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The study of a separated boundary value problem for a fractional equation involving a derivative of lower order at the Schauder fixed point

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Abstract:
In this study, we explore a new category of separated boundary value problems for non-linear fractional differential equations where the non-linear term ***f*** relies on a lower-order fractional derivative given by:

$$ {}^{c}D_{0^+}^{\alpha} x(t) = f\left(t, x(t), {}^{c}D_{0^+}^{\beta} x(t)\right) \quad , \quad t\epsilon[0,T], 1 < \alpha \leq 2, 0 < \beta \leq 1 $$

subject to separated fractional boundary conditions :

$$ a_1 x(0) + b_1 \left({}^{c}D_{0^+}^{\gamma} x(0)\right) = c_1, a_2 x(T) + b_2 \left({}^{c}D_{0^+}^{\gamma} x(T)\right) = c_2 \quad , \ 0 < \gamma < 1 $$

Here ${}^{c}D_{0^+}^{\alpha}$, is the Caputo fractional derivative, *f*: [0,T] $\times R^2 \rightarrow R$is a continuous function, and a_i, b_i, c_i ,i = 1, 2 are real constants satisfying: $a_1 \neq 0$ and T > 0.
The aim of this work is to study the existence and uniqueness of solutions to the separated fractional boundary value problem using the Schauder fixed point theorem, which is a topological theorem that asserts a relatively compact map has a fixed point that is not necessarily unique. Therefore, it is not necessary to establish estimates on the function, but rather on its compactness.
To conclude, an example is presented to illustrate the application of this theorem.

Keywords: fractional differential equation, separated fractional boundary conditions, Schauder fixed point theorem, existence.

1. Introduction:

The study of fractional calculus is a discipline almost as ancient as differential calculus itself, and traces its roots to the pioneering work of Leibniz, Gauss, and Newton[1-6].
Nonlinear fractional order differential equations are a natural outgrowth of ordinary differential equations, and have come to occupy a crucial position in the wider field of mathematics, with applications in numerous areas of science and engineering[7-9,2,4]- such as physics, chemistry, and biology.
In this study, we will examine the existence and uniqueness of solutions for a class of nonlinear fractional differential equations, wherein the term***f***depends on the lower-order function derivative of the unknown function,

$$ {}^{c}D_{0^+}^{\alpha} x(t) = f\left(t, x(t), {}^{c}D_{0^+}^{\beta} x(t)\right) \quad , \quad t\epsilon[0,T], 1 < \alpha \leq 2, 0 < \beta \leq 1 \quad .....(1) $$

under separate fractional boundary conditions.

$$ a_1 x(0) + b_1 \left({}^{c}D_{0^+}^{\gamma} x(0)\right) = c_1, a_2 x(T) + b_2 \left({}^{c}D_{0^+}^{\gamma} x(T)\right) = c_2 \quad , \ 0 < \gamma < 1....(2) $$

Here ${}^{c}D_{0^+}^{\alpha}$ is the Caputo fractional derivative,*f*: [0,T] $\times R^2 \rightarrow R$is a continuous function and a_i, b_i, c_i , i = 1, 2 are real constants with:$a_1 \neq 0$ and T > 0.
To do this, we will apply Schauder's fixed point theorem.

2. Some Background Material:

Definition 2.1. (Gamma function)

For any complex number $\mathbf{z}$ such that Re($\mathbf{z}$) > 0, we define the following function called Gamma and denoted by the Greek letter "Γ ".

$$\boldsymbol{\Gamma} : \mathbf{R}^{*+} \rightarrow \mathbf{R}$$

$$\mathbf{z} \rightarrow \boldsymbol{\Gamma}(\mathbf{z}) = \int_0^{+\infty} \mathbf{t}^{\mathbf{z}-1} \exp^{-\mathbf{t}} \mathbf{dt}.$$

Definition 2.2.([12]) (Fractional derivative in Caputo's sense)

Let $\mathbf{f}: [\mathbf{0}, +\infty[\rightarrow \mathbf{R}, \boldsymbol{\alpha} > \mathbf{0}$ *and* $\boldsymbol{n} = [\boldsymbol{\alpha}] + \mathbf{1}$, fractional derivative in Caputo's sense to the right of 0 of order $\boldsymbol{\alpha} > \mathbf{0}$ is defined by :

$$\mathbf{c}_{\mathbf{D}_{0^+}}^{\alpha} \mathbf{f}(\mathbf{t}) = \frac{\mathbf{1}}{\boldsymbol{\Gamma}(\mathbf{n}-\boldsymbol{\alpha})} \int_{\mathbf{0}}^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\mathbf{n}-\boldsymbol{\alpha}-\mathbf{1}} \mathbf{f}^{(\mathbf{n})}(\mathbf{s}) \mathbf{ds}$$

$$= \mathbf{I}_{\mathbf{0}^+}^{\mathbf{n}-\boldsymbol{\alpha}} \left(\frac{\mathbf{d}^{\mathbf{n}}}{\mathbf{dt}^{\mathbf{n}}} \mathbf{f}(\mathbf{t}) \right).$$

Theorem 2.1.([10]) Schauder's fixed point theorem:

Let K be a non-empty closed and convex subset of X (X is a Banach space).

Let F: K → K be a continuous and relatively compact operator. If F(K) is contained in K, then F has at least one fixed point in K : $\exists u \in K$ such that $\mathrm{F}u = u$.

Let us first study the linear boundary problem.

3. The presentation of the problem

Our focus will be on determining the solution's existence for a boundary value problem of fractional order differential equations:

$$c_{D_{0^+}^{\alpha}} x(t) = f\left(t, x(t), c_{D_{0^+}^{\beta}} x(t)\right) \quad , \quad t\epsilon[0,T], \quad 1 < \alpha \leq 2 , 0 < \beta \leq 1$$

subject to separated fractional boundary conditions :

$$a_1 x(0) + b_1 \left(c_{D_{0^+}^{\gamma}} x(0)\right) = c_1, a_2 x(T) + b_2 \left(c_{D_{0^+}^{\gamma}} x(T)\right) = c_2 \qquad 0 < \gamma < 1$$

The Caputo fractional derivative is represented by$c_{D_{0^+}^{\alpha}}$,$\boldsymbol{f}$: [0,T] $\times R^2 \rightarrow$ Ris a continuous function, and a_i, b_i, c_i, i = 1, 2 are real constants such that :$a_1 \neq 0$ and T > 0.

Our presentation will provide an existence result pertaining to this problem, which relies on Schauder's fixed point theorem.

Lemma 3.1: Let be a given function and consider the linear equation:

$c_{D_{0^+}^{\alpha}} x(t) = y(t)$

Then the linear fractional differential equation:

$$c_{D_{0^+}^{\alpha}} x(t) = y(t) \quad , \quad t\epsilon[0,T], \quad 1 < \alpha \leq 2 , 0 < \beta \leq 1 \qquad \ldots(3)$$

subject to separated fractional boundary conditions :

$$a_1 x(0) + b_1 \left(c_{D_{0^+}^{\gamma}} x(0)\right) = c_1, a_2 x(T) + b_2 \left(c_{D_{0^+}^{\gamma}} x(T)\right) = c_2 \qquad 0 < \gamma < 1 \qquad \ldots.(4)$$

Has a unique solution given by:

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds - \frac{t}{v_1} \left\{ a_2 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s) ds \right\} + v_2 t + \frac{c_1}{a_1}$$

Such that: $v_1 = \frac{a_2 \, T\Gamma(2-\gamma) + b_2 T^{1-\gamma}}{\Gamma(2-\gamma)}$, $v_2 = \frac{a_1 c_2 - a_2 c_1}{a_1 v_1}$

4. Results and discussions

According to Schauder's fixed point theorem, we consider the existence of the solution of the boundary problem (1).

Let: I = [0,T] et C(I) be the space of all continuous real functions defined I and defining the space:

$$\mathbf{X} = \left\{\mathbf{x(t): x(t)} \in \mathbf{C([0,T])} \textbf{and } {}^{c}D_{0^+}^{\beta}\,\mathbf{x(t)} \in \mathbf{C([0,T])}\ , \mathbf{0 < \beta \leq 1}\right\}$$

Muni's Norm: $\|\mathbf{x}\|_{\mathbf{X}} = \|\mathbf{x}\|_\infty + \left\| {}^{c}D_{0^+}^{\beta}\,\mathbf{x(t)}\right\|_\infty$

Here: $\|\mathbf{x}\|_\infty = \max_{t\in[0,T]} |\mathbf{x(t)}|$ and $\left\| {}^{c}D_{0^+}^{\beta}\right\|_\infty = \max_{t\in[0,T]} \left|{}^{c}D_{0^+}^{\beta}\,\mathbf{x(t)}\right|$

(X, $\|.\|$) is a Banach space.

Therefore, the solution of the boundary problem (1)-(2) is equivalent to (3)-(4).

We define the operator F: X → X as follows:

$$\begin{aligned}\mathbf{Fx(t)} &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x(s), {}^{c}D_{0^+}^{\beta}\,x(s)\right) ds \\ &\quad - \frac{t}{v_1}\left\{a_2 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x(s), {}^{c}D_{0^+}^{\beta}\,x(s)\right) ds \right. \qquad \ldots\ldots(5) \\ &\quad \left. + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f\left(s, x(s), {}^{c}D_{0^+}^{\beta}\,x(s)\right) ds \right\} + v_2 t + \frac{c_1}{a_1}\end{aligned}$$

Thus, each solution of the boundary problem (1)-(2) is also a solution of the boundary problem (5).

Therefore, we must show that F is relatively compact so that problem (1)-(2) has at least one solution.

Remark 4.1: ([12]) $I_{0^+}^{\alpha} f(t)$ is the Riemann-Liouville inequality of order α such that :

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$$

Theorem 4.2:

Let :*f*: [0, T] × R → R , $\mathbf{m} \in L^{\frac{1}{\tau}}([0,T], \mathbf{R}^+)$, $\tau \in (0, \alpha - 1)$, satisfying :

$|f(t, x, y)| \leq m(t) + d_1|x|^{\rho_1} + d_2|y|^{\rho_2}$,

Such that: $d_i \geq 0$, $0 \leq \rho_i \leq 1$, i=1,2 .

Thus, the problem (1)-(2) has at least one solution.

Demonstration :

Let us note : $\|\mathbf{m}\| = \left(\int_0^T |\mathbf{m(s)}|^{\frac{1}{\tau}} ds\right)^{\tau}$, Let the set be defined by a closed, bounded, and convex B_r set:

$B_r = \{x \in X : \|x\| < r\}$, $r > 0$.

We set: $\mathbf{Fx = F_1x + F_2x}$, such that: $(F_1x)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Nx)(s) ds$, $(F_2x)(t) = -K_x t + \frac{c_1}{a_1}$

With : $K_x = \frac{1}{v_1}\left\{a_2 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} (Nx)(s) ds + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} (Nx)(s) ds\right\} v_2$

And $(Nx)(t) = f\left(t, x(t), {}^{c}D_{0^+}^{\beta}\,x(t)\right)$, $t \in [0, T]$.

We will prove that **F** maps the set B_r to itself, i.e: $\mathbf{F}(B_r) \subset B_r$.

Calculating : $|(F_1x)(t)|$

$$|(F_1x)(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(Nx)(s)ds\right|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)|\,ds + \frac{d_1|x|^{\rho_1} + d_2\left|{}^{c}D_{0^+}^{\beta}x\right|^{\rho_2}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds$$

Hence**:**

$$\|F_1x\|_\infty \leq \frac{\|m\|}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} + \frac{(d_1r^{\rho_1} + d_2r^{\rho_2})T^\alpha}{\Gamma(\alpha+1)}$$

Calculating:$|(F_2x)(t)|$

$$|(F_2x)(t)| \leq |K_x|T + \frac{c_1}{a_1}$$

$$\leq \frac{T}{v_1}\left\{|a_2|\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left(|m(s)| + d_1|x|^{\rho_1} + d_2\left|{}^{c}D_{0^+}^{\beta}x\right|^{\rho_2}\right)ds + |b_2|\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\left(|m(s)| + d_1|x|^{\rho_1} + d_2\left|{}^{c}D_{0^+}^{\beta}x\right|^{\rho_2}\right)ds\right\} + \frac{|c_1|}{|a_1|} + T|v_2|$$

Hence:

$$\|F_2x\|_\infty \leq \frac{T}{v_1}\left\{\frac{|a_2|\|m\|}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau}T^{\alpha-\tau} + \frac{T^\alpha|a_2|}{\Gamma(\alpha+1)}(d_1r^{\rho_1} + d_2r^{\rho_2}) + \frac{|b_2|\|m\|}{\Gamma(\alpha-\gamma)}\left(\frac{1-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau}T^{\alpha-\gamma-\tau} + \frac{T^{\alpha-\gamma}|b_2|}{\Gamma(\alpha-\gamma+1)}(d_1r^{\rho_1} + d_2r^{\rho_2})\right\} + \frac{|c_1|}{|a_1|} + T|v_2|$$

We then obtain:

$$\|Fx\|_\infty \leq \frac{|c_1|}{|a_1|} + T|v_2| + \frac{\|m\|T^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau}\left(1 + \frac{|a_2|T}{|v_1|}\right) + \frac{|b_2|\|m\|T^{\alpha-\gamma-\tau}}{|v_1|\Gamma(\alpha-\gamma)}\left(\frac{1-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau} + (d_1r^{\rho_1} + d_2r^{\rho_2}) \times \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{|a_2|T^{\alpha+1}}{|v_1|\Gamma(\alpha+1)} + \frac{|b_2|T^{\alpha-\gamma+1}}{|v_1|\Gamma(\alpha-\gamma+1)}\right)$$

Now, let us calculate:$\left\|{}^{c}D_{0^+}^{\beta}F_x\right\|_\infty$

$$\left|{}^{c}D_{0^+}^{\beta}F_x(t)\right| = \left|I_{0^+}^{\alpha-\beta}(Nx)(t) - K_x\frac{t^{1-\beta}}{\Gamma(2-\beta)}\right|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left(|m(s)| + d_1|x|^{\rho_1} + d_2\left|{}^{c}D_{0^+}^{\beta}x\right|^{\rho_2}\right)ds + |K_x|\frac{T^{1-\beta}}{\Gamma(2-\beta)}$$

$$\leq \frac{\|m\|T^{\alpha-\beta-\tau}}{\Gamma(\alpha-\beta)}\left(\frac{1-\tau}{\alpha-\beta-\tau}\right)^{1-\tau} + \frac{(d_1r^{\rho_1} + d_2r^{\rho_2})T^{\alpha-\beta}}{\Gamma(\alpha-\beta-\tau)} + |K_x|\frac{T^{1-\beta}}{\Gamma(2-\beta)}$$

$$\leq \frac{\|\mathbf{m}\|\mathbf{T}^{\alpha-\beta-\tau}}{\Gamma(\alpha-\beta)}\left(\frac{1-\tau}{\alpha-\beta-\tau}\right)^{1-\tau} + \frac{\mathbf{T}^{1-\beta}|\mathbf{v}_2|}{\Gamma(2-\beta)} + \frac{\|\mathbf{m}\|\mathbf{T}^{1-\beta}}{\Gamma(2-\beta)|\mathbf{v}_1|}\left(\frac{|\mathbf{a}_2|\mathbf{T}^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} + \frac{|\mathbf{b}_2|\mathbf{T}^{\alpha-\gamma-\tau}}{\Gamma(\alpha-\gamma)}\left(\frac{1-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau}\right) + (\mathbf{d}_1\mathbf{r}^{\rho_1} + \mathbf{d}_2\mathbf{r}^{\rho_2})\left\{\frac{\mathbf{T}^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\mathbf{T}^{1-\beta}}{|\mathbf{v}_1|\Gamma(2-\beta)}\left(\frac{|\mathbf{a}_2|\mathbf{T}^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\mathbf{b}_2|\mathbf{T}^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right)\right\}$$

Finally, we have:

$$\|\mathbf{Fx}\|_X \leq \frac{|c_1|}{|a_1|} + \mathbf{T}|\mathbf{v}_2| + \frac{\|\mathbf{m}\|\mathbf{T}^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau}\left(1 + \frac{|\mathbf{a}_2|\mathbf{T}}{|\mathbf{v}_1|}\right) + \frac{|\mathbf{b}_2|\|\mathbf{m}\|\mathbf{T}^{\alpha-\gamma-\tau}}{|\mathbf{v}_1|\Gamma(\alpha-\gamma)}\left(\frac{1-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau} + (d_1 r^{\rho_1} + d_2 r^{\rho_2}) \times \left(\frac{\mathbf{T}^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\mathbf{a}_2|\mathbf{T}^{\alpha+1}}{|\mathbf{v}_1|\Gamma(\alpha+1)} + \frac{|\mathbf{b}_2|\mathbf{T}^{\alpha-\gamma+1}}{|\mathbf{v}_1|\Gamma(\alpha-\gamma+1)}\right)$$

$$+\frac{\|\mathbf{m}\|\mathbf{T}^{\alpha-\beta-\tau}}{\Gamma(\alpha-\beta)}\left(\frac{1-\tau}{\alpha-\beta-\tau}\right)^{1-\tau} + \frac{\mathbf{T}^{1-\beta}|\mathbf{v}_2|}{\Gamma(2-\beta)} + \frac{\|\mathbf{m}\|\mathbf{T}^{1-\beta}}{\Gamma(2-\beta)|\mathbf{v}_1|}\left(\frac{|\mathbf{a}_2|\mathbf{T}^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} + \frac{|\mathbf{b}_2|\mathbf{T}^{\alpha-\gamma-\tau}}{\Gamma(\alpha-\gamma)}\left(\frac{1-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau}\right) + (\mathbf{d}_1\mathbf{r}^{\rho_1} + \mathbf{d}_2\mathbf{r}^{\rho_2})\left\{\frac{\mathbf{T}^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\mathbf{T}^{1-\beta}}{|\mathbf{v}_1|\Gamma(2-\beta)}\left(\frac{|\mathbf{a}_2|\mathbf{T}^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\mathbf{b}_2|\mathbf{T}^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right)\right\}$$

Let us note :

$$\mathrm{L} = \frac{|c_1|}{|a_1|} + \mathrm{T}|v_2| + \frac{\|m\|T^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau}\left(1 + \frac{|a_2|T}{|v_1|}\right) + \frac{\mathbf{T}^{1-\beta}|\mathbf{v}_2|}{\Gamma(2-\beta)} + \frac{\|\mathbf{m}\||\mathbf{b}_2|\mathbf{T}^{\alpha-\gamma-\tau+1}}{|\mathbf{v}_1|\Gamma(\alpha-\gamma)}\left(\frac{1-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau}$$

$$+ \frac{\|\mathbf{m}\|\mathbf{T}^{\alpha-\beta-\tau}}{\Gamma(\alpha-\beta)}\left(\frac{1-\tau}{\alpha-\beta-\tau}\right)^{1-\tau} + \frac{\|m\|\mathbf{T}^{1-\beta}}{|\mathbf{v}_1|\Gamma(2-\beta)}\left(\frac{|\mathbf{a}_2|\mathbf{T}^{\alpha}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} + \frac{|\mathbf{b}_2|\mathbf{T}^{\alpha-\gamma}}{\Gamma(\alpha-\gamma)}\left(\frac{1-\tau}{\alpha-\gamma-\tau}\right)^{1-\tau}\right)$$

$$\mathbf{M} = \frac{\mathbf{T}^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\mathbf{a}_2|\mathbf{T}^{\alpha+1}}{|\mathbf{v}_1|\Gamma(\alpha+1)} + \frac{|\mathbf{b}_2|\mathbf{T}^{\alpha-\gamma+1}}{|\mathbf{v}_1|\Gamma(\alpha-\gamma+1)} + \frac{\mathbf{T}^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\mathbf{T}^{1-\beta}}{|\mathbf{v}_1|\Gamma(2-\beta)}\left(\frac{|\mathbf{a}_2|\mathbf{T}^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\mathbf{b}_2|\mathbf{T}^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right)$$

Now, let**r**be a positive number such that:

$$r \geq \max\left\{3L, (3d_1\, M)^{\frac{1}{1-\rho_1}}, (3d_2\, M)^{\frac{1}{1-\rho_2}}\right\}$$

Then:

$$\|\mathrm{Fx}\|_X \leq \mathrm{L} + \mathrm{M}(\mathbf{d}_1\mathbf{r}^{\rho_1} + \mathbf{d}_2\mathbf{r}^{\rho_2}) .$$

$$\leq \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r .$$

Finally: $\mathrm{F}(\boldsymbol{B}_r) \subset \boldsymbol{B}_r$.

We demonstrate that**F**is compact.
It is clear that the operator**F**is uniformly bounded(demonstrated previously).
Let$B_r \subset X$, a bounded and continuous set $f \in X$ then$f$ is continuous

$$\left|f\left(\mathrm{t}, \mathrm{x(t)}, {}^{c}D_{0^+}^{\beta}\, \mathrm{x(t)}\right)\right| \leq N, \quad t \in [0,T] \;\; et \;\; x \in B_r.$$

We show that **F** is equicontinuous :

For $t_1, t_2 \in [0,T] : 0 \le t_1 < t_2 \le T < 1$, for each $x \in B_r$.
Calculating : $\|(\mathbf{F_1x})(\boldsymbol{t_2}) - (\mathbf{F_1x})(\boldsymbol{t_1})\|_\infty$

$$|(\mathbf{F_1x})(\boldsymbol{t_2}) - (\mathbf{F_1x})(\boldsymbol{t_1})| = \left|\int_0^{t_1} \frac{(\boldsymbol{t_1} - \mathbf{s})^{\alpha-1}}{\Gamma(\alpha)}(\mathbf{Nx})(\mathbf{s})\mathbf{ds} - \int_0^{t_2} \frac{(\boldsymbol{t_2} - \mathbf{s})^{\alpha-1}}{\Gamma(\alpha)}(\mathbf{Nx})(\mathbf{s})\mathbf{ds}\right|$$

$$= \left|\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}(\mathbf{Nx})(\mathbf{s})\mathbf{ds} - \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}(\mathbf{Nx})(\mathbf{s})\mathbf{ds} - \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}(\mathbf{Nx})(\mathbf{s})\mathbf{ds}\right|$$

$$\le \int_{t_1}^{t_2} \frac{(\boldsymbol{t_1} - \mathbf{s})^{\alpha-1}}{\Gamma(\alpha)}|(\mathbf{Nx})(\mathbf{s})|\mathbf{ds} - \int_0^{t_1} \frac{(\boldsymbol{t_2} - \mathbf{s})^{\alpha-1} - (\boldsymbol{t_1} - \mathbf{s})^{\alpha-1}}{\Gamma(\alpha)}|(\mathbf{Nx})(\mathbf{s})|\mathbf{ds}$$

$$\le \frac{(\boldsymbol{t_2} - \boldsymbol{t_1})^{\alpha}\boldsymbol{N}}{\Gamma(\alpha+1)} + \frac{(|(\boldsymbol{t_2} - \boldsymbol{t_1})^{\alpha} - \boldsymbol{t_2}^{\alpha} + \boldsymbol{t_1}^{\alpha}|)\boldsymbol{N}}{\Gamma(\alpha+1)}$$

$$\le \frac{2\boldsymbol{N}(\boldsymbol{t_2} - \boldsymbol{t_1})^{\alpha}}{\Gamma(\alpha+1)} + \frac{\boldsymbol{N}|\boldsymbol{t_2}^{\alpha} - \boldsymbol{t_1}^{\alpha}|}{\Gamma(\alpha+1)}$$

Calculating: $\|(\mathbf{F_2x})(\boldsymbol{t_2}) - (\mathbf{F_2x})(\boldsymbol{t_1})\|_\infty$

$$|(\mathbf{F_1x})(\boldsymbol{t_2}) - (\mathbf{F_1x})(\boldsymbol{t_1})| = \left|-\mathbf{t_2K_x} + \frac{\mathbf{c_1}}{\mathbf{a_1}} + \mathbf{t_1K_x} - \frac{\mathbf{c_1}}{\mathbf{a_1}}\right|$$

$$\le |\mathbf{K_x}||\mathbf{t_2} - \mathbf{t_1}|$$

$$\le \left\{\frac{|a_2|}{|v_1|}\int_0^T \frac{(T-s)^{\alpha-1}N}{\Gamma(\alpha)}ds + \frac{|b_2|}{|v_1|}\int_0^T \frac{(T-s)^{\alpha-\gamma-1}N}{\Gamma(\alpha-\gamma)}ds + |v_2|\right\}|\mathbf{t_2} - \mathbf{t_1}|$$

$$\le \left(\frac{N}{|v_1|}\left(\frac{|a_2|T^{\alpha}}{\Gamma(\alpha+1)} + \frac{|b_2|T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right) + |v_2|\right)|\mathbf{t_2} - \mathbf{t_1}|$$

Hence:

$$\|(\mathrm{F}\mathbf{x})(\boldsymbol{t_2}) - (\mathrm{F}\mathbf{x})(\boldsymbol{t_1})\|_\infty \le \frac{2N(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)} + \frac{N|t_2{}^{\alpha} - t_1{}^{\alpha}|}{\Gamma(\alpha+1)} + \left(\frac{|a_2|T^{\alpha}N}{|v_1|\Gamma(\alpha+1)} + \frac{|b_2|T^{\alpha-\gamma}}{|v_1|\Gamma(\alpha-\gamma+1)} + |v_2|\right)|\mathbf{t_2} - \mathbf{t_1}|$$

Now let us calculate : $\left\|\left({}^{c}D_{0^+}^{\beta}\,\mathrm{Fx}\right)(\boldsymbol{t_2}) - \left({}^{c}D_{0^+}^{\beta}\,\mathrm{F}\mathbf{x}\right)(\boldsymbol{t_1})\right\|_\infty$

$$\left|({}^{c}D_{0^+}^{\beta}\,\mathrm{Fx})(\boldsymbol{t_2}) - ({}^{c}D_{0^+}^{\beta}\,\mathrm{F}\mathbf{x})(\boldsymbol{t_1})\right| = \left|(I_{0^+}^{\alpha-\beta}\mathrm{Nx})(t_2) - \frac{K_x t_2{}^{1-\beta}}{\Gamma(2-\beta)} - (I_{0^+}^{\alpha-\beta}\mathrm{Nx})(t_1) + \frac{K_x t_1{}^{1-\beta}}{\Gamma(2-\beta)}\right|$$

$$\leq \frac{|K_x|}{\Gamma(2-\beta)}\left|t_2{}^{1-\beta}-t_1{}^{1-\beta}\right| + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|(Nx)(s)|ds$$

$$+\int_0^{t_1} \frac{(t_2-s)^{\alpha-\beta-1}-(t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|(Nx)(s)|ds$$

$$\leq \frac{1}{\Gamma(\alpha-\beta)}\left(\frac{N}{|v_1|}\left\{\frac{|a_2|T^{\alpha}N}{|v_1|\Gamma(\alpha+1)}+\frac{|b_2|T^{\alpha-\gamma}}{|v_1|\Gamma(\alpha-\gamma+1)}\right\}+|v_2|\right)$$

$$\times\left|t_2{}^{1-\beta}-t_1{}^{1-\beta}\right| + \frac{2N(t_2-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{N\left|t_2{}^{\alpha-\beta}-t_1{}^{\alpha-\beta}\right|}{\Gamma(\alpha-\beta+1)}.$$

Finally, we obtain:

$$\|(Fx)(t_2)-(Fx)(t_1)\|_X \leq \frac{2N(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)} + \frac{N|t_2{}^{\alpha}-t_1{}^{\alpha}|}{\Gamma(\alpha+1)}$$

$$+\left(\frac{|a_2|T^{\alpha}N}{|v_1|\Gamma(\alpha+1)}+\frac{|b_2|T^{\alpha-\gamma}}{|v_1|\Gamma(\alpha-\gamma+1)}+|v_2|\right)|\mathbf{t_2}-\mathbf{t_1}|$$

$$+\frac{1}{\Gamma(\alpha-\beta)}\left(\frac{N}{|v_1|}\left\{\frac{|a_2|T^{\alpha}N}{|v_1|\Gamma(\alpha+1)}+\frac{|b_2|T^{\alpha-\gamma}}{|v_1|\Gamma(\alpha-\gamma+1)}\right\}+|v_2|\right)$$

$$\times\left|t_2{}^{1-\beta}-t_1{}^{1-\beta}\right| + \frac{2N(t_2-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{N\left|t_2{}^{\alpha-\beta}-t_1{}^{\alpha-\beta}\right|}{\Gamma(\alpha-\beta+1)},$$

When$t_2 \to t_1$and as$\alpha > 1$, $\alpha-\beta > 0$ and $1-\beta \geq 0$ then $|(F\mathbf{x})(\boldsymbol{t_2})-(F\mathbf{x})(\boldsymbol{t_1})| \to 0$

And since*f*is continuous**F** is also continuous. This implies that$f$ isequicontinuous.

We have proved that**F** is equicontinuous and uniformly bounded, according to the Ascoli-Arzela theorem,**F** is relatively compact.

According to Schauder's fixed point theorem, problem (1)-(2) has at least one solution.

Corollary 4.3: we Suppose there exists a constant $\tau \in]0, \alpha-1[$, $f$and a function,
$m \in L^{\frac{1}{\tau}}([0,T], R^{+})$ such that :

$|\boldsymbol{f}\ (\mathbf{t}, \mathbf{x},\mathbf{y})| \leq \mathbf{m(t)} + \boldsymbol{d_1}|\boldsymbol{x}| + \boldsymbol{d_2}|\boldsymbol{y}|$, $d_i \geq 0$, i=1,2 .

If$(\boldsymbol{d_1}+\boldsymbol{d_2})M< 1$, then problem (1) -(2) has at least one solution.
Moving on to the practice.

5.Example :

Be the problem of nonlinear fractional differential equations:

$$ {}^{c}D_{0^+}^{\frac{5}{3}} x(t) = (5t^2 - 3t)e^{-x^2(t)} + \frac{1}{2\pi}|x(t)|^{\frac{1}{3}} + \left(\frac{\left| {}^{c}D_{0^+}^{\frac{3}{4}} x(t) \right|}{1 + |\sin^2 x(t)|} \right)^{\frac{1}{2}}, t\epsilon[0,T] \dots\dots (6) $$

Subject to separated fractional boundary conditions:

$$ x(0) + b_1 \left({}^{c}D_{0^+}^{\frac{1}{2}} x(0) \right) = 2.5, 2x(1) + \frac{1}{5}\left({}^{c}D_{0^+}^{\frac{1}{2}} x(1) \right) = \pi. \quad \dots\dots(7) $$

Let : $\alpha = \frac{5}{3}$, $\beta = \frac{3}{4}$, $\gamma = \frac{1}{2}$, $T = 1$, $a_1 = 1$, $c_1 = 2.5$, $a_2 = 2$, $b_2 = \frac{1}{5}$ et $c_2 = \pi$,
In this case, we have :

$$ f(t,x,y) = (5t^2 - 3t)e^{-x^2(t)} + \frac{1}{2\pi}|x(t)|^{\frac{1}{3}} + \left(\frac{|y|}{1 + \sin^2 x} \right)^{\frac{1}{2}}, $$

Which implies:

$$ |f(t,x,y)| \leq \left|5t^2 - 3t\right| + \frac{1}{2\pi}|x|^{\frac{1}{3}} + |y|^{\frac{1}{2}} $$

Such that: $d_1 = \frac{1}{2\pi}$, $d_2 = 1$, $\rho_1 = \frac{1}{3}$, $\rho_2 = \frac{1}{2}$ and $m(t) = \left|5t^2 - 3t\right| \in L^{\infty}[0,1]$.
It is easy to verify that the conditions of theorem (1) are satisfied, therefore the problem (6)-(7) has at least one solution in [0,1].

6 .Conclusion

Using Schauder's fixed point theorem, we were able to determine the existence of at least one solution to the separated boundary value problem.

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