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On the conjectures of Liu, Laine and Zhang, Kang, Liao

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Abstract

In the paper, we deal with the uniqueness problem of entire function sharing an entire function with its higher order difference operators and obtain two results which confirm the conjectures posed by Liu and Laine [8] and Zhang, Kang and Liao [10] respectively. Also we exhibit some relevant examples to fortify our results.

Keywords: Difference operators, sharing values, entire functions.

1. Introduction

In this paper, a meromorphic function f always means it is meromorphic in the whole complex plane C. We assume that the reader is familiar with standard notation and main results of Nevanlinna Theory (see, e.g., [3, 11]). By $S(r, f)$ we denote any quantity that satisfies the condition $S(r, f) = o(T(r, f))$ as r possibly outside of an exceptional set of finite logarithmic measure. A meromorphic function a is said to be a small function of f if $T(r, a) = S(r, f)$. Moreover, we use notations $\rho(f)$, $\rho1(f)$, $\mu(f)$ and $\lambda(f)$ for the order, the hyper-order, the lower order and the exponent of convergence of zeros of a meromorphic function f respectively. If $\mu(f) = \rho(f)$, we say that f is of regular growth. We know that if $f = eg$, where g is a polynomial, then $\rho(f) = \mu(f) = \deg(g)$. Let f and g be two non-constant meromorphic functions in C such that $\rho(f) < \mu(g)$. Then we know that $T(r, f) = o(T(r, g)$ ($r \to \infty$) (see Theorem 1.18 [11]). As usual, the abbreviation CM means counting multiplicities, while IM means ignoring multiplicities.

We now introduce some notations. Let $c \in C \setminus \{0\}$. Then the forward difference $\Delta_c^n f$ for each integer $n \in N$ is defined in the standard way by

$$\Delta_c^1 f(z) = \Delta_c f(z) = f(z+c) - f(z)$$
$$\Delta_c^n f(z) = \Delta_c\left(\Delta_c^{n-1} f(z)\right) = \Delta_c^{n-1} f(z+c) - \Delta_c^{n-1} f(z), \quad n \geq 2.$$

Moreover

$$\Delta_c^n f(z) = \sum_{j=0}^{n} (-1)^{n-j} C_n^j f(z+jc),$$

where C_n^j is a combinatorial number. If an equation includes shifts or differences of f , then the equation is called difference equation.

We recall the following conjecture proposed by Bru¨ck [1].

Conjecture A. Let f be a non-constant entire function such that ρ1(f) f∈ N ∪ {∞}. If f and f' share one finite value a CM, then f' − a = c(f − a), where c ∈ C \ {0}.

Though the conjecture is not settled in its full generality, it gives rise to a long course of research on the uniqueness of entire and meromorphic functions sharing a single value with its derivatives.

Meromorphic solutions of complex difference equations, and the value distribution and the uniqueness of complex differences have become an area of current interest and the study is based on the Nevanlinna value distribution of difference operators established by Halburd and Korhonen [4] and by Chiang and Feng [2] respectively. Recently, many authors (see [5, 6, 8, 10]) have started to consider the sharing values problems of meromorphic functions with their difference operators or shifts. Also it is well known that Δcf can be regarded as the difference counterpart of f'. Now, we recall the following result due to Heittokangas et al. [5], which is difference analogue of the Bru¨ck conjecture.

Theorem A. [5] Let f be a non-constant meromorphic function with ρ(f) < 2 and c ∈ C. If f (z) and f (z+c) share the values a ∈ C and ∞, then f (z+c) - a = τ (f (z) - a) holds for some constant τ.

In the same paper, they also exhibited the following example to show that the condition "ρ(f) < 2" can not be relaxed to "ρ(f) ≤ 2".

Example A. Let $f(z) = e^{z^2} + 1$ and c ∈ C \ {0}. Clearly f (z) and f (z + c) share 1 and ∞ CM,

$$\frac{f(z+c)-1}{f(z)-1} = e^{2cz+c^2} \neq \ constant.$$

Using the same restrictive condition “$\rho(f) < 2$” in Theorem A, Liu and Laine [8] obtained the following result.

Theorem B. [8] Let f be a transcendental entire function of order $\rho(f) < 2$ not having period c. If f and $\Delta_c^n f$ share the value 0 CM, then $\Delta_c^n f = \tau f$, where $\tau \in C \setminus \{0\}$.

Next we consider the following example.

Example B. [8] Let $f(z) = Ae^{z \log(c+1)} - ((1 - c)/c)$, where $c \in R \setminus \{0\}$, $c > -1$ and A is an arbitrary constant. Then we get $(\Delta_1 f(z) - 1) = c(f(z) - 1)$.

Therefore Example A suggests that $(\Delta^n f - a)/(f - a)$ may reduce to a non-zero constant, at least if $\rho(f) = 1$ and $N(r, 1/f) \neq S(r, f)$. Now motivated by Example A, Liu and Laine [8] posed the following conjecture (see Remark 1.4 [8]).

Conjecture B. Let f be a transcendental entire function of order $1 < \rho(f) < \infty$ not having period c. If f and $\Delta_c^n f$ share the value $a \in C\setminus\{0\}$ CM, then $\Delta_c^n f - a = \tau(f - a)$, where $\tau \in C \setminus \{0\}$.

In 2015, Zhang, Kang and Liao [10] settled Conjecture B partially under the conditions “$\rho(f) < 2$” and “$\lambda(f - a) < \rho(f)$”. Actually they obtained the following result.

Theorem C. [10] Let f be a transcendental entire function such that $\rho(f) < 2$ and $\alpha(f \not\equiv 0)$ be an entire function such that $\rho(\alpha) < \rho(f)$ and $\lambda(f - \alpha) < \rho(f)$. If $f - \alpha$ and $\Delta_1^n f - \alpha$ share 0 CM, then α is a polynomial of degree at most $n - 1$ and f must be of form $f(z) = \alpha(z) + H(z)e^{dz}$, where H is a polynomial such that $cH = -\alpha$ and $c, d \in C \setminus \{0\}$ such that $c^d = 1$.

In the same paper, Zhang, Kang and Liao [10] made the following comment:

“We are not sure whether the assumption $\alpha \not\equiv 0$ in Theorem 1.2 [10] is necessary or not.”

Thereby Jhang, Kang and Liao [10] posed the following conjecture.

Conjecture C. Let f be a transcendental entire function such that $\rho(f) < 2$ and $\lambda(f) < \rho(f)$. If f and $\Delta^n f$ share 0 CM, then f must be of the form $f(z) = ce^{dz}$, where $c, d \in C \setminus \{0\}$.

To the knowledge of authors Conjectures B and C are not still confirmed. In the paper, we have settled both Conjectures B and C. On the other hand we have improved Theorems B and C by removing the condition “$\rho(f) < 2$”

We now state our main results.

Theorem 1.1. Let f be a transcendental entire function such that $\rho(f) < +\infty$, $c \in C \setminus \{0\}$ such that $\Delta_c^n f \not\equiv 0$ and $\alpha(\not\equiv 0)$ be an entire function such that $\rho(\alpha) < \rho(f)$ and $\lambda(f - \alpha) < \rho(f)$. If $f - \alpha$ and $\Delta_c^n f - \alpha$ share 0 CM, then α is a polynomial of degree at most $n - 1$ and f must be of form $f(z) = \alpha(z) + H(z)e^{dz}$, where H is a polynomial such that $cH = -\alpha$ and $c, d \in C \setminus \{0\}$ such that $c^d = 1$.

Theorem 1.2. Let f be a transcendental entire function such that $\rho(f) < +\infty$, $c \in C \setminus \{0\}$ such that $\Delta_c^n f$ $f\equiv 0$ and $\lambda(f) < \rho(f)$. If f and $\Delta_c^n f$ share 0 CM, then f must be of form $f(z) = c0e^{dz}$, where $c0, d \in C \setminus \{0\}$.

Following example shows that condition “$\lambda(f - \alpha) < \rho(f)$” in Theorem 1.1 is sharp.

Example 1.1. Let $f(z) = e^z$ and $c = \log 2$. Note that

$$\begin{aligned}
\Delta_c^n f(z) = \sum_{j=0}^{n}(-1)^j C_n^j f(z+(n-j)c) &= e^z \sum_{j=0}^{n}(-1)^j C_n^j e^{(n-j)c} \\
&= \left(e^{nc} - C_n^1 e^{(n-1)c} + \ldots + (-1)^n\right) e^z \\
&= (e^c - 1)^n e^z = e^z.
\end{aligned}$$

Therefore $\Delta_c^n f \equiv f$ and so f and $\Delta_c^n f$ share 1 CM. Since 0 and ∞ are the Borel exceptional values of f, it follows that 1 can not be a Borel exceptional value of f and so $\lambda(f - 1) = \rho(f)$. Clearly f does not satisfies the conclusion of Theorem 1.1.

Following examples show that condition “$\lambda(f) < \rho(f)$” in Theorem 1.2 is sharp.

Example 1.2. Let $f(z) = (\exp z - 1) \exp \log \left(\frac{\log(1+\tau)}{c} z\right)$, where log denotes the principal branch of the logarithm and $c = 2\pi i$ such that $\log(1 + \tau)$ $f\neq c$. Note that

$$\begin{aligned}
\Delta_c f(z) &= (\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c}(z+c)\right) - (\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c} z\right) \\
&= (\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c} z\right)(\exp(\log(1+\tau)) - 1) \\
&= \tau(\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c} z\right) \\
&= \tau f(z).
\end{aligned}$$

Clearly f and $\Delta_c f$ share 0 CM. On the other hand, we see that $\rho(f) \leq 1$ and $\lambda(f) = \lambda(\exp z - 1) = 1$. Since $\lambda(f) \leq \rho(f)$, it follows that $\lambda(f) = \rho(f)$. Also it is clear that f does not satisfies the conclusion of Theorem 1.2.

Example 1.3. Let $f(z) = \sin z$. Note that $\lambda(f) = \rho(f)$ and $\Delta_\pi(f)$ and f share 0 CM, but f does not satisfy Theorem 1.2.

It is easy to see that condition “$\rho(\alpha) < \rho(f)$” in Theorem 1.1 is sharp.

Example 1.4. Let $f(z) = e^z + 1$, $\alpha(z) = e^z - 1$ and $c = \log 2$. Note that $\rho(\alpha) = \rho(f)$ and $\Delta_c f(z) = e^z$. Clearly $\lambda(f - \alpha) = 0 < \rho(f)$ and $f - \alpha$ and $\Delta_c f - \alpha$ share 0, but f does not satisfies the conclusion of Theorem 1.1.

Following example shows that condition “$\rho(f) < +\infty$” in Theorem 1.1 is necessary.

Example 1.5. Let $f(z) = e^z(e^{s(z)} - 1)$, where $s(z)$ is a periodic function with period $c = \log 2$ and $\alpha(z) = -e^z$. Clearly $\rho(f) = +\infty$. Note that $\Delta_c f = f$ and so $f - \alpha$ and $\Delta_c f - \alpha$ share 0 CM. On the other hand, we see that $\lambda(f - \alpha) = 0 < \rho(f)$, but f does not satisfies the conclusion of Theorem 1.1.

Following example shows that condition “$\rho(f) < +\infty$” in Theorem 1.2 is necessary.

Example 1.6. Let $f(z) = e^z e^{s(z)}$, where $s(z)$ is a periodic function with period $c = \log 2$. Clearly $\rho(f) = +$. Note that $\Delta_c f = f$ and so f and $\Delta_c f$ share 0 CM. On the other hand, we see that $\lambda(f) = 0 < \rho(f)$, but f does not satisfies the conclusion of Theorem 1.2.

2. Auxiliary lemmas

For the proof of our main results, we make use of five key lemmas. We just recall these lemmas here.

Lemma 2.1. [2] Let f be a meromorphic function of finite order ρ and $\eta1, \eta2 \in C$ such that $\eta1 \neq \eta2$. Then for any $\varepsilon > 0$ we have

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\rho-1+\varepsilon}).$$

Lemma 2.2. [2] Let $\eta1, \eta2$ be two arbitrary complex numbers and f be a meromorphic function of finite order σ. Let $\varepsilon > 0$ be given. Then there exists a subset $E \subset (0, \infty)$ with finite logarithmic measure such that for all $r \notin E \cup [0, 1]$ we have

$$\exp(-r^{\sigma-1+\varepsilon}) \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp(r^{\sigma-1+\varepsilon}).$$

Lemma 2.3. [7] Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form U (z, f)P (z, f) = Q(z, f), where U (z, f), P (z, f), Q(z, f) are difference polynomials such that the total degree deg (U (z, f)) = n in f and its shifts and deg (Q(z, f)) $\leq$ n. Moreover we assume that U (z, f) contains just one term of maximal total degree in f and its shifts. Then for each $\varepsilon > 0$ we have

$$m(r, P(z, f)) = O(r^{\rho-1+\varepsilon}) + S(r, f)$$

possible outside of an exceptional set of finite logarithmic measure.

Remark 2.1. From the proof of Lemma 2.3, we can see that if the coefficients of U (z, f), P (z, f), Q(z, f), say a_λ satisfy $m(r, a_\lambda) = S(r, f)$, then the same conclusion still holds.

Lemma 2.4. [9] Let g be a non-constant meromorphic solution of the linear difference equation $b_k g(z + kc) + b_{k-1} g(z + (k-1)c) + \dots + b_0 g(z) = R(z)$, where R is polynomial and bi's for i = 0, 1, . . . , k are complex constants with $b_k b_0 \neq 0$, $c \in C \setminus \{0\}$ and $k \in N$. Then either $\rho(g) \geq 1$ or g is a polynomial. In particular if $b_k \neq \pm b_0$, then $\rho(g) \geq 1$.

Lemma 2.5. [11] Suppose that f1, f2, . . . , fn (n $\geq$ 2) are meromorphic functions and g1, g2, . . . , gn are entire functions satisfying the following conditions

(i) $\sum_{j=1}^{n} f_j e^{g_j} = 0$

(ii) $g_i - g_j$ *is non-constant for* $1 \leq i < j \leq n$;

(iii) $T(r, f_j) = o(T(r, e^{g_h - g_k}))$ $(r \to \infty, r \notin E)$ *for* $1 \leq j \leq n$, $1 \leq h < k \leq n$.

Then $f_j \equiv 0$ *for* $j = 1, 2, \dots, n$.

3. Proofs of main results

Proof of Theorem 1.1. By the given conditions, we have $\lambda(f - \alpha) < \rho(f)$. Then there exist an entire function $H (\not\equiv 0)$ and a polynomial P such that

$$f = \alpha + He^{P}, \qquad (3.1)$$

where $\rho(H) < \rho(f - \alpha)$ and $\deg(P) = \rho(f - \alpha)$. Since $\rho(\alpha) < \rho(f)$, it follows that $\rho(H) < \rho(f)$ and $\deg(P) = \rho(f)$. Let

$$P(z) = a_s z^s + a_{s-1} z^{s-1} + \cdots + a0, \qquad (3.2)$$

where $a_s (\neq 0), a_{s-1}, \dots, a_0 \in C$ and $s \in N$. Also from (3.1), we deduce that

$$\Delta_c^n f = \Delta_c^n \alpha + H_n e^P, \tag{3.3}$$

Where

$$H_n(z) = \sum_{i=0}^{n} c_i H(z+ic) e^{P(z+ic)-P(z)} \text{ and } c_i = (-1)^{n-i} C_n^i. \tag{3.4}$$

Now we divide the following two cases.

Case 1. Suppose ρ(f) < 2. In this case, from Theorem C we easily conclude that α is a polynomial of degree at most n−1 and f must be of the form f (z) = α(z)+H(z)e^{dz}, where H is a polynomial such that cH = −α and c, d ∈ C \ {0} such that c^d = 1.

Case 2. Suppose ρ(f) ≥ 2. Since f − α and $\Delta_c^n f$ − α share 0 CM, then there exists a polynomial function Q such that

$$\Delta_c^n f - \alpha = (f - \alpha) e^Q. \tag{3.5}$$

Since ρ(H) < ρ(f), we have ρ(H(z + ic)) < ρ(f) for i = 0, 1, . . . , n. Note that deg (P (z + ic) - P (z)) s - 1 = ρ(f) - 1. Then from (3.4), we deduce that ρ(Hn) < ρ(f). Also we see that $\rho(\Delta_c^n \alpha) \leq \rho(\alpha)$. Consequently from (3.3), one can easily conclude that $\rho\left(\Delta_c^n f\right) \leq \rho(f)$. Now from (3.5), we deduce that deg(Q) ρ(f) = s.

Now we divide the following two sub-cases.

Sub-case 2.1. Suppose deg(Q) = 0. Then from (3.5), we may assume that

$$\Delta_c^n f - \alpha = c_0 (f - \alpha), \tag{3.6}$$

where c0 ∈ C \ {0}. Now from (3.1), (3.3) and (3.6), we deduce that

$$\Delta_c^n \alpha - \alpha = (c_0 H - H_n) e^P. \tag{3.7}$$

Note that $\rho\left(\Delta_c^n \alpha - \alpha\right) < \rho(f) = \rho\left(e^P\right) = \mu\left(e^P\right)$ and $\rho\left(c_0 H - H_n\right) < \rho(f) = \mu\left(e^P\right)$. Consequently $T\left(r, \Delta_c^n \alpha - \alpha\right) = S\left(r, e^P\right)$ and $T\left(r, c_0 H - H_n\right) = S\left(r, e^P\right)$. Now from Lemma 2.5 and (3.7), we deduce that $\Delta_c^n \alpha \equiv \alpha$ and $c_0 H \equiv H_n$.

Let $h(z) = e^{sca_s z^{s-1}}$. Then $\rho(h) = \mu(h)$ and

$$T(r, h) = m(r, h) = \frac{|s||c||a_s|}{\pi}(1 + o(1)) r^{s-1}. \tag{3.8}$$

Now for 0 ≤ j ≤ n, we have

$$e^{P(z+jc)-P(z)} = e^{jsca_s z^{s-1}} e^{P_j(z)} = h^j(z) e^{P_j(z)},$$

where P_j is polynomial with degree at most $s - 2$. Note that

$$\rho(e^{P_j}) = \deg(P_j) \le s - 2 < s - 1 = \rho(h) = \mu(h)$$

and so T $r, e^{P_j} = S(r, h)$ and so $T(r, e^{P_j}) = S(r, hj)$ for $j = 0, 1, 2, \ldots, n$. Now from

Lemma 2.1, we have

$$m\left(r, \frac{H(z+jc)}{H(z)}\right) = O\left(r^{\rho(H)-1+\varepsilon}\right), \tag{3.9}$$

where $\varepsilon > 0$ is arbitrary. Since $\rho(H) < \rho(e^P)$, we chose $\varepsilon > 0$ such that $\rho(H) - 1 + 2\varepsilon < \rho(e^P) - 1 = \rho(h) = \mu(h)$. By the definition of the lower order, there exists a $R > 0$ such that $T(r, h) > r^{\mu(h)-\varepsilon} = r^{\rho(h)-\varepsilon}$ holds for $r \ge R$. Then from (3.9), we deduce that

$$\lim_{r\to\infty} \frac{m\left(r, \frac{H(z+jc)}{H(z)}\right)}{T(r, h)} = 0$$

and so

$$m\left(r, \frac{H(z+jc)}{H(z)}\right) = S(r, h) \tag{3.10}$$

for $j = 0, 1, \ldots, n$. Similarly we can prove that

$$m\left(r, \frac{H(z)}{H(z+jc)}\right) = S(r, h) \tag{3.11}$$

for $j = 0, 1, \ldots, n$. Let

$$b_{n-j}(z) = c_j \frac{H(z+jc)}{H(z)} e^{P_j(z)}, \tag{3.12}$$

for $j = 0, 1, 2, \ldots, n$ and

$$F_n(h) = \sum_{j=0}^{n} b_{n-j} h^j. \qquad (3.13)$$

Since $T\ (r, e^{Pj}) = S(r, h)$, for $j = 0, 1, \ldots, n$, from (3.10), (3.11) and (3.12), one can easily conclude that

$$m(r, b_j) + m\left(r, \frac{1}{b_j}\right) = S(r, h),\ \text{for } j = 0, 1, \ldots, n. \qquad (3.14)$$

Now from the proof of Theorem 2.1 [9], we can easily conclude that

$$n\, m(r,h) + S(r,h) = m(r, Fn(h)) + S(r,h)$$

and so from (3.8), we have

$$m(r, F_n(h)) = n\frac{|s||c||a_s|}{\pi}(1 + o(1))r^{s-1} + S(r, h) = n\, T(r, h) + S(r, h). \qquad (3.15)$$

Now since $c_0 H \equiv H_n$, from (3.4), we conclude that

$$F_n(h) = \sum_{j=0}^{n} b_{n-j} h^j = c_0$$

and so $m(r, F_n(h)) = O(1)$. Then from (3.15), we deduce that $T\ (r, h) = S(r, h)$, which is impossible.

Sub-case 2.2. Suppose $\deg(Q) \geq 1$. Now by taking logarithmic differentiation on (3.5), we get

$$\frac{(\Delta_c^n f)' - \alpha'}{\Delta f - \alpha} - \frac{f' - \alpha'}{f - \alpha} = Q'$$

and so from (3.1) and (3.3), we deduce that

$$\left(H_n' H - H_n H' - H_n H Q'\right) e^P = (\Delta_c^n \alpha - \alpha)\left(H' | + HP' + HQ'\right) - (\Delta\alpha - \alpha)' H. (3.16)$$

Now we divide following two sub-cases.

Sub-case 2.2.1. Suppose $\Delta_c^n \alpha \not= \alpha$. Since $\rho(H) < \rho(e^P)$, $\rho(H_n) < \rho(e^P)$ and $\rho(\alpha) < \rho(e^P)$, from Lemma 2.5 and (3.16), we conclude that

$$H_n'H - H_nH' - H_nHQ' = 0 \tag{3.17}$$

and

$$(\Delta_c^n\alpha - \alpha)(H' + HP' + HQ') - (\Delta_c^n\alpha - \alpha)'H = 0. \tag{3.18}$$

Then from (3.17), we get

$$\frac{H_n'}{H_n} - \frac{H'}{H} - Q' = 0$$

and so from (3.18), we have

$$\frac{(\Delta_c^n\alpha - \alpha)'}{(\Delta_c^n\alpha - \alpha)} = \frac{H_n'}{H_n} + P'.$$

On integration, we have

$$\Delta_c^n\alpha - \alpha = c_1H_ne^P \quad (c_1 \in \mathbb{C} \setminus \{0\}). \tag{3.19}$$

Since $\rho(H_n) < \rho(e^P)$, it follows that $\rho\ (Hn^{eP}) = \rho(e^P)$. Again since $\rho\ (\Delta_c^n\alpha - \alpha) < \rho(e^P)$, we arrive at a contradiction from (3.19).

Sub-case 2.2.2. Suppose $\Delta_c^n\alpha = \alpha$. In this case, from (3.16), we have $\frac{H_n'}{H_n} - \frac{H'}{H} - Q' \equiv 0$ and so $H_n \equiv c_2He^Q$, where $c2 \in C \setminus \{0\}$. Also from (3.5), we have

$$F_n(h) = \sum_{j=0}^{n} b_{n-j}h^j = e^Q, \tag{3.20}$$

where bj's are given by (3.12). Now from (3.20), we conclude that $F_n(h)$ is an entire function and so $T\ (r, F_n(h)) = m(r, F_n(h))$. Then from (3.15), we get $T\ (r, F_n(h)) = n\ T\ (r, h) + S(r, h)$ and so from (3.20), we have $n\ T\ (r, h) + S(r, h) = T\ (r, e^Q)$. Clearly $\rho(h) = \rho(e^Q)$. Since $\rho(h) = s - 1$, it follows that $\rho(e^Q) = s - 1$ and so $\deg(Q) = s - 1$.

Let

$$Q(z) = d_{s-1}z^{s-1} + d_{s-2}z^{s-2} + \cdot\cdot\cdot + d0. \tag{3.21}$$

Now from (3.5), we have

$$\sum_{j=1}^{n} c_j \frac{H(z+jc)}{H(z)} e^{R_j(z)} + (-1)^n - e^{Q(z)} = 0, \tag{3.22}$$

where Rj(z) = P (z + jc) - P (z) (j = 1, 2, . . . , n). Then from (3.2), we may assume that

$$R_j(z) = jsa_s cz^{s-1} + P_{s-2,j}(z), \tag{3.23}$$

where $P_{s-2,j}(z)$ is a polynomial with degree at most s - 2. Clearly $\deg(R_j) = s - 1$ for j = 1, 2, . . . , n.

Now we divide the following two sub-cases.

Sub-case 2.2.2.1. Suppose n = 1. Then from (3.22), we have

$$c_1 \frac{H(z+c)}{H(z)} e^{R_1(z)} - 1 = e^{Q(z)}. \tag{3.24}$$

From (3.24), it is clear that $\frac{H(z+c)}{H(z)}$ is an entire function. Now from (3.9), we deduce that

$$T\left(r, \frac{H(z+c)}{H(z)}\right) = m\left(r, \frac{H(z+c)}{H(z)}\right) = O\left(r^{\rho(H)-1+\varepsilon}\right)$$

and so $\rho\left(\frac{H(z+c)}{H(z)}\right) = \rho(H) - 1 < \rho(f) - 1 = s - 1 = \rho(e^{R_1})$. Therefore it is easy to conclude that 0 is a Borel exceptional value of the entire function $c_1 \frac{H(z+c)}{H(z)} e^{R_1(z)}$.

Consequently 1 is not a Borel exceptional of $c_1 \frac{H(z+c)}{H(z)} e^{R_1(z)}$ and so $c_1 \frac{H(z+c)}{H(z)} e^{R_1(z)} - 1$ must have infinitely many zeros. But in this case, we arrive at a contradiction from (3.24).

Sub-case 2.2.2.2. Suppose n ≥ 2. Then from (3.21) and (3.23), we see that

$$R_j(z) - Q(z) = (jsa_s c - d_{s-1})z^{s-1} + \dots,$$

where j = 1, 2, . . . , n. Now we divide following two sub-cases:

Sub-case 2.2.2.2.1. Suppose there exists j0($1 \le j0 \le n$) such that $j_0 sa_s c = d_{s-1}$. Therefore $\deg(Rj_0 - Q) \le s - 2$. In this case from (3.22), we have

$$\left(\sum_{\substack{1\le j\le n\\ j\ne j_0}} c_j\frac{H(z+cj)}{H(z)}e^{P(z+jc)-P(z+c)}+B_{j_0}e^{P(z+j_0c)-P(z+c)}\right)e^{R_1(z)}=(-1)^{n+1}, \quad (3.25)$$

where

$$B_{j_0}(z)=c_{j_0}\frac{H(z+j_0c)}{H(z)}-e^{Q(z)-R_{j_0}(z)}. \quad (3.26)$$

Let $Q_1 = e^{R1}$. Note that

$$\begin{aligned} Q_1(z+(j-1)c)Q_1(z+(j-2)c)\dots Q_1(z+c) &= e^{\left(\sum\limits_{i=2}^{j} P(z+ic)-P(z+(i-1)c)\right)} \\ &= e^{P(z+jc)-P(z+c)} \end{aligned}$$

for j = 2, 3, . . . , n. Then (3.25) can be written as

$$U(z, Q_1)Q_1 = (-1)^{n+1}, \quad (3.27)$$

where

$$\begin{aligned} U(z,Q_1(z)) &= \sum_{\substack{1\le j\le n\\ j\ne j_0}} c_j\frac{H(z+jc)}{H(z)}Q_1(z+(j-1)c)Q_1(z+(j-2)c)\cdots Q_1(z+c) \\ &\quad +B_{j_0}(z)Q_1(z+(j_0-1)c)Q_1(z+(j_0-2)c)\cdots Q_1(z+c) \end{aligned}$$

if $j_0 \ge 2$ and

$$\begin{aligned} U(z,Q_1(z)) &= \sum_{2\le j\le n} c_j\frac{H(z+jc)}{H(z)}Q_1(z+(j-1)c)Q_1(z+(j-2)c)\cdots Q_1(z+c) \\ &\quad +B_{j_0}(z) \end{aligned}$$

if j0 = 1.

From (3.27), it is clear that U (z, Q1) $\not\equiv$ 0 and deg(U (z, Q1)) = n - 1 $\ge$ 1. Now we want to prove that if aλ is a coefficient of U (z, Q_1), then m(r, aλ) = S(r, Q_1). Note that $\mu(e^{R1}) = \rho(e^{R1}) = \deg(R_1)$ = s − 1 and $\rho(e^{Q-Rj0}) = \deg(Q - R_{j0}) \le s - 2 < s - 1 = \mu(e^{R1})$. Consequently

$$T(r, e^{Q-Rj0}) = S(r, e^{R1}) = S(r, Q_1). \quad (3.28)$$

Also it is easy to prove from (3.9) that

$$m\left(r,\frac{H(z+jc)}{H(z)}\right)=S(r,e^{R_1})=S(r,Q_1)\ (j=1,2,\ldots,n). \tag{3.29}$$

Now from (3.26), (3.28) and (3.29), we see that

$$m(r,B_{j_0}(z))\le m\left(r,\frac{H(z+j_0c)}{H(z)}\right)+m\left(r,e^{Q(z)-R_{j_0}(z)}\right)\le S(r,Q_1).$$

Then in view of Remark 2.1 and using Lemma 2.3, one can easily conclude from (3.27) that m(r, Q_1) = S(r, Q_1). Therefore T (r, Q_1) = m(r, Q_1) = S(r, Q_1), which is a contradiction.

Sub-case 2.2.2.2.2. Suppose jsa_sc ≠ ds 1, 1 j n. In this case, (3.22) can be rewriten as

$$e^{Q(z)}=e^{d_{s-1}z^{s-1}}e^{\tilde{P}_{s-2}(z)}=\sum_{j=0}^{n}c_j\frac{H(z+jc)}{H(z)}e^{R_j(z)}, \tag{3.30}$$

where

$$\tilde{P}_{s-2}(z)=Q(z)-d_{s-1}z^{s-1}=d_{s-2}z^{s-2}+d_{s-3}z^{s-3}+\cdots+d_0. \tag{3.31}$$

Again from (3.23) and (3.30), we have

$$e^{Q(z)}=e^{d_{s-1}z^{s-1}}e^{\tilde{P}_{s-2}(z)}=\sum_{j=1}^{n}c_j\frac{H(z+jc)}{H(z)}e^{jsa_scz^{s-1}}e^{P_{s-2,j}(z)}+(-1)^n. \tag{3.32}$$

Note that ns $|a_sc|$ > (n − 1)s$|ca_s|$ > · · · > s$|a_sc|$ and either $|d_{s-1}|$ ∈ {js$|a_sc|$: j = 1, 2, . . . , n} or $|d_{s-1}|$ ∈ {js $|a_sc|$: j = 1, 2, . . . , n}. Therefore if we compare $|d_{s-1}|$ with ns |asc|, (n - 1)s |asc|, . . ., s $|a_sc|$, then it is enough to compare $|d_{s-1}|$ with ns $|a_sc|$.

Now we divide following two sub-cases.

Sub-case 2.2.2.2.2.1. Suppose ns$|a_sc|$ ≤ $|d_{s-1}|$. Let arg d_{s-1} = θ_1 and arg(a_sc) = θ_2. Take θ_0 such that cos((s − 1)θ_0 + θ_1) = 1. Then using Lemma 2.2, we see that for any given ε (0 < ε < s − ρ(H)), there exists a set E ⊂ (1, ∞) of finite logarithmic measure such that for all z = $re^{i\theta_0}$ satisfying |z| = r ∉ [0, 1] ∪ E we have

$$\exp\left(-r^{\rho(H)-1+\varepsilon}\right)\le\left|\frac{H(z+jc)}{H(z)}\right|\le\exp\left(r^{\rho(H)-1+\varepsilon}\right)\ (j=1,2,\ldots,n). \tag{3.33}$$

Note that

$$\begin{aligned} & \left|\exp\left(d_{s-1}z^{s-1}\right)\right| \qquad (3.34) \\ = & \left|\exp\left(|d_{s-1}|r^{s-1}\left(\cos((s-1)\theta_0+\theta_1)\right)+i\sin((s-1)\theta_0+\theta_1)\right)\right| = \exp\left(|d_{s-1}|r^{s-1}\right). \end{aligned}$$

Similarly we can show that

$$\left|\exp\left(jsa_s cz^{s-1}\right)\right| = \exp\left(js|a_s c|r^{s-1}\cos((s-1)\theta_0+\theta_2)\right), \; j = 1, 2, \ldots, n. \qquad (3.35)$$

If possible suppose deg($P_{s-2,j}$) = s - 2 > 0. Set $P_{s-2,j}$ = c0,j +c1,jz +. . .+cs−2,j^{zs-2}, where $c_{k,j}$ = $q_{k,j}$(cos $\alpha_{k,j}$ + i sin $\alpha_{k,j}$). Then

$$P_{s-2,j}(z) = \sum_{k=0}^{s-2} \varrho_{k,j} r^k \{\cos(\alpha_{k,j}+k\theta_0) + i\sin(\alpha_{k,j}+k\theta_0)\}$$

and so

$$\left|\exp\left(P_{s-2,j}(z)\right)\right| = \exp\left(\sum_{k=0}^{s-2} \varrho_{k,j} r^k \cos(\alpha_{k,j}+k\theta_0)\right). \qquad (3.36)$$

Note that

$$\left|\sum_{k=0}^{s-2} \varrho_{k,j} r^k \cos(\alpha_{k,j}+k\theta_0)\right| \le \sum_{k=0}^{s-2} \varrho_{k,j} r^k = \varrho_{s-2,j} r^{s-2}\left(1 + \frac{\varrho_{s-3,j}}{\varrho_{s-2,j}}\frac{1}{r} + \ldots + \frac{\varrho_{0,j}}{\varrho_{s-2,j}}\frac{1}{r^{s-2}}\right).$$

We see that $\frac{\varrho_{s-3,j}}{\varrho_{s-2,j}}\frac{1}{r} + \ldots + \frac{\varrho_{0,j}}{\varrho_{s-2,j}}\frac{1}{r^{s-2}} \to 0$ as $r \to \infty$ and so for ε_0 in $(0,1)$ and $r > r(\varepsilon_0)$, we have $\frac{\varrho_{s-3,j}}{\varrho_{s-2,j}}\frac{1}{r} + \ldots + \frac{\varrho_{0,j}}{\varrho_{s-2,j}}\frac{1}{r^{s-2}} < \varepsilon_0$. Consequently from (3.36), we get $\left|\exp\left(P_{s-2,j}(z)\right)\right| < \exp\left(|c_{s-2,j}|r^{s-2}(1+\varepsilon_0)\right)$ for $r > r(\varepsilon_0)$. Let $|C_0| = \max\{|c_{s-2,j}| : j = 1, \ldots, n\}$ and $\varepsilon_0 = \frac{1}{2}$. Then we have

$$\left|\exp\left(P_{s-2,j}(z)\right)\right| < \exp\left(\frac{3|C_0|}{2}r^{s-2}\right), \; j = 1, 2, \ldots, n \qquad (3.37)$$

for sufficiently large values of r. If $P_{s-2,j}$ is a constant, then (3.37) also holds.

Similarly we can prove that

$$\left|\exp\left(-\tilde{P}_{s-2}(z)\right)\right| < \exp\left(\frac{3|d_{s-2}|}{2}r^{s-2}\right) \qquad (3.38)$$

for sufficiently large values of r. Clearly from (3.33), (3.35) and (3.37), we get

$$\begin{aligned}
&\left|\frac{H(z+jc)}{H(z)}e^{jsa_scz^{s-1}}e^{P_{s-2,j}(z)}\right| &(3.39)\\
\leq\ & \exp\left(js|a_sc|r^{s-1}\cos((s-1)\theta_0+\theta_2))+r^{\rho(H)-1+\varepsilon}+\frac{3|C_0|}{2}r^{s-2}\right)\\
=\ & \exp(jx+y),
\end{aligned}$$

where

$$x=s|a_sc|r^{s-1}\cos((s-1)\theta_0+\theta_2)),\quad y=r^{\rho(H)-1+\varepsilon}+\frac{3|C_0|}{2}r^{s-2}$$

and j = 1, 2, . . . , n. Now from (3.32), (3.34), (3.38) and (3.39), we have

$$\begin{aligned}
\exp\left(|d_{s-1}|r^{s-1}\right) &= \left|\exp(d_{s-1}z^{s-1})\right| &(3.40)\\
&= \left|\frac{\exp(Q(z))}{\exp(\tilde{P}_{s-2}(z))}\right|\\
&\leq \left|\sum_{j=1}^{n}c_j\frac{H(z+jc)}{H(z)}e^{jsa_scz^{s-1}}e^{P_{s-2,j}(z)}+(-1)^n\right|\left|\exp(-\tilde{P}_{s-2}(z))\right|\\
&\leq \left(\sum_{j=1}^{n}|c_j|\exp\left(jx+y\right)+1\right)\exp\left(\frac{3|d_{s-2}|}{2}r^{s-2}\right).
\end{aligned}$$

First we suppose $x \leq 0$, i.e., $\cos((s-1)\theta_0+\theta_2)) \leq 0$. Let $\cos((s-1)\theta_0+\theta_2)) = -x_1$, where $0 \leq x_1 \leq 1$. If $x_1 = 0$, then from (3.40), one can easily conclude that

$$\exp\left(|d_{s-1}|r^{s-1}\right) < C_1\exp\left(r^{\rho(H)-1+\varepsilon}+\frac{3(|C_0|+|d_{s-2}|)}{2}r^{s-2}\right), \qquad (3.41)$$

where $C_1 > 0$. Since $\rho(H)-1+\varepsilon < s-1$, (3.41) leads to a contradiction. Next we suppose $0 < x_1 \leq 1$. In this case, since $\rho(H)-1+\varepsilon < s-1$, we see that

$$\sum_{j=1}^{n}|c_j|\exp\left(-js|a_sc|x_1r^{s-1}+r^{\rho(H)-1+\varepsilon}+\frac{3|C_0|}{2}r^{s-2}\right)\rightarrow 0$$

as $r \rightarrow \infty$ and so for ε_1 in (0, 1) and $r > r(\varepsilon_1)$, we have

$$\sum_{j=1}^{n} |c_j| \exp\left(-js|a_s c|x_1 r^{s-1} + r^{\rho(H)-1+\varepsilon} + \frac{3|C_0|}{2} r^{s-2}\right) < \varepsilon_1.$$

Consequently from (3.40), we get

$$\exp\left(|d_{s-1}|r^{s-1}\right) < (1+\varepsilon_1)\exp\left(\frac{3|d_{s-2}|}{2} r^{s-2}\right),$$

which leads to a contradiction.

Next we suppose $x > 0$, i.e., $\cos((s - 1)\theta_0 + \theta_2)) > 0$. In this case, we have $\exp(jx + y) < \exp(nx + y)$ for $j = 0, \ldots, n$ and so from (3.40), we get

$$\begin{aligned}
& \exp\left(|d_{s-1}|r^{s-1}\right) && (3.42)\\
\le\ & \left(\sum_{j=1}^{n} |c_j| \exp(jx+y) + 1\right) \exp\left(\frac{3|d_{s-2}|}{2} r^{s-2}\right) \\
\le\ & \left(\sum_{j=0}^{n} |c_j|\right) \exp(jx+y) \exp\left(\frac{3|d_{s-2}|}{2} r^{s-2}\right) \\
=\ & B\exp\left(ns|a_s c|r^{s-1}\cos((s-1)\theta_0+\theta_2)) + r^{\rho(H)-1+\varepsilon} + \frac{3(|C_0|+|d_{s-2}|)}{2} r^{s-2}\right),
\end{aligned}$$

Where $B = \sum_{j=0}^{n} |c_j|.$ since $p(H) - 1 + \varepsilon < s - 1$ and $B = \exp(\log B) = o(r^{s-1})$, from (3.42), we deduce that

$$\exp\left(|d_{s-1}|r^{s-1}\right) \le \exp\left(ns|a_s c|\cos((s-1)\theta_0+\theta_2) r^{s-1} + o(r^{s-1})\right). \qquad (3.43)$$

By assumption, we have $d_{s-1} \neq nsa_sc$ and $ns|a_sc| \le |d_{s-1}|$. First we suppose $ns|a_sc| = |d_{s-1}|$. In that case $\cos((s - 1)\theta_0 + \theta_2) \neq 1$ and so $\cos((s - 1)\theta_0 + \theta_2) < 1$.

Therefore $ns|a_sc| \cos((s - 1)\theta_0 + \theta_2) < ns|a_sc| = |d_{s-1}|$. Next we suppose $ns|a_sc| < |d_{s-1}|$. Then obviously $ns|a_sc| \cos((s - 1)\theta_0 + \theta_2) \le ns|a_sc| < |d_{s-1}|$.

Therefore in either case, we have $ns|asc| \cos((s - 1)\theta_0 + \theta_2) < |d_{s-1}|$. Then there exists $\varepsilon 2 > 0$ such that $ns|asc| \cos((s- 1)\theta_0 + \theta_2) + 2\varepsilon_2 < |d_{s-1}|$ and so from (3.43), we

$$\exp\left(|d_{s-1}|r^{s-1}\right) \leq \exp\left(\left(|d_{s-1}| - 2\varepsilon_2\right) r^{s-1}\right),$$

which is a contradiction.

Sub-case 2.2.2.2.2.2.2. Suppose $ns|a_sc| > |d_{s-1}|$. In this case (3.32) can be rewritten as

$$\begin{aligned} & e^{nsa_scz^{s-1}} e^{P_{s-2,n}(z)} \\ = & \sum_{j=0}^{n-1} \frac{c_j}{c_n} \frac{H(z+jc)}{H(z+nc)} e^{jsa_scz^{s-1}} e^{P_{s-2,j}(z)} - \frac{1}{c_n} \frac{H(z)}{H(z+nc)} e^{d_{s-1}z^{s-1}} e^{\tilde{P}_{s-2}(z)}. \end{aligned} \tag{3.44}$$

Let $\arg(a_sc) = \theta_1$ and $\arg d_{s-1} = \theta_2$. Take θ_0 such that $\cos((s-1)\theta_0 + \theta_1) = 1$. Then using Lemma 2.2, we see that for any given ε ($0 < \varepsilon < s - \rho(H)$), there exists a set $E \subset (1, \infty)$ of finite logarithmic measure such that for all $z = re^{i\theta 0}$ satisfying $|z| = r \notin [0, 1] \cup E$ we have

$$\exp\left(-r^{\rho(H)-1+\varepsilon}\right) \leq \left|\frac{H(z+jc)}{H(z+nc)}\right| \leq \exp\left(r^{\rho(H)-1+\varepsilon}\right) \ (j = 0, 1, \ldots, n-1). \tag{3.45}$$

In this case, we easily get

$$\left|\exp\left(jsa_scz^{s-1}\right)\right| = \exp\left(js|a_sc|r^{s-1}\right), \ j = 0, 1, \ldots, n \tag{3.46}$$

and

$$\left|\exp\left(d_{s-1}z^{s-1}\right)\right| = \left|\exp\left(|d_{s-1}|r^{s-1}\cos((s-1)\theta_0 + \theta_2)\right)\right| \leq \left|\exp\left(|d_{s-1}|r^{s-1}\right)\right|. \tag{3.47}$$

Also we easily obtain

$$|\exp(\pm P_{s-2,j}(z))| < \exp\left(\frac{3|C_0|}{2} r^{s-2}\right) \text{ and } \left|\exp(\tilde{P}_{s-2}(z))\right| < \exp\left(\frac{3|d_{s-2}|}{2} r^{s-2}\right) \tag{3.48}$$

For sufficiently large values of r. Then from (3.45), (3.46) and (3.48) we get

$$\begin{aligned} \left|\frac{H(z+jc)}{H(z+nc)} e^{jsa_scz^{s-1}} e^{P_{s-2,j}(z)}\right| & \leq \exp\left(js|a_sc|r^{s-1} + r^{\rho(H)-1+\varepsilon} + \frac{3|C_0|}{2} r^{s-2}\right) \quad (3.49) \\ & \leq \exp\left((n-1)s|a_sc|r^{s-1} + r^{\rho(H)-1+\varepsilon} + \frac{3|C_0|}{2} r^{s-2}\right), \end{aligned}$$

for j = 1, ..., n - 1. Again from (3.45), (3.47) and (3.48) we get

$$\left|\frac{H(z)}{H(z+nc)}e^{d_{s-1}z^{s-1}}e^{\tilde{P}_{s-2}(z)}\right| \leq \exp\left(|d_{s-1}|r^{s-1}+r^{\rho(H)-1+\varepsilon}+\frac{3|d_{s-2}|}{2}r^{s-2}\right). \quad (3.50)$$

Now from (3.44), (3.46), (3.48)-(3.50), we obtain

$$\begin{aligned} &\exp\left(ns|a_s c|r^{s-1}\right) \qquad (3.51)\\ = &\left|\sum_{j=0}^{n-1}\frac{c_j}{c_n}\frac{H(z+jc)}{H(z+nc)}e^{jsa_s cz^{s-1}}e^{P_{s-2,j}(z)}-\frac{1}{c_n}\frac{H(z)}{H(z+nc)}e^{d_{s-1}z^{s-1}}e^{\tilde{P}_{s-2}(z)}\right|\left|e^{P_{s-2,n}(z)}\right|\\ \leq &\ B\exp\left(x_2 r^{s-1}+r^{\rho(H)-1+\varepsilon}+\frac{6|C_0|+3|d_{s-2}|}{2}r^{s-2}\right), \end{aligned}$$

Where $B=\sum_{j=0}^{n}|c_j|$ and x2 = max {(n – 1)s|a_sc|, |d_{s-1}|}. Since |d_{s-1}| < ns|a_sc|, it follows that x2 < ns|a_sc|. Also we have p(H) – 1 + ∈ < s – 1. Consequently from (3.51), one can easily arrive at a contradiction. This completes the proof.

Proof of Theorem 1.2. By the given conditions, we have λ(f) < ρ(f). Then there exist an entire function H($\not\equiv$ 0) and a polynomial P such that f = He^P , where ρ(H) < ρ(f) and deg(P) = ρ(f). Now we divide the following two cases.

Case 1. Suppose ρ(f) < 2. Since ρ(f) = deg(P) < 2, it follows that ρ(f) = deg(P) = 1. Again since λ(H) < ρ(f), we have λ(H) < 1. Therefore we can assume that

$$f(z) = H(z)e^{dz}, \qquad (3.52)$$

where d ∈ C \ {0} and λ(H) < 1. Now from (3.52), we deduce that

$$\Delta_c^n f(z) = H_n(z)e^{dz}, \qquad (3.53)$$

where

$$H_n(z)=\sum_{i=0}^{n}c_i H(z+ic)e^{icd} \text{ and } c_i| = (-1)^{n-i}C_n^i. \qquad (3.54)$$

Since f and $\Delta_c^n f$ share 0 CM, then by Theorem B, we have

$$\Delta_c^n f = af, \tag{3.55}$$

where $a \in$ C $\backslash\{0\}$. Then from (3.52)-(3.55), we deduce that

$$\sum_{i=1}^{n} c_i e^{icd} H(z+ic) | + ((-1)^n - a)H(z) = 0. \tag{3.56}$$

Now we divide following two sub-cases.

Sub-case 1.1. Suppose a $\neq$ $(-1)^n$. Then using Lemma 2.4, we conclude from (3.56) that H is a non-zero polynomial. We claim that H is a non-zero constant. If not suppose deg(H) = k ∈ N. Let H(z) = ak^{zk} + $a_{k-1}z^{k-1}$ + + a1z + a0 (ak$\neq$ 0). Then comparing the coefficients of z^k and z^{k-1} from (3.56), we have respectively

$$\sum_{i=1}^{n} c_i e^{icd} + ((-1)^n - a) = (e^{cd} - 1)^n - a = 0 \tag{3.57}$$

and

$$\sum_{i=1}^{n} c_i e^{icd}(ika_k + a_{k-1}) + ((-1)^n - |a)a_{k-1} = 0. \tag{3.58}$$

Now from (3.57) and (3.58), we immediately have

$$\sum_{i=1}^{n} c_i i e^{icd} = 0. \tag{3.59}$$

Again from (3.57), we have $a = (e^{cd} - 1)^n$. Since $a \neq 0$, we have $e^{cd} \neq 1$. Note that

$$C_n^i i = \frac{n!i}{(n-i)!i!} = \frac{n(n-1)!}{((n-1)-(i-1))!(i-1)!} = nC_{n-1}^{i-1}$$

and so

$$\begin{aligned}\sum_{i=1}^{n} c_i i e^{icd} = \sum_{i=1}^{n} (-1)^{n-i} C_n^i i e^{icd} &= n e^{cd} \sum_{i=1}^{n} (-1)^{n-i} C_{n-1}^{i-1} e^{(i-1)cd} \\ &= n e^{cd}(e^{cd}-1)^{n-1} \neq 0.\end{aligned} \tag{3.60}$$

Therefore from (3.59) and (3.60), we arrive at a contradiction. Hence H is a non-zero constant. Then from (3.52), we can assume that $f(z) = c_0 e^{dz}$, where $c_0, d \in \mathbb{C} \setminus \{0\}$.

Sub-case 1.2. Suppose $a = (-1)^n$. Then from (3.56), we have

$$\sum_{i=1}^{n} c_i e^{icd} H(z+ic) = 0. \tag{3.61}$$

If n = 1 then we immediately arrive a contradiction from (3.61). Hence $n \geq 2$.

If we let w = z + c, then from (3.61), we have

$$\sum_{i=0}^{n-1} c_{i+1} e^{(i+1)cd} H(w+ic) = 0. \tag{3.62}$$

Note that $\rho(H) < 1$. Now using Lemma 2.4, one can easily conclude from (3.62) that H(w) is a non-zero polynomial, i.e., H(z + c) is non-zero polynomial. Consequently, H(z) is a non-zero polynomial. We claim that H is a non-zero constant. If not suppose $\deg(H) = m \in \mathbb{N}$. Let $H = b_m z^m + b_{m-1} z^{m-1} + \dots + b_0 (b^m \neq 0)$. Now comparing the coefficient of zm from (3.62), we have

$$\sum_{i=1}^{n} c_i e^{icd} = 0. \tag{3.63}$$

Again comparing the coefficient of z^{m-1} from (3.62) and using (3.63), we have

$$\sum_{i=1}^{n} c_i e^{icd}(mib_m + b_{m-1}) = 0 \Rightarrow mb_m \sum_{i=1}^{n} c_i i e^{icd} = 0. \tag{3.64}$$

Then form (3.63), we have

$$\sum_{i=1}^{n} (-1)^{n-i} C_n^i e^{icd} + (-1)^n - a = 0 \Rightarrow a = (e^{cd}-1)^n \neq 0. \tag{3.65}$$

Consequently from (3.60), (3.64) and (3.65), we have a contradiction. So H is a non-zero constant. Hence f is of the form $f(z) = c_0e^{dz}$, where $c_0, d \in C \setminus \{0\}$.

Case 2. Suppose $\rho(f) \geq 2$. In this case, proceeding in the same way as done in Case 2 in the proof of Theorem 1.1, we arrive at a contradiction.

References

[1] R. Bru¨ck, On entire functions which share one value CM with their first derivative, Results Math., 30 (1-2) (1996), 21-24.
[2] Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic of f (z +η) and difference equations in the complex plane, Ramanujian J., 16 (1) (2008), 105-129.
[3] W. K. Hayman, Meromorphic Functions, Clarendon Press, Oxford (1964).
[4] R. G. Halburd and R. J. Korhonen, Nevanlinna theory for the difference operator, Ann. Acad. Sci. Fenn. Math., 31 (2) (2006), 463-478.J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and J. Zhang, Value sharing results for shifts of meromorphic functions and sufficient conditions for periodicity, J Math. Anal. Appl., 355 (2009), 352-363.
[5] J. Heittokangas, R. Korhonen, I. Laine and J. Rieppo, Uniqueness of meromorphic functions sharing values with their shifts, Complex Var. Elliptic Equ., 56 (1-4) (2011), 81-92.
[6] Laine and C. C. Yang, Clunie theorems for difference and q-difference polynomials, L. Lond. Math. Soc., 76 (3) (2007), 556-566.
[7] K. Liu and I. Laine, A note on value distribution of difference polynomials, Bull. Aust. Math. Soc., 81 (2010), 353-360.
[8] F. Lu¨, W. Lu¨, C. Li and J. Xu, Growth and uniqueness related to complex differential and difference equations, Results Math., 74 (30) (2019), 1-18.
[9] J. Zhang, H. Y. Kang and L. W. Liao, Entire functions sharing a small entire function with their difference operators, Bull. Iranian Math. Soc., 41 (5) (2015), 1121-1129.
[10] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Kluwer Academic Pub- lishers, Dordrecht/Boston/London, 2003.