UtilitasMathematica
ISSN 0315-3681 Volume 120, 2023

An exponential Diophantine equation involving Narayana cow's numbers

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Abstract

Let (Cn)n≤0 be the Narayana cow's sequence given by C0 = Cfi = C2 = fi and Cn‡3 = Cn‡2 ‡ Cn for n ≤ O. In this paper, we are interested in finding all powers of two which are sums of two Narayana cow's numbers, i.e.,
we detemine all solutions of the exponential Diophantine equation Cm ‡ Cn = Xs in nonnegative integers n, m, and s ≤ X.

Keywords: Padovan numbers, Narayana cow' s numbers, Linear form in logarithms, Reduction method.

1. Introduction

Narayana's cows sequence $(C_n)_{n \leq O}$ originated from a herd of cows and calves problem, proposed by the Indian mathematician Narayana Pandit in his book Ganita Kaumudi [fi]. It is the sequence A000930 in the OEIS [fi2] satisfying the recurrence relation

$$C_{n+3} = C_{n+2} + C_n, \quad n \leq 1$$

for initial values $C_O = C_1 = C_2 = 1$.

In the literature, there are several results dealing with Diophantine equations involving factorials, repdigits, and recurrence sequences. In the recent past, the study of Narayana's cows sequence has been a source of attraction for many authors. For instance, Bravo et al. [4] searched for the presence of repdigits in Narayana's cows sequence. They also obtained results on the existence of Mersenne prime numbers and numbers with distinct blocks of digits in this sequence.

The determination of perfect powers of Lucas and Fibonacci sequences does not date from today. The real contribution of determination of perfect powers of Lucas and Fibonacci sequences began in 2006. By classical and modular approaches of Diophantine equations, Bugeaud, Mignotte, and Siksek [8] defined powers of Lucas and Fibonacci sequences by solving the equations $F_n = y^p$ and $G_n = y^p$ prespectively. From there, many researchers tackled similar problems. It is in the same thought that, others have determined the powers of 2 of the sum/difference of two Lucas numbers [6], powers of 2 of the sum/difference of Fibonacci numbers [F], powers of 2 and of 3 of the product of Pell numbers and Fibonacci numbers.

In this note, we move our interest on the Narayana cows sequence, in an other word, we are interested in finding all powers of two which are sums of two Narayana numbers, i.e., we study the exponential Diophantine equation (1.1) Cn + Cm = 2s,

in nonnegative integers n, m and s ≥ 2. The proof of our main theorem uses lower bounds for linear forms in logarithms, properties of continued fractions, and a version of the Baker-Davenport reduction method in Diophantine approximation.

2. Preliminaries

First, we recall some facts and properties of the Narayana cow sequence (Cn)n≥O, which will be used later. One can also see [1]. The characteristic equation

x3 — x2 — 1 = 0

It has one real root α and two complex roots β and *y* = β. More precisely

$$\begin{aligned}
\alpha &= \frac{1}{3}\left(\sqrt[3]{\frac{1}{2}\left(29-3\sqrt{93}\right)+\frac{1}{2}\left(29+3\sqrt{93}\right)}+1\right),\\
\beta &= \frac{1}{3}-\frac{1}{6}\left(1-i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(29-3\sqrt{93}\right)}+\frac{1}{6}\left(1+i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(29+3\sqrt{93}\right)},\\
\gamma &= \frac{1}{3}-\frac{1}{6}\left(1+i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(29-3\sqrt{93}\right)}+\frac{1}{6}\left(1-i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(29+3\sqrt{93}\right)}.
\end{aligned}$$

For all n ≥ 0, the Narayana's cows sequence satisfying the following "Binet-like" formula

$$C_n = c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n, \tag{2.1}$$

Where

$$c_\alpha = \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)},\ c_\beta = \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)},\ c_\gamma = \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)}.$$

One can easy see that

$$\begin{aligned}
1.46 &< \alpha < 1.47,\\
0.82 &< |\beta| = |\gamma| < 0.83,\\
0.61 &< c_\alpha < 0.62,\\
0.57 &< |c_\beta| = |c_\gamma| < 0.58.
\end{aligned}$$

From the fact that $\beta = \alpha^{-1/2}e^{i\theta}$ and $\gamma = \alpha^{-1/2}e^{-i\theta}$, for some $\theta \in (0, 2\pi)$. One can see that

$$|C_n - c_\alpha \alpha^n| < \frac{1}{\alpha^{n/2}}, \tag{2.2}$$

holds for all n ≥ 1:

By the induction method, it is not di¢ cult to prove the following lemma.

Lemma 1. For all n ≥ 2; we have

$$\alpha^{n-2} < C_n < \alpha^{n-1}. \tag{2.3}$$

The formula (2.1) can be also written as

$$C_n = c'_\alpha \alpha^{n+2} + c'_\beta \beta^{n+2} + c'_\gamma \gamma^{n+2}, \tag{2.4}$$

where

$$c'_x = \frac{1}{x^3+1}, \quad x \in \{\alpha, \beta, \gamma\}.$$

DIOPHANTINE EQUATION Cn+Cm= 2^s

Furtheremore, we have

$$c'_\alpha \approx 0.1942 \cdots.$$

The next tools are related to the transcendental approach to solve Diophantine equations. For any non-zero algebraic number In order to prove our main result, we use a few times a Baker-type lower bound for a non-zero linear forms in logarithms of algebraic numbers. We state a result of Matveev [fifi] about the general lower bound for linear forms in logarithms, but first, recall some basic notations from algebraic number theory. Let 4 be an algebraic number of degree d with minimal primitive polynomial

$$f(X) := a_0 + a_1 X + ... + a_d X^d = a_d \prod_{i=1}^{d} \left(X - \eta^{(i)}\right) \in \mathbb{Z}[X].$$

where the a_i are relatively prime integers, $a_d > 0$ and and the $n^{(i)}$'s are conjugates of n. We denote by

$$h(\eta) = \frac{1}{d}\left(\log a_d + \sum_{i=1}^{d} \log\max\left(1, \left|\eta^{(i)}\right|\right)\right),$$

its absolute logarithmic height.

Lemma 2. For an algebraic numbers n and λ; we have

$$h(\eta \cdot \lambda) \leq h(\eta) + h(\lambda),$$

and

$$h(\eta + \lambda) \leq h(\eta) + h(\lambda) + \log 2,$$

Moreover, for any algebraic number $\xi \neq 0$ and for any m $\in$ Z; we have

$$h(\xi^m) = |m|\, h(\xi).$$

To prove Theorems 1 and 2, we use lower bounds for linear forms in logarithms to bound the index n appearing in equations (1.1) and (2). We need the following general lower bound for linear forms in logarithms due to Matveev [11].

LEMMA 3. (Matsees). Assume that n1, ..., nj are positive algebraic numbers in a rea1 algebaric fle1d k of degvee D, b1, ..., bj are rational integers, and

$$\Lambda := \eta_1^{b_1}, ..., \eta_j^{b_j} - 1$$

is no zero. Then

$$|\Lambda| \geq \exp\left(-1.4.30^{j+3}.j^{4.5}.D^2.A_1...A_j(1 + \log D)\right)(1 + \log B),$$

where

$$B \geq \max\left(|b_1|, ..., |b_1|\right)$$

and

$$A_i \geq \max\left(Dh\left(\eta_i\right), \log|b_i|, 0.16\right).$$

The next step is to reduce the bound of n, which is generally too large. To this end, we present a variant of the reduction method of Baker and Davenport, which was introduced by de Weger [13].

Let

$$\Gamma = \vartheta_1 x_1 + \vartheta_2 x_2 + \beta \tag{2.5}$$

where $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ are given and x1, x2 $\in$ Z are unknowns. Set X = max {$|x_1|$, $|x_1|$} and X0; Y be positive. Assume that

$$Y < X \leq X_0 \tag{2.6}$$

and

(2.7) $|T| < c\exp^{(-\delta \cdot Y)},$

Where $c.\, \delta$ be positive constants. When $\beta \neq 0$ in (2:5), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2.$ Then we have

$$\frac{\Gamma}{\vartheta_2} = \psi - \vartheta x_1 + x_2.$$

Let p/q be a convergent of 9 with q > X_0. For a real number x, we let ||x|| = min {|x – n| , n $\in \mathbb{Z}$} for the distance from x to the nearest integer. We may use the following Davenport Lemma.

LEMMA 4. Suppose that

$$||q\psi|| > \frac{2X_0}{q}.$$

Then the solution of (2:6) and (2:7) satisfy

$$Y < \frac{1}{\delta}\log\left(\frac{q^2 c}{|\vartheta_2| \cdot X_0}\right).$$

3. Main theorem

Our main result can be stated in the following theorem.

Theorem 1. The only solutions (n; m; s) of the exponential Diophantine equation

$$C_n + C_m = 2^s,$$

in nonnegative integers n $\geq$ m and s $\geq$ 2 are given by

(n, m, s) $\in$ {(3, 3, 2) (4, 0, 2) , (4, 1, 2) , (4, 1, 2) , (6, 3, 3) , (5, 5, 3) , (8, 4, 4) , (9, 8, 5) , (10, 5, 5) , (12, 5, 6)} .

First, we study the case n = m, next we assume n > m and study the case n $\leq$ 100 with Mathematicia in the range 0 $\leq$ m < n $\leq$ 100 and finally we study the case n > 100. Assume throughout that equation (1.1) holds.

In the case when s = 1, we get the solutions (n, m) with n, m $\in$ (0, 1, 2}. Then we assume from now that s $\geq$ 2.

Step 1: First of all, we assume that n = m, then the original equation (1.1) becomes

(3.1) $C_n = 2^{s-1},$

we check in the range n $\leq$ 100, we get the solutions (n, s) $\in$ {(3, 2), (5, 3)}. Now we assume that n > 100, let explore a relationship between n and s. For all n $\geq$ 2, we have

$$\alpha^{n-2} < C_n = 2^{s-1} < \alpha^{n-1} < 2^{n-1}, \tag{3.2}$$

Now takin Logarithm on both sides of the inequality (3.2), we obtain

$$(n-2)\frac{\log \alpha}{\log 2} + 1 < s < n.$$

Next, we rewrite equation (3.1) taking the Binet's formula (2.1) and taking the absolute values as

$$\left|2^{s-1} - c_\alpha \alpha^n\right| < \frac{1}{\alpha^{n/2}},$$

Dividing both sides of the last inequality by $c_\alpha \alpha^n$, we get

$$\left|2^{s-1} c_\alpha^{-1} \alpha^{-n} - 1\right| < \frac{1}{c_\alpha \alpha^{3n/2}} < \frac{2}{\alpha^{3n/2}} \tag{3.3}$$

Let put us

$$\Gamma_1 := 2^{s-1} c_\alpha^{-1} \alpha^{-n} - 1$$

One can see that

$$\log|\Gamma_1| \leq -3n \log \alpha \tag{3.4}$$

Now, we need to check T1 6$\neq$ 0: If T1 = 0; then

$$|c_\beta \beta^n| = 2^{s-1},$$

but $|c_\beta \beta^n| < |c_\beta| \approx 0407506... < 1$, whereas $2^{s-1} \geq 1$. Which is false, then $\Gamma_1 \neq 0$. We take

$$\eta_1 = \log 2,\ \eta_2 = \log \alpha,\ \ \eta_3 = \log\left(1/c_\alpha\right),\ \ b_1 = s-1,\ b_2 = -n,\ \ b_3 = -1.$$

where $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\alpha)$ and so we take $D = 3$. Also, $B = \max(s-1, n, 1) = n$. Furthermore,

$$h(\eta_1) = \log 2,\ h(\eta_2) = \frac{\log \alpha}{3},$$

The minimal polynominal of c_α is $31X^3 - 31X^2 + 9X - 1$. Therefore h(n3) = $h(c_\alpha) = \frac{\log 31}{3}$. Thus, we can take

$$A_1 = 2.08,\ A_2 = 0.39,\ A_3 = 3.44.$$

Applying Lemma (3), we get

$$\log|\Gamma_1| \leq -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2(1+\log 3)(1+\log n)(2.08)(0.39)(3.44).$$

Combining the above inequality with (3.4), it follows that

$$\frac{n}{1+\log n} < 1.3 \cdot 10^{13}.$$

We calculate, we get

$$n < 4.5 \,.\, 10^{14}.$$

In order to reduce the bound in n, we use Lemma (4). Recall that

$$\Gamma_1 := 2^{s-1} c_\alpha^{-1} \alpha^{-n} - 1 = e^{\Lambda_1} - 1$$

where

$$\Lambda_1 = (s-1)\log 2 - n\log\alpha - \log(c_\alpha).$$

The inequality (3:3) can be expressed as

$$\left|e^{\Lambda_1} - 1\right| = |\Gamma_1| \leq \frac{1}{\alpha^{3n/2}}. \tag{3.5}$$

One can see that $\Lambda_1 \neq 0$: Since n > 100; the right side of (3:5) is smaller than 1/2: Thus, we get

$$|\Lambda_1| \leq \frac{2}{\alpha^{3n/2}}$$

wich implies that

$$|(s-1)\log 2 - n\log\alpha - \log(c_\alpha)| < 2\exp\left(-1.5n\log\alpha\right),$$

now, we are in a situation to apply Lemma (4), for this, we take

$$X_0 = 4.5 \,.\, 10^{14}.$$

we also have

$$c = 2,\ \ \delta = \log\alpha,\ \ \beta = -\log\left(c_\alpha\right),$$

and

$$(\vartheta_1, \vartheta_2) = (-\log\alpha, \log 2), \ \vartheta = \frac{\log\alpha}{\log 2}, \quad \psi = \frac{-\log(c_\alpha)}{\log 2}.$$

We find $\frac{p_{38}}{q_{38}} = \frac{453285617800432}{821969096806723}$ is the 38th convergent of $\vartheta|$ satisfies $q_{38} > X_0$ and q_{39} = 3140144568890233 satisfies $||q\psi|| > \frac{2X_0}{q}$. Furthermore, Lemma (4) implies that

$$Y < \frac{1}{0.385\,262\,400\,8}\log\left(\frac{3140144568890233^2\cdot 2}{0.555\,816\,155\,1\cdot 4.5\cdot 10^{14}}\right) \leq 100.$$

which contradicts our assumption that n > 100:

Step 2: If n $\leq$ 100, the search with Mathematicia in the range 0 < m < n $\leq$ 100 gives the solutions cited in theorem fi. Now, we assume that n > 100, let first get a relation between s and n which is important for our purpose. Combining (1.1) and the right inequality of (2.3), we get:

$$2^{\frac{m}{2}} < 2\left(\sqrt{2}\right)^{m-1} < \alpha^{n-1} + \alpha^{m-1} < 2^s = C_n + C_m < \alpha^{n-1} + \alpha^{m-1} < 2\alpha^{n-1} < 2^n.$$

Taking Logarithm on both sides of the above inequality, we get

$$\frac{m}{2} < s < n.$$

Rewriting equation (1:1) as

$$c'_\alpha\alpha^{n+2} - 2^s = -C_m - c'_\beta\beta^{n+2} - c'_\gamma\gamma^{n+2}, \tag{3.6}$$

but, we have for all n $\geq$ 1;

$$\left|c'_\alpha\alpha^{n+2} - 2^s\right| \leq C_m + \frac{1}{2}$$

so the inequality (3:6) becomes

$$\left|c'_\alpha\alpha^{n+2} - 2^s\right| \leq C_m + \frac{1}{2}$$

Dividing both sides of above inequality, we have

$$\begin{aligned}\left|1-\frac{2^s}{c'_\alpha\alpha^{n+2}}\right| &\leq \frac{C_m}{c'_\alpha\alpha^{n+2}}+\frac{1}{2c'_\alpha\alpha^{n+2}}<\frac{\alpha^{m-1}}{c'_\alpha\alpha^{n+2}}+\frac{1}{2c'_\alpha\alpha^{n+2}}\\ &< \frac{5.1479\alpha^{m-1}}{\alpha^{n+2}}+\frac{2.5739}{\alpha^{n+2}}.\end{aligned}$$

Since $\alpha > 1.45$, then $5.1479\alpha^{m-1} + 2.5739 < 7\alpha^{m-1}$ for all $m > 0$, and so

$$\left|2^s\cdot\left(\frac{1}{c'_\alpha}\right)\cdot\alpha^{-(n+2)}-1\right|<\frac{9}{\alpha^{n-m}}. \tag{3.7}$$

We put

$$\Gamma_2 = 2^s\cdot\left(\frac{1}{c'_\alpha}\right)\cdot\alpha^{-(n+2)}-1$$

Similarly as in the case of T1 above, we show that T2 $\neq 0$: Now, we apply Matveev's result Lemma 4, we take j = 3; and

$$\eta_1=\log 2,\ \eta_2=\log\alpha,\ \ \eta_3=\log\left(1/c'_\alpha\right),\ \ b_1=s,\ b_2=-\left(n+2\right),\ \ b_3=1.$$

where $\eta_1,\eta_2,\eta_3\in\mathbb{Q}\left(\alpha\right)$ and so we take $D=3$. Also, $B=\max\left(s,n+2,1\right)=n+2$. Furthermore,

$$h(\eta_1)=\log 2,\ h(\eta_2)=\frac{\log\alpha}{3},$$

The minimal polynomial of c'_α is $31X^3-31X^2+9X-1$. Therefore, $h(\eta_3)=h(c'_\alpha)=\dfrac{\log 31}{3}$. Thus, we can take

$$A_1=2.08,\ A_2=0.39,\ A_3=3.44.$$

Applying Lemma (3), we get

$$\begin{aligned}\text{Log }|T1| &\geq -7.55.10^{12}\,(1+\log(n+2))\\ &> -7.55.10^{12}\,(1+\log n)\end{aligned}$$

Combining the above inequality with (3.7), it follows that

$$\frac{n-m}{1+\log n}<2.2\cdot 10^{12}. \tag{3.8}$$

Let us find a second linear form in Logarithm. For this, we rewrite equation (1.1) as follows

$$\begin{aligned} 2^s &= |C_m + C_n| = |C_m - c_\alpha\alpha^m + C_n - c_\alpha\alpha^n + c_\alpha\alpha^m + c_\alpha\alpha^n| \\ &\leq |C_m - c_\alpha\alpha^m| + |C_n - c_\alpha\alpha^n| + c_\alpha(\alpha^m + \alpha^n) \\ &\leq \frac{1}{\alpha^{m/2}} + \frac{1}{\alpha^{n/2}} + c_\alpha(\alpha^m + \alpha^n) \\ &\leq \frac{2}{\alpha^{m/2}} + 2c_\alpha\alpha^n \end{aligned}$$

Taking the absolute value in above inequality, we get

$$(3.9) \qquad \left|2^{s-1} \cdot (1/c_\alpha) \cdot \alpha^{-n} - 1\right| \leq \frac{1}{\alpha^{n+m/2}} \text{ i.e } |\Gamma_3| \leq \frac{1}{\alpha^{n+m/2}},$$

where

$$\Gamma_3 = 2^{s-1} \cdot (1/c_\alpha) \cdot \alpha^{-n} - 1.$$

All the conditions are now met to apply a Matveev's theorem (Lemma 4), where

$$\eta_1 = \log 2, \eta_2 = \log|\alpha, \ \eta_3 = \log(1/c_\alpha), \ b_1 = s - 1, \ b_2 = -n, \ b_3 = 1.$$

where $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\alpha)$ and so we take $D = 3$. Also, $B = \max(s, n, 1) = n$. Furthermore,

$$h(\eta_1) = \log 2, \ h(\eta_2) = \frac{\log \alpha}{3}, \ h(c_\alpha) = \frac{\log 31}{3}$$

Thus, we can take

A1 = 2:08; A2 = 0:39; A3 = 3:44:

Now, Lemma 3 gives us the following estimate:

$$\log|\Gamma_3| \geq -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2 (1 + \log 3)(1 + \log n)(2.08)(0.39)(3.44)|$$

it follows

$$\log|\Gamma_3| \geq -7.546\,790\,878 \times 10^{12} \cdot (1 + \log n)$$

Combining the above inequality with (3.9), we get

$$(3.10) \qquad \frac{2n + m}{1 + \log(n+2)} < 3.94 \cdot 10^{13}$$

By (3.8) and (3.10), we have

$$\frac{3n}{1+\log n} < 4.16 \times 10^{13}.$$

and so

$$n < 4.83 \cdot 10^{14}.$$

Now, we can apply the Lemma 4, similary as in step 1. For this, we take

$$X_0 = 4.83 \cdot 10^{14},$$

we also have

$$c = 2,\ \delta = \log \alpha,\ \beta = -\log (c_\alpha),$$

and

$$(\vartheta_1, \vartheta_2) = (-\log \alpha, \log 2),\ \vartheta = \frac{\log \alpha}{\log 2},\quad \psi = \frac{-\log (c_\alpha)}{\log 2}.$$

We find $\frac{p_{38}}{q_{38}} = \frac{453285617800432}{821969096806723}$ is the 38th convergent of ϑ satisfies q38 > X_0 and q39= 3140144568890233 satisfies $||q\psi|| > \frac{2X_0}{q}$. Furthermore, Lemma (4) implies that

$$Y < \frac{1}{0.385\,262\,400\,8} \log \left(\frac{3140144568890233^2 \cdot 2}{0.555\,816\,155\,1 \cdot 4.83 \cdot 10^{14}} \right) \leq 100.$$

which contradicts our assumption that n > 100: This proves our theorem.

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