

Proof. From Lemma 2.3, we have

$$\begin{aligned} |\det(A+\mathrm{i}B)| &= \begin{vmatrix} A & -B \\ B & A \end{vmatrix}^{\frac{1}{2}} \\ &= \det{}^{\frac{1}{2}} A . \det{}^{\frac{1}{2}} \left(A + BA^{-1}B\right) \\ &\leq \det\left(A + \frac{BA^{-1}B}{2}\right), \end{aligned}$$

where we have used Young's inequality to obtain the last inequality. This completes the proof.

Remark 2.1. We would like to mention that the inequality (2.8) can be regarded as an improvement the left side of (2.7). When B ≤ 2A, we get

$$|\det(A+iB)| \leq \det\left(A+\frac{BA^{-1}B}{2}\right) \leq \det(A+B).$$

Let A, B ∈ Mn be positive definite. It is well known that (e.g., [5, p. 511])

$$\det(A + B) \geq \det A + \det B. \qquad (2.9)$$

Haynsworth obtained the following refinement of this inequality.

Theorem 2.4. [4, Theorem 3] Suppose A, B ∈ Mn are positive definite. Let Ak and Bk , k = 1, 2, ..., n−1, denote the k-th principal submatrices of A and B respectively. Then

$$\det(A+B) \geq \left(1+\sum_{k=1}^{n-1}\frac{\det B_k}{\det A_k}\right)\det A + \left(1+\sum_{k=1}^{n-1}\frac{\det A_k}{\det B_k}\right)\det B. \qquad (2.10)$$

Hartel [3] proved an improvement of (2.10) as follows:

$$\det(A+B) \geq \left(1+\sum_{k=1}^{n-1}\frac{\det B_k}{\det A_k}\right)\det A + \left(1+\sum_{k=1}^{n-1}\frac{\det A_k}{\det B_k}\right)\det B + (2^n-2n)\sqrt{\det AB}.$$

As a direct result, Hartel also presented the following inequality

$$\det(A + B) \geq \det A + \det B + (2n - 2)\sqrt{\det AB}. \qquad (2.11)$$

Now, we present the complex version of (2.11).

Theorem 2.5. Let A, B ∈ Mn be positive definite. Then

$$|\det(A + iB)|^2 \geq \det^2 A + \det^2 B + (2^n - 2) \det AB.$$

Proof. By Lemma 2.3 and (2.11), we have

$$\begin{aligned}|\det(A+iB)|^2 &= \det A.\det\left(A+BA^{-1}B\right)\\ &\geq \det A.\left(\det A+\det\left(BA^{-1}B\right)+(2^n-2)\det^{\frac{1}{2}}A.\det^{\frac{1}{2}}\left(BA^{-1}B\right)\right)\\ &= \det^2 A+\det^2 B+(2^n-2)\det AB.\end{aligned}$$

The desired conclusion follows.

Remark 2.2. If n ≥ 2, then (2n − 2) ≥ 2. Therefore by Theorem2.5,

$$\begin{aligned}|\det(A+iB)|^2 &\geq \det^2 A+\det^2 B+(2^n-2)\det AB\\ &\geq \det^2 A+\det^2 B+2\det AB\\ &= (\det A+\det B)^2.\end{aligned}$$

This implies the superadditive inequality

$$|\det(A + iB)| \geq (\det A + \det B),$$

which is also an improvement of (2.9).

In the following result, we will present a Haynsworth-Hartfiel-type result for A + iB. For this, we will need the following two lemmas [13].

Lemma 2.4. Let A ∈ Mn be positive definite and Ai be a principal submatrix of A. Then

$$(A^{-1})_i \geq (A_i)^{-1}.$$

Lemma 2.5. Let A ∈ Mn be positive definite. Then for any B ∈ Mn,

$$(B^*)_i (A_i)^{-1} (B^*)_i \leq \left(B^* A^{-1} B\right)_i.$$

Now we are ready for the following main result.

Theorem 2.6. Let A, B ∈ Mn be positive definite. Then

$$\begin{aligned} |\det(A + \mathrm{i}B)|^2 \geq & \left(1 + \sum_{i=1}^{n-1} \frac{\det^{-1}\left(B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}}\right)_i}{\det\left(B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}}\right)_i}\right) \det^2 B \\ & + \left(1 + \sum_{i=1}^{n-1} \frac{\det^{-1}\left(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\right)_i}{\det\left(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\right)_i}\right) \det^2 A \\ & + (2^n - 2n) \,.\, \det A \det B. \end{aligned}$$

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Proof. We can write

$$\begin{aligned}|\det(A+\mathrm{i}B)|^2 &= \det A \det\left(A+BA^{-1}B\right)\\ &= \det A \det B \det\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}+B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)\\ &\geq \det A \det B\left(1+\sum_{i=1}^{n-1}\frac{\det\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)_i}{\det\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)_i}\right)\\ &\quad\cdot\det\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)+\left(1+\sum_{i=1}^{n-1}\frac{\det\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)_i}{\det\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)_i}\right)\\ &\quad\cdot\det\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)+\left(2^n-2n\right).\end{aligned}$$

Thus,

$$\begin{aligned}|\det(A+\mathrm{i}B)|^2 \geq &\left(1+\sum_{i=1}^{n-1}\frac{\det^{-1}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)_i}{\det\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)_i}\right)\det^2 B\\ &+\left(1+\sum_{i=1}^{n-1}\frac{\det^{-1}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)_i}{\det\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)_i}\right)\det^2 A\\ &+\left(2^n-2n\right).\det A\det B,\end{aligned}$$

as desired. □

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Compliance with Ethical Standards

Conflict of interest

All authors declare that they have no conflict of interest.

Ethical approval

This article does not contain any studies with human participants or animals performed by any authors.

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