UtilitasMathematica
ISSN 0315-3681 Volume 120, 2023

Solvability of a coupled hemivariational system in anti planes hear

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Abstract

In this research paper, we present some results on the solvability of the evolutionary system constituting two types of hemivariational inequalities in the study of a piezoelectric frictional contact problem with specific subdifferential boundary conditions. Here, the modeling of quasistatic contact processes is described with linear electro-viscoelastic behavior in the case of antiplane shear deformation, where the foundation is assumed conductive. The novelty of this modeling is the dependence of the friction force on the total slip-rate with electrical effects, while the conductivity coefficient only depends on the total slip rate. We use a variational formula to express this friction model, which is a coupled system composed of two hemivariational inequalities, one of which is specific to the displacement field and the other is time-dependent for the electric potential field. Then, we show that this mathematical model has a unique weak solution, the proof is based on results of evolutionary and history-dependent inclusions under certain assumptions.

Keywords: Antiplane shear; evolutionary hemivariational inequality; Clarke subdifferential; electro-viscoelastic body;total slip-rate dependent.

1 Introduction

This paper focuses on the generalized class of variational inequalities, using Clarke's notion of the generalized gradient of a class of locally Lipschitz functionals in the study of hemivariational inequalities which are concerned with nondifferentiable, nonsmooth and nonconvex energy functions. Typically, hemivariational studies enable us to solve various mechanical problems through multivalued mathematical models describing the behavior of the weak solution under different conditions. In this study, we will concentrate on the formulation of hemivariational inequalities for modeling contact problems in piezoelectric material systems characterized by the coupling between mechanical and electrical properties. The aim of this work is to further study the difficulties of frictional contact in the context of antiplane shear with electro-viscoelastic materials, using standard types of hemivariational inequalities, but with distinct forms because the problems under consideration have different boundary conditions, and the solvability of the corresponding contact problem depends on the types of subdifferential inclusions.
Our article presents a continuation of the other researcher's antiplane shear study in [16], [25], [8],[22], [5], [13]and its references. It's devoted to the study of a new linear mathematical system describing multivalued forms of boundary conditions. The presence of electrical effects and the total slip rates in subdifferential boundary conditions, due to the conductivity of the foundation, make the novelty of this mathematical system.

The paper's content is structured as follows. In Section 2, we first explain the physical process of an electro-viscoelastic cylinder in contact with a conductive foundation, then, we derive a mathematical framework that makes sense. In Section 3, the problem is formulated in a variational form as a linear system coupling two hemivariational inequalities of displacement and electric potential fileds, under a smallness assumption on the data. In Sections 4, 5 and 6, we mention our most important results in solving contact problems, which we rely on in the next section. The proof of principal theorem 3.1 is presented in Section 7.

Now, we recall the definition of the generalized subdifferential of locally Lipschitz function $\phi: X \longrightarrow \mathbb{R}$ by

$$\partial\phi(x) = \{\zeta \in X' : \phi^0(x; v) \geq <\zeta, v>_{X' \times X} \ \forall v \in X\}$$

where $\phi^0(x; v)$is the generalized directional derivative of ϕat $x \in X$in the direction$v \in X$, with X is a Banach space and that X' its dual space. For more details see [3].

2 The physical setting of antiplane shear

For further detailson antiplane shear deformations, we direct the reader to the references [1], [24], [25], [9].
We consider a sufficiently long piezoelectric cylinder $\mathrm{B} = \Omega \times (-\infty, +\infty)$, with generators parallel to the x_3-axes having a cross section that is a regular region $\Omega \subset \mathbb{R}^2$ in the x_1, x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system.
To describe the boundary conditions, we denote by Γ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, such that meas$\Gamma_1 > 0$. We assume that the cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and it's in contact with a rigid foundation on $\Gamma_3 \times (-\infty, +\infty)$.
On the one hand, we consider a partition of $\Gamma_1 \times (-\infty, +\infty) \cup \Gamma_2 \times (-\infty, +\infty)$ into two open parts $\Gamma_a \times (-\infty, +\infty)$ and $\Gamma_b \times (-\infty, +\infty)$, such that meas $\Gamma_a > 0$.
The cylinder is subjected to time-dependent volume forces of density f_0 in $\Omega \times (-\infty, +\infty)$ and to time-dependent surface tractions of density f_2 on $\Gamma_2 \times (-\infty, +\infty)$. We also assume that the electrical potential vanishes in $\Gamma_a \times (-\infty, +\infty)$, a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (-\infty, +\infty)$ and a free volume electric charge of density q_0 is prescribed in bounded domain Ω.
We are interested in this body's deformation on the time interval of interest $[0, T]$, with $T > 0$. Everywhere in this paper, the dots represent the derivatives with respect to time, i.e. $\dot{u} = \frac{\partial u}{\partial t}$ and the index that follows a comma represents the partial derivative with respect to the corresponding spatial variable, i.e. $u_{,i} = \frac{\partial u}{\partial x_i}$, $i = 1,2$.
We use $\mathcal{S}^3$ for the linear space of second order symmetric tensors on $\mathbb{R}^3$ or, equivalently, the space of symmetric matrices of order 3, and " $\cdot$ ", $\|\cdot\|$ will represent the inner products and the Euclidean norm on $\mathbb{R}^3$ and $\mathcal{S}^3$; we have:

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i, \quad \| \mathbf{v} \| = (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}} \textit{ for all} \mathbf{u} = (u_i),\ \mathbf{v} = (v_i) \in \mathbb{R}^3,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij},\ \| \boldsymbol{\tau} \| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \textit{ for all } \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathcal{S}^3.$$

We assume that the forces and the electric charges are given by

$$\mathbf{f}_0 = (0,0,f_0) \quad \textit{with} f_0 = f_0(x_1, x_2, t): \Omega \times [0, T] \to \mathbb{R}, \tag{1}$$
$$\mathbf{f}_2 = (0,0,f_2) \textit{with} f_2 = f_2(x_1, x_2, t): \Gamma_2 \times [0, T] \to \mathbb{R}, \tag{2}$$
$$q_0 = q_0(x_1, x_2, t): \Omega \times [0, T] \to \mathbb{R}, \tag{3}$$
$$q_2 = q_2(x_1, x_2, t): \Gamma_b \times [0, T] \to \mathbb{R}. \tag{4}$$

The displacement $\mathbf{u}$ and the electric potential field φ which are independent on x_3 and have the form

$$\mathbf{u} = (0,0,u) \textit{with } u = u(x_1, x_2, t): \Omega \times [0, T] \to \mathbb{R}, \tag{5}$$
$$\varphi = \varphi(x_1, x_2, t): \Omega \times [0, T] \to \mathbb{R}. \tag{6}$$

The forms ofthe strain tensor $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ and the electric field $\mathbf{E}(\varphi) = (E_i(\varphi))$ are

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}), \quad E_i(\varphi) = -\varphi_{,i}.$$

The stress tensor $\boldsymbol{\sigma} = (\sigma_{ij})$ and the electric displacement field $\mathbf{D} = (D_i)$ are modeled by linear constitutive laws

$$\boldsymbol{\sigma} = 2\theta\varepsilon(u) + \zeta \mathrm{tr}(\varepsilon(u))I + 2\mu\varepsilon(u) + \lambda(\mathrm{tr}\varepsilon(u))I - \mathcal{E}^* E(\varphi), \tag{7}$$
$$\mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \beta\mathbf{E}(\varphi), \tag{8}$$

where θ and ζ are viscosity coefficients which $\theta > 0$ and $\zeta \geq 0$, λ and μ are the Lamé coefficients, $\mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}) =$

$\varepsilon_{ii}(\mathbf{u})$, $\mathbf{I}$ is the unit tensor in $\mathbb{R}^3$, β is the electric permittivity constant, $\mathcal{E}$ represents the third-order piezoelectric tensor and $\mathcal{E}^*$ is its transpose. We assume that

$$\mathcal{E}\boldsymbol{\varepsilon} = \begin{pmatrix} e(\varepsilon_{13}+\varepsilon_{31}) \\ e(\varepsilon_{23}+\varepsilon_{32}) \\ e\varepsilon_{33} \end{pmatrix} \quad \forall \boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathcal{S}^3, \tag{9}$$

where e is a piezoelectric coefficient. We also assume that the coefficients θ, μ, β and e depend on the spatial variables x_1,x_2, but are independent on the spatial variable x_3. Since $\mathcal{E}\boldsymbol{\varepsilon} \cdot \mathbf{v} = \boldsymbol{\varepsilon} \cdot \mathcal{E}^*\mathbf{v}$ for all $\boldsymbol{\varepsilon} \in \mathcal{S}^3$,$\mathbf{v} \in \mathbb{R}^3$, it follows from (9) that

$$\mathcal{E}^*\boldsymbol{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix} \quad \forall \boldsymbol{v} = (v_i) \in \mathbb{R}^3. \tag{10}$$

In the antiplane context (5), (6), using the constitutive equations (7), (8) and equalities (9), (10) it follows that the stress field and the electric displacement field are given by

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{pmatrix}$$

$$\begin{cases} \sigma_{13}(t) = \sigma_{31}(t) = \theta\dot{u}_{,1}(t) + \mu u_{,1}(t) + e\varphi_{,1}(t), \\ \sigma_{23}(t) = \sigma_{32}(t) = \theta\dot{u}_{,2}(t) + \mu u_{,2}(t) + e\varphi_{,2}(t). \end{cases} \tag{11}$$

$$\boldsymbol{D} = \begin{pmatrix} eu_{,1} - \beta\varphi_{,1} \\ eu_{,2} - \beta\varphi_{,2} \\ 0 \end{pmatrix}. \tag{12}$$

The balance equations reduce to the following scalar equations

$$Div\boldsymbol{\sigma} + \boldsymbol{f}_0 = 0, \quad divD - q_0 = 0 \quad in B \times (0,T),$$

where $Div\boldsymbol{\sigma} = (\sigma_{ij,j})$ represents the divergence of the tensor field $\boldsymbol{\sigma}$ and ρ denotes the density of mass. Taking into account (11), (12), (5), (6), (1) and (3), the equilibrium equations above reduce to the following scalar equations

$$Div\big(\theta\nabla\dot{u}(t) + \mu\nabla u(t) + e\nabla\varphi(t)\big) + f_0(t) = 0, \quad in \Omega \times (0,T), \tag{13}$$

$$div\big(e\nabla u(t) - \beta\nabla\varphi(t)\big) = q_0(t), \qquad in \Omega \times (0,T). \tag{14}$$

During the process the cylinder is clamped on $\Gamma_1 \times (-\infty,+\infty) \times (0,T)$ and the electric potential vanishes on $\Gamma_a \times (-\infty,+\infty) \times (0,T)$; thus, (5) and (6) imply that

$$u = 0 \quad on \Gamma_1 \times (0,T), \tag{15}$$

$$\varphi = 0 \quad on \Gamma_a \times (0,T). \tag{16}$$

The Cauchy stress vector and the normal component of the electric displacement field are given by

$$\boldsymbol{\sigma}(t) = \big(0,0,\theta\partial_\nu\dot{u}(t) + \mu\partial_\nu u(t) + e\partial_\nu\varphi(t)\big), \quad \boldsymbol{D}\cdot\boldsymbol{\nu}(t) = e\partial_\nu u(t) - \beta\partial_\nu\varphi(t) \tag{17}$$

where $\boldsymbol{\nu}$ denote the unit normal on $\Gamma \times (-\infty,+\infty)$ defined by

$$\boldsymbol{\nu} = (\nu_1,\nu_2,0) \quad with \nu_i = \nu_i(x_1,x_2): \Gamma \to \mathbb{R}, \quad i = 1,2. \tag{18}$$

Taking into account (2), (4) and (17), the traction condition on Γ_2 and the electric conditions on Γ_b are given by

$$\theta\partial_\nu\dot{u}(t) + \mu\partial_\nu u(t) + e\partial_\nu\varphi(t) = f_2(t) on \Gamma_2 \times (0,T), \tag{19}$$

$$e\partial_\nu u(t) - \beta\partial_\nu\varphi(t) = q_2(t) on \Gamma_b \times (0,T). \tag{20}$$

The boundary conditions on $\Gamma_3 \times (-\infty,+\infty) \times (0,T)$ are

$$\boldsymbol{u}_\tau(t) = (0,0,u(t)), \quad \boldsymbol{\sigma}_\tau(t) = (0,0,\mu\partial_\nu u(t) + \theta\partial_\nu\dot{u}(t) + e\partial_\nu\varphi(t)). \tag{21}$$

We assume that the foundation is conductive. The friction and electric contact subdifferential boundary conditions are represented by the expression

$$-(\mu\partial_\nu u(t)+\theta\partial_\nu\dot{u}(t)+e\partial_\nu\varphi(t))\in h(S_t(\dot{u}),\varphi(t)-\varphi_F)\partial j(\dot{u}(t)) \quad (22)$$
$$e\partial_\nu u(t)-\beta\partial_\nu\varphi(t)\in h_e\big(S_t(\dot{u})\big)\partial j_e(\varphi(t)-\varphi_F) \quad (23)$$

where $S_t(\dot{u})=\int_0^t|\dot{u}(s)|ds$, for all $t\in[0,T]$, represents the total slip rate, φ_F represents the electric potential
of the foundation assumed to be given and the electric charges on the contact surface are proportional to the difference of potential$(\varphi-\varphi_F)$ with a total slip rate dependent proportionality coefficient. Here and below j,j_e,h and h_e are prescribed functions which may depend on x_1 and x_2 but do not depend on x_3,∂j and ∂j_e denote the Clarke subdifferentials of the functions j and j_e respectively.
Finally, we prescribe the initial displacement by

$$u(0)=u_0 in\Omega, \quad (24)$$

where u_0 is given function in Ω.

Problem$\mathcal{P}$: Find the displacement field $u:\Omega\times[0,T]\to\mathbb{R}$ and the potential electric field $\varphi:\Omega\times[0,T]\to\mathbb{R}$ such that

$$\begin{cases} div\ (\theta\nabla\dot{u}(t)+\mu\nabla u(t)+e\nabla\varphi(t))+f_0(t)=0 & in\Omega\\ div\ (e\nabla u(t)+\beta\nabla\varphi(t))=q_0(t) & in\Omega\\ u(t)=0 & on\Gamma_1\\ \varphi(t)=0 & on\Gamma_a\\ \theta\partial_\nu\dot{u}(t)+\mu\partial_\nu u(t)+e\partial_\nu\varphi(t)=f_2(t) & in\Gamma_2\\ e\partial_\nu u(t)-\beta\partial_\nu\varphi(t)=q_2(t) & in\Gamma_b\\ \left.\begin{array}{l}-(\theta\partial_\nu\dot{u}(t)+\mu\partial_\nu u(t)+e\partial_\nu\varphi(t))\in h(S\dot{u}(t),\varphi(t)-\varphi_F)\partial j(\dot{u}(t))\\ e\partial_\nu u(t)-\beta\partial_\nu\varphi(t)\in h_e(S\dot{u}(t))\partial j_e(\varphi(t)-\varphi_F)\end{array}\right\} & on\Gamma_3\\ u(0)=u_0 & in\Omega \end{cases} \quad (25)$$

for all $t\in[0,T]$.
We introduce the function spaces to derive a variational formulation to this problem

$$V=\{v\in H^1(\Omega): v=0\ \ on\Gamma_1\},\quad W=\{\psi\in H^1(\Omega):\psi=0\ \ on\Gamma_a\}$$

where, here and below, we write w for the trace γw of a function $w\in H^1(\Omega)$ on Γ. Since $meas\Gamma_1>0$ and $meas\Gamma_a>0$, it is well known that V and W are real Hilbert spaces with the inner products

$$(u,v)_V=\int_\Omega\nabla u\cdot\nabla v dx\quad\forall u,v\in V,\quad(\varphi,\psi)_W=\int_\Omega\nabla\varphi\cdot\nabla\psi dx\quad\forall\varphi,\psi\in W.$$

Moreover, the associated norms

$$\|v\|_V=\|\nabla v\|_{L^2(\Omega;\mathbb{R}^2)}\quad\forall v\in V,\quad\|\psi\|_W=\|\nabla\psi\|_{L^2(\Omega;\mathbb{R}^2)}\quad\forall\psi\in W. \quad (26)$$

are equivalent on V and W, respectively, with the usual norm $\|\cdot\|_{H^1(\Omega)}$. By Sobolev's trace theorem we deduce that there exist two positive constants $c_V>0$ and $c_W>0$ such that

$$\|v\|_{L^2(\Gamma_3)}\le c_V\|v\|_V\quad\forall v\in V,\quad\|\psi\|_{L^2(\Gamma_3)}\le c_W\|\psi\|_W\quad\forall\psi\in W. \quad (27)$$

In the study of problem $\mathcal{P}$ we assume that the viscosity, the Lamé, the electric permittivity and the piezoelectric coefficients satisfy

$$\boldsymbol{H}(c):\begin{cases} a)\theta\in L^\infty(\Omega)\ and\ there\ exists\ \ \theta^*>0\ \ such\ that\ \ \theta(\boldsymbol{x})\ge\theta^*\\ \quad a.e.\ \ \boldsymbol{x}\in\Omega.\\ b)\mu\in L^\infty(\Omega).\\ c)\beta\in L^\infty(\Omega)\ and\ there\ exists\ \ \beta^*>0\ \ such\ that\ \ \beta(\boldsymbol{x})\ge\beta^*\\ \quad a.e.\ \ \boldsymbol{x}\in\Omega.\\ d)e\in L^\infty(\Omega).\end{cases}$$

The forces and surface free charge densities have the regularity
$\boldsymbol{H}(f)$: $f_0\in W^{1,2}(0,T,L^2(\Omega)),\quad f_2\in W^{1,2}(0,T,L^2(\Gamma_2))$.
$\boldsymbol{H}(q)$: $q_0\in W^{1,2}(0,T,L^2(\Omega)),\quad q_2\in W^{1,2}(0,T,L^2(\Gamma_b))$.
The functions h_e,j_e, h and j satisfy

$$\boldsymbol{H}(h_e):\begin{cases} h_e: \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}; \\ (a) \exists L_{h_e} \geq 0 \quad such that \quad |h_e(x,r_1) - h_e(x,r_2)| \leq L_{h_e}|r_1 - r_2|, \\ \quad \forall r_1, r_2 \in \mathbb{R} \quad a.e. \quad x \in \Gamma_3; \\ (b) \forall r \in \mathbb{R}, \quad k(.,r) \ is Lebesgue measurable on \Gamma_3, h_e(.,0) \in L^1(\Gamma_3). \\ (c) 0 \leq h_e(x,r) \leq \overline{h}_e, \forall r \in \mathbb{R} \quad a.e. \quad x \in \Gamma_3, \ with \overline{h}_e > 0. \end{cases}$$

$$\boldsymbol{H}(j_e):\begin{cases} j_e: \Gamma_3 \times \mathbb{R} \to \mathbb{R} \ and , \\ (a) \forall r \in \mathbb{R}, j_e(.,r) \ is Lebesgue measurable on \Gamma_3, j_e(.,0) \in L^1(\Gamma_3) \\ (b) j_e(x,.) \ is locally Lipschitz for a.e. x \in \Gamma_3, with coefficient L_{j_e} > 0. \\ (c) |\partial j_e(x,r)| \leq c_{0_e} + c_{1_e}|r| \ for all r \in \mathbb{R} \ for a.e. x \in \Gamma_3, c_{0_e}, c_{1_e} \geq 0 \\ (d) j_e{}^0(x,r;-r) \leq d_e(1+|r|) \, for all r \in \mathbb{R} \ for a.e. x \in \Gamma_3 with d_e \geq 0 \\ (e) \ either j_e(x,.) \ or \ -j_e(x,.) \ is regular for a.e. x \in \Gamma_3 \\ (f)(p_1 - p_2)(r_1 - r_2) \geq -m_e|r_1 - r_2|^2 for all p_i \in \partial j_e(x,r_i), r_i \in \mathbb{R}, \\ i = 1,2, \ a.e. \ x \in \Gamma_3 with m_e \geq 0. \end{cases}$$

$$\boldsymbol{H}(h):\begin{cases} h: L^2(\Gamma_3) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \ and , \\ (a) h(.,r_1,r_2) \, is measurable for \Gamma_3 for all r_1, r_2 \in \mathbb{R} \\ (b) h(x,.,.) \ is locally Lipschitz on \ \mathbb{R} \times \mathbb{R} \ for a.e. x \in \Gamma_3, \\ with coefficient L_h > 0. \\ (c) 0 \leq h(x,r_1,r_2) \leq \overline{h} for all r_1, r_2 \in \mathbb{R}, \quad a.e. \ x \in \Gamma_3 with \overline{h} > 0. \end{cases}$$

$$\boldsymbol{H}(j):\begin{cases} j: \Gamma_3 \times \mathbb{R} \to \mathbb{R} \ and , \\ (a) \forall r \in \mathbb{R}, j(.,r) \ is Lebesgue measurable on \Gamma_3 and j(.,0) \in L^1(\Gamma_3) \\ (b) j(x,.) \ is locally Lipschitz for a.e. x \in \Gamma_3, with coefficient L_j > 0. \\ (c) |\partial j(x,r)| \leq c_0 + c_1|r| \ for all r \in \mathbb{R} \ for a.e. x \in \Gamma_3, c_0, c_1 \geq 0 \\ (d) j^0(x,r;-r) \leq d(1+|r|) \, for all r \in \mathbb{R} \ for a.e. x \in \Gamma_3 with d \geq 0 \\ (e) \ either j(x,.) \ or \ -j(x,.) \ is regular for a.e. x \in \Gamma_3 \\ (f)(p_1 - p_2)(r_1 - r_2) \geq -m|r_1 - r_2|^2 for all p_i \in \partial j(x,r_i), r_i \in \mathbb{R}, \\ i = 1,2, a.e. \ x \in \Gamma_3 with m \geq 0. \end{cases}$$

The initial data and the electric potential of the foundation are given by

$$\boldsymbol{H}(0): \ u_0 \in V.$$
$$\boldsymbol{H}(\varphi_F): \ \varphi_F \in L^2(\Gamma_3).$$

Next, we define the bilinear forms $a_\theta: V \times V \to \mathbb{R}$,$a_\mu: V \times V \to \mathbb{R}$,$a_e: V \times W \to \mathbb{R}$,$a_e^*: W \times V \to \mathbb{R}$, and $a_\beta: W \times W \to \mathbb{R}$, by equalities

$$a_\theta(u,v) = \int_\Omega \theta \nabla u \cdot \nabla v dx, \tag{28}$$
$$a_\mu(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v dx, \tag{29}$$
$$a_e(u,\varphi) = \int_\Omega e \nabla u \cdot \nabla \varphi dx = a_e^*(\varphi, u), \tag{30}$$
$$a_\beta(\varphi,\psi) = \int_\Omega \beta \nabla \varphi \cdot \nabla \psi dx, \tag{31}$$

for all $u, v \in V$ and $\varphi, \psi \in W$.

Assumptions $\mathbf{H}(c)$ imply that the integrals above are well defined and, using (26) and (27), it follows that the symmetrical forms a_θ, a_μ, a_e, a_e^* and a_β are continuous; moreover, the forms$a_\theta$and $a_\beta$are$V$-elliptic and W-elliptic, respectively, such that

$$a_\theta(v,v) \geq \theta^* \parallel v \parallel_V^2 \quad \forall v \in V, \tag{32}$$
$$a_\beta(\psi,\psi) \geq \beta^* \parallel \psi \parallel_W^2 \quad \forall \psi \in W. \tag{33}$$

By assumptions $\mathbf{H}(f)$ and $\mathbf{H}(q)$, We can define the mappings $f: [0,T] \to V'$ and $q: [0,T] \to W'$ respectively, by

$$(f(t),v)_{V'\times V} = \int_{\Omega} f_0(t)vdx + \int_{\Gamma_2} f_2(t)vda, \tag{34}$$

$$(q(t),\psi)_{W'\times W} = \int_{\Omega} q_0(t)\psi dx - \int_{\Gamma_b} q_2(t)\psi da, \tag{35}$$

for all $v \in V$, $\psi \in W$, a.e. $t \in (0,T)$, also, V' and W' denote the dual space of V and W respectively. $(.,.)_{X'\times X}$ represents the duality paring of space X.

The assumptions $\mathbf{H}(f)$ and $\mathbf{H}(q)$ combined with the trace theorem implies that f and q have the regularity

$$f \in W^{1,2}(0,T;V'), \tag{36}$$

$$q \in W^{1,2}(0,T;W'). \tag{37}$$

For every $t \in [0,T]$ we need to consider the operator S defined by

$$S_t: L^2(0,T;V) \to L^2(0,T;L^2(\Gamma_3)),$$

$$S_t(v)(x) = \int_0^t |\gamma v(s)(x)| ds\, a.e. \quad x \in \Omega \cup \Gamma_3. \tag{38}$$

From (38), it follows that the for all $v_1, v_2 \in L^2(0,T;V)$ and $t \in [0,T]$, the following inequality holds

$$\| S_t(v_1) - S_t(v_2) \|_{L^2(\Gamma_3)} \leq \sqrt{t} \| \gamma \| \| v_1 - v_2 \|_{L^2(0,t;V)}. \tag{39}$$

3 Variational formulation

We turn now to the variational formulation of the contact problem $\mathcal{P}$. Let $v \in V$, we use standard arguments based on the equilibrium equation (13), the boundary conditions (15),(16), the forms (28)–(31) and use the Green formula to find that

$$a_\theta(\dot{u}(t),v) + a_\mu(u(t),v) + a_e^*(\varphi(t),v) = \int_\Omega f_0(t)vdx + \int_{\Gamma_2} f_2(t)vda + \int_{\Gamma_3} (\theta\partial_\nu\dot{u}(t) + \mu\partial_\nu u(t) + e\partial_\nu\varphi(t))\, vda. \tag{40}$$

The subdifferential inclusions (22) we give

$$-(\theta\partial_\nu\dot{u}(t) + \mu\partial_\nu u(t) + e\partial_\nu\varphi(t))v \leq h(S_t(\dot{u}),\varphi(t) - \varphi_F)j^0(\dot{u}(t);v)$$

which implies

$$\int_{\Gamma_3} (\theta\partial_\nu\dot{u}(t) + \mu\partial_\nu u(t) + e\partial_\nu\varphi(t))vda \geq -\int_{\Gamma_3} h(S_t(\dot{u}),\varphi(t) - \varphi_F)\, j^0(\dot{u}(t);v)da. \tag{41}$$

Using the inequality (41) with the defintion of the element $f \in V'$ given by (34) to see that the previous inequality (40) becomes

$$a_\theta(\dot{u}(t),v) + a_\mu(u(t),v) + a_e^*(\varphi(t),v) + \int_{\Gamma_3} h(S_t(\dot{u}),\varphi(t) - \varphi_F)\, j^0(\dot{u}(t);v)da \geq (f(t),v)_{V'\times V}. \tag{42}$$

Similarly, for every $\psi \in W$, from (14),(16),(20), (23), (35) and Green formula, we deduce that

$$a_\beta(\varphi(t),\psi) - a_e(u(t),\psi) + \int_{\Gamma_3} h_e(S_t(\dot{u}))j_e^0(\varphi(t) - \varphi_F;\psi)\, da \geq (q(t),\psi)_{W'\times W} \tag{43}$$

We have the following variational formula of Problem $\mathcal{P}$.

Problem$\mathcal{P}^{\mathcal{V}}$: Find the displacement $u: [0,T] \to V$ and the electric potential $\varphi: [0,T] \to W$ such that

$$a_\theta(\dot{u}(t),v) + a_\mu(u(t),v) + a_e^*(\varphi(t),v) + \int_{\Gamma_3} h(S_t(\dot{u}),\varphi(t) - \varphi_F)j^0(\dot{u}(t);v)da \geq (f(t),v)_{V'\times V}, \quad \forall v \in V, \;\; a.e. \;\; t \in (0,T), \tag{44}$$

$$a_\beta(\varphi(t),\psi) - a_e(u(t),\psi) + \int_{\Gamma_3} h_e(S_t(\dot{u}))j_e^0(\varphi(t) - \varphi_F;\psi)\, da \geq (q(t),\psi)_{W'\times W}, \forall\psi \in W, a.e. \;\; t \in (0,T). \tag{45}$$

$$u(0) = u_0. \tag{46}$$

Problem $\mathcal{P}^{\mathcal{V}}$ represents a system of hemivariational inequalities. One of the main features of this system arises in the strong coupling between the unknowns u and φ. Our main results in the study of problem $\mathcal{P}$ are presented in Theorem 4.1 and Proposition5.1 .

Theorem 3.1 Assume that **H**(c), **H**(f), **H**(q), **H**(j),**H**(j_e),**H**(h), **H**(h_e), **H**(φ_F), **H**(0), (36), (37) hold and, moreover, assume that

$$\theta^* > m\overline{h} \parallel \gamma \parallel^2. \quad (47)$$

$$\beta^* > m_e\overline{h}_e \parallel \gamma \parallel^2. \quad (48)$$

$$min\{\theta^*, \beta^*\} > max\{m\overline{h}, m_e\overline{h}_e\} \parallel \gamma \parallel^2 + L_h\tilde{c}_2 \parallel \gamma \parallel^2, with\tilde{c}_2 > 0. \quad (49)$$

Then, there exists a unique solution of problem $\mathcal{P}^{\mathcal{V}}$ which satisfies

$$(u, \varphi) \in W^{1,2}(0, T; V \times W). \quad (50)$$

The proof of Theorem 3.1 will be carried out in several steps.

4 Main results

Here and bellow c represents a changingconstant.
We consider the space $Z = H^{1/2}(\Omega; \mathbb{R}^2)$, the trace operator $\gamma: Z \to L^2(\Gamma; \mathbb{R}^2)$ and we denote by $\parallel \gamma \parallel = \parallel \gamma \parallel_{\mathcal{L}(Z;(L^2(\Gamma);\mathbb{R}^2))}$ and we put $\mathcal{Z} = L^2(0, T; Z)$ with $\mathcal{Z}' = L^2(0, T; Z')$ the dual of $\mathcal{Z}$.

We put the product space $Y = V \times W \subset H^1(\Omega; \mathbb{R}^2)$, which is a Hilbert space endowed with the inner product

$$(y, z)_Y = (w, v)_V + (\varphi, \psi)_W forally = (w, \varphi) \in Y, z = (v, \psi) \in Y$$

and the associated norm $\parallel . \parallel_Y$

$$\parallel z \parallel_Y^2 = \parallel v \parallel_V^2 + \parallel \psi \parallel_W^2 \ forallz = (v, \psi) \in Y.$$

Combine the two inequality (44) and (45), to find

$$a_\theta(\dot{u}(t), v) + a_\mu(u(t), v) + a_e^*(\varphi(t), v) + \int_{\Gamma_3} h(S_t(\dot{u}), \varphi(t) - \varphi_F)\, j^0(\dot{u}(t); v) da$$
$$+ a_\beta(\varphi(t), \psi) - a_e(u(t), \psi) + \int_{\Gamma_3} h_e(S_t(\dot{u})) j_e^0(\varphi(t) - \varphi_F; \psi)\ da$$
$$\geq (f(\mathrm{t}), v)_{V' \times V} + (q(t), \psi)_{W' \times W}, \forall (v, \psi) \in Y, a.e.\ \ t \in (0, T). \quad (51)$$

We define also the functional $J_t: \Sigma_3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$J_t(x, r_1, r_2) = \int_{\Gamma_3} h(S_t(r_1), r_2 - \varphi_F)\, j(r_1) da + \int_{\Gamma_3} h_e(S_t(r_1)) j_e(r_2 - \varphi_F)\, da, \quad (52)$$

for all $r_1, r_2 \in \mathbb{R}$, such that $\Sigma_3 = \Gamma_3 \times (0, T)$.
Under assumptions **H**(j),**H**(j_e),**H**(h), **H**(h_e) and **H**(φ_F), the functional defined by (52) satisfies

$$J_t^0(x, r_1, r_2; \tilde{z}) = \int_{\Gamma_3} h(S_t(r_1), r_2 - \varphi_F)\, j^0(r_1; \tilde{v}) da + \int_{\Gamma_3} h_e(S_t(r_1)) j_e^0(r_2 - \varphi_F; \tilde{\psi})\, da \quad (53)$$

for all $r_1, r_2 \in \mathbb{R}$ and $\tilde{z} = (\tilde{v}, \tilde{\psi}) \in Y$.
The inequality (51) becomes

$$a_\theta(\dot{u}(t), v) + a_\mu(u(t), v) + a_e^*(\varphi(t), v) + a_\beta(\varphi(t), \psi) - a_e(u(t), \psi)$$
$$+ J_t^0(x, \dot{u}, \varphi; z) \geq (f(t), v)_{V' \times V} + (q(t), \psi)_{W' \times W}, \forall z = (v, \psi) \in Y, a.e.\ \ t \in (0, T) \quad (54)$$

On the other hand, we defined

$$((f(t), q(t)), z)_{Y' \times Y} = (f(t), v)_{V' \times V} + (q(t), \psi)_{W' \times W}, \forall z = (v, \psi) \in Y, a.e.\ \ t \in (0, T). \quad (55)$$

where $(f, q) \in W^{1,2}(0, T; Y')$ is given by (36) and (37) such that $Y' = V' \times W'$ the dual space of Y.
Now, we introduce the operator $A: Y \to Y'$ defined by

$$(Ay(t), z)_{Y' \times Y} = a_\theta(w(t), v) + a_\beta(\varphi(t), \psi), \quad \forall z \in Y \quad (56)$$

such that $y(t) = (w(t), \varphi(t)) \in Y$, a.e. $t \in (0, T)$.
Consider the functions $h_i, j_i: \Sigma_3 \times \mathbb{R}^2 \to \mathbb{R}$ for i = 1,2 given by

$$h_1(x, r_1, r_2) \quad = h(x, S_t(r_1), r_2 - \varphi_F(x)).$$
$$h_2(x, r_1, r_2) \quad = h_e(x, S_t(r_1)).$$
$$j_1(x, r_1, r_2) \quad = j(x, r_1).$$
$$j_2(x, r_1, r_2) \quad = j_e(x, r_2 - \varphi_F(x)).$$

for all $(r_1, r_2) \in \mathbb{R}^2$ and a.e. $x = (x_1, x_2, t) \in \Sigma_3$.

Under the notation above, we associate the following hemivariational inequality:

$$\left.\begin{aligned} &Find y(t) = (w(t), \varphi(t)) \in Y such that \\ &(Ay(t), z)_{Y' \times Y} + \int_{\Gamma_3} \sum_{i=1}^{2} h_i(\gamma y(t)) \; j_i^0(\gamma y(t); \gamma z) \; da \geq ((f(t), q(t)), z)_{Y' \times Y} \\ &for all z \in Y, \;\; a.e. t \in (0, T) \end{aligned}\right\} \quad (57)$$

Theorem 4.1 Assume that **H**(c)(a,c), **H**(f),**H**(q), **H**(j),**H**(j_e),**H**(h), **H**(h_e), **H**(φ_F), **H**(0), (36) and (37) hold. Then problem (57) has at least one solution.

Proof.
(**i**) Under hypotheses **H**(c)(a, c), the operator A is pseudomonotone. Indeed, by **H**(c)(a, c) and the continuity of forms a_θ, a_β, we have

$$|(Ay(t), z)_{Y' \times Y}| \leq c\{\| w(t) \|_V + \| \varphi(t) \|_W\}\{\| v \|_V + \| \psi \|_W\}$$

for all $y(t) = (w(t), \varphi(t))$, $z = (v, \psi) \in Y$ a.e. $t \in (0, T)$, and $c = max\{m_\theta, m_\beta\} > 0$, such that $m_\theta > 0$ and $m_\beta > 0$ the coefficients obtained from continuity of the forms a_θ and a_β, respectively. Using the inequality $xy \leq \frac{x^2+y^2}{2}$ to obtain

$$|(Ay(t), z)_{Y' \times Y}| \leq c \| y(t) \|_Y \| z \|_Y.$$

This gives $\| Ay(t) \|_{Y'} \leq c \| y(t) \|_Y$ for all $y(t) \in Y$, a.e. $t \in (0, T)$and implies the boundedness of A.
To show the strong monotonicity of A, we use the hypotheses (32) and (33), to see that

$$\begin{aligned} (Ay_1(t) - Ay_2(t), y_1(t) - y_2(t))_{Y' \times Y} &\geq c\{\| w_1(t) - w_2(t) \|_V^2 + \| \varphi_1(t) - \varphi_1(t) \|_W^2\} \\ &= c \| y_1(t) - y_2(t) \|_Y^2 \geq 0 \end{aligned} \quad (58)$$

for all $y_i(t) = (w_i(t), \varphi_i(t)) \in Y$, $i = 1,2$, a.e. $t \in (0, T)$and $c = min\{\theta^*, \beta^*\} > 0$.
By the assumption of continuity of bilinear forms a_θ and a_β, we infer that the operator A is continuous.
The operator A is bounded, monotone, and hemicontinuous, then A is pseudomonotone.
(**ii**) The operator A is coercive. Indeed, by hypotheses (32) and (33), we get

$$(Ay(t), y(t))_{Y' \times Y} \geq \theta^* \| w(t) \|_V^2 + \beta^* \| \varphi(t) \|_W^2 \geq min\{\theta^*, \beta^*\} \| y(t) \|_Y^2$$

for all $y(t) = (w(t), \varphi(t)) \in Y$, a.e. $t \in (0, T)$. It follows that the operator A is coercive with constant $c = min\{\theta^*, \beta^*\} > 0$.
(**iii**) Now we verify the hypotheses on j_i and h_i.
We note that $j_1(x, r_1, r_2) = j(x, Nr_1)$, for all $(r_1, r_2) \in \mathbb{R}^2$, a.e.$x = (x, t) \in \Sigma_3$, with the operator $N \in L^\infty(\Gamma, \mathcal{L}(\mathbb{R}; \mathbb{R}))$ given by $Nr = r$, for all $r \in \mathbb{R}$, this operator depend on the spatial variable $x \in \Gamma \times (0, T)$.
From **H**(j) (a,b), **H**(j_e)(a,b), it is easy to observe that $j_i(., r_1, r_2), i = 1,2$, are measurable on Σ_3 for all $(r_1, r_2) \in \mathbb{R}^2$ and $j_i(x, ., .)$ are locally Lipschitz on $\mathbb{R}^2$ for a.e. $x \in \Sigma_3$,such that $j_i(., e(.)) \in L^1(\Sigma_3)$ for all $e \in L^2(\Sigma_3; \mathbb{R}^2), i = 1,2$.
The definition of the generalized directional derivative of $j_i(x, ., .)$, we give

$$\left.\begin{aligned} j_1^0(x, r_1, r_2; \varrho, \kappa) &\leq j^0(x, r_1; \varrho) \\ j_2^0(x, r_1, r_2; \varrho, \kappa) &\leq j_e^0(x, r_2 - \varphi_F(x); \kappa) \end{aligned}\right\} \quad (59)$$

for all $(r_1, r_2), (\varrho, \kappa) \in \mathbb{R}^2$, a.e. $x \in \Sigma_3$. Using **H**(j)(d),**H**(j_e)(d) to find

$$\left.\begin{aligned} &j_1^0(x, r_1, r_2; -r_1, -r_2) \leq j^0(x, r_1; -r_1) \leq d_1(1 + \| (r_1, r_2) \|_{\mathbb{R}^2}) \\ &j_2^0(x, r_1, r_2; -r_1, -r_2) \leq j_e^0(x, r_2 - \varphi_F(x); -r_2) \leq j_e^0\big(x, r_2 - \varphi_F; -r_2 + \varphi_F(x)\big) + j_e^0\big(x, r_2 - \varphi_F; -\varphi_F(x)\big) \\ &\qquad \leq d_2(1 + \| (r_1, r_2) \|_{\mathbb{R}^2}) \end{aligned}\right\}$$

for all $(r_1, r_2) \in \mathbb{R}^2$, a.e. $x \in \Sigma_3$and $d_1, d_2 \geq 0$.

From **H**(j)(e), **H**(j_e)(e) we infer that either $j_i(x,.,.)$ or $-j_i(x,.,.)$ for $i=1,2$ are regular for a.e. $x \in \Sigma_3$, so the generalized gradient of $j_i$verifies the following inclusion

$$\partial j_i(x,r_1,r_2) \subseteq \partial_{r_1} j_i(x,r_1,r_2) \times \partial_{r_2} j_i(x,r_1,r_2) \qquad (60)$$

for all $(r_1,r_2) \in \mathbb{R}^2$, a.e.$x \in \Sigma_3$, and $\partial_{r_1} j_i$,$\partial_{r_2} j_i$are the partial generalized gradients of $j_i(x,.,r_2)$ and $j_i(x,r_1,.)$, respectively. Therefore, we have

$$\left.\begin{array}{l} \partial j_1(x,r_1,r_2) \subseteq \partial j(x,r_1) \times \{0\} \\ \partial j_2(x,r_1,r_2) \subseteq \{0\} \times \partial j_e(x,r_2-\varphi_F(x)) \end{array}\right\}$$

for all $(r_1,r_2) \in \mathbb{R}^2$, a.e. $x \in \Sigma_3$.
On the other hand, the norms of ∂j_1 and ∂j_2 are given by the following estimates

$$\| \partial j_1(x,r_1,r_2) \|_{\mathbb{R}^2} \leq |\partial j(x,r_1)|_{\mathbb{R}} \leq c_0 + c_1|r_1| \leq c_0 + c_1 \| (r_1,r_2) \|_{\mathbb{R}^2}$$

$$\| \partial j_2(x,r_1,r_2) \|_{\mathbb{R}^2} \leq |\partial j_e(x,r_2-\varphi_F(x))|_{\mathbb{R}} \leq c_{0_e} + c_{1_e}|r_2-\varphi_F(x)|$$

$$\leq c_{0_e} + c_{1_e}|\varphi_F(x)| + c_{1_e} \| (r_1,r_2) \|_{\mathbb{R}^2}$$

for all $(r_1,r_2) \in \mathbb{R}^2$, a.e. $x \in \Sigma_3$.

Now, we turn to the hypotheses on $h_i, i=1,2$. From **H**(h), **H**(h_e), it is easy to observe that $h_i(.,r_1,r_2)$ are measurables on Σ_3, for all $(r_1,r_2) \in \mathbb{R}^2$, $h_i(x,.,.)$ are continuous on $\mathbb{R}^2$ for a.e. $x \in \Sigma_3$ and $0 \leq h_i(x,r_1,r_2) \leq \overline{\overline{h}}$, for all $(r_1,r_2) \in \mathbb{R}^2$, a.e. $x \in \Sigma_3$with $\overline{\overline{h}} > 0$.
Finally, we deduce that (57) has at least one solution $y(t) = (w(t),\varphi(t)) \in Y$, a.e. $t \in (0,T)$, which completes the proof. □

Next, we choose $z=(v,0) \in Y$ in (57) and using (59) with (55), (56) to obtain the inequality

$$a_\theta(w(t),v) + \int_{\Gamma_3} h(S_t(w),\varphi(t)-\varphi_F)\, j^0(w(t);v)\, da \geq (f(t),v)_{V'\times V}, \forall v \in V, \text{ a.e. } t \in (0,T). (61)$$

When we choose $z=(0,\psi) \in Y$ with (59), we obtain the inequality

$$a_\beta(\varphi(t),\psi) + \int_{\Gamma_3} h_e(S_t(w)) j_e^0(\varphi(t)-\varphi_F;\psi)\ da \geq (q(t),\psi)_{W'\times W}, \forall \psi \in W, \text{a.e.}\, t \in (0,T). \quad (62)$$

It follows that the pair of functions $\big(w(t),\varphi(t)\big) \in Y, a.e. t \in (0,T)$ represents a solution to system (61)-(62). By theorem 4.1, we deduce that this problem has at least one solution.

From definition (52), the problem (61)-(62) is equivalent to the following inclusion of subdifferential type.
Problem$\mathcal{P}^{\mathcal{V}^*}$**:** Find $y(t) \in Y$ such that

$$Ay(t) + \gamma^* \partial J_t(\gamma y(t)) \ni (f(t),q(t)) \quad a.e. t \in (0,T). \qquad (63)$$

Let $y(t) \in Y$ be a solution to problem (61)-(62). Using the definition (53), (55) and (56), we have

$$(Ay(t),z)_{Y'\times Y} + J_t^0(\gamma y(t);\gamma z) \geq ((f(t),q(t)),z)_{Y'\times Y} for all z \in Y, a.e. t \in (0,T)$$

thus

$$((f(t),q(t)) - Ay(t),z)_{Y'\times Y} \leq J_t^0(\gamma y(t);\gamma z) = (J_t o \gamma)^0(y(t);z)$$

for all $z \in Y$, a.e. $t \in (0,T)$. By the definition of Clarke's subdifferential, we have

$$(f(t),q(t)) - Ay(t) \in \partial(J_t o\gamma)(y(t)) = \gamma^* \partial J_t(\gamma y(t)) \quad a.e.\ \ t \in (0,T).$$

We find that $y(t) \in Y$ solves the inclusion (63).
Now, let $y(t) \in Y$ be a solution to (63). Then $Ay(t) + \gamma^*\xi(t) = (f(t),q(t))$, $\xi(t) \in \partial J_t(\gamma y(t))$, a.e. $t \in (0,T)$. Let $z \in Y$, by (53) we have

$$(\gamma^*\xi(t),z)_{Z'\times Z} = (\xi(t),\gamma z)_{L^2(\Gamma_3;\mathbb{R}^2)} \leq J_t^0(\gamma y(t);\gamma z).$$

Such that

$$J_t^0(\gamma y(t);\gamma z) = \int_{\Gamma_3} h(S_t(w),\varphi(t)-\varphi_F)\, j^0(w(t);v)\, da + \int_{\Gamma_3} h_e(S_t(w)) j_e^0(\varphi(t)-\varphi_F;\psi)\ da$$

By definition (55) and (56), we find

$$(f(t),v)_{V'\times V} + (q(t),\psi)_{W'\times W} = (Ay(t),z)_{Y'\times Y} + (\gamma^*\xi(t),z)_{Z'\times Z}$$

$$\leq a_\theta(w(t),v)+a_\beta(\varphi(t),\psi)+\int_{\Gamma_3} h(S_t(w),\varphi(t)-\varphi_F)$$
$$j^0(w(t);v)\,da+\int_{\Gamma_3} h_e(S_t(w))j_e^0(\varphi(t)-\varphi_F;\psi)\ da.$$

This shows that $y(t)\in Y$, a.e. $t\in(0,T)$ is a solution to problem (61)-(62). As we found above, we conclude that problem (61)-(62) and (63) are equivalent.

Let $\eta=(\eta_1,\eta_2)\in L^2(0,T;Y')$. We consider the following intermediate problem.

Problem$\mathcal{P}_\eta^{\mathcal{V}}$**:** Find the displacement $u_\eta:[0,T]\to V$ and the electric potential $\varphi_\eta:[0,T]\to W$ such that

$$a_\theta(\dot{u}_\eta(t),v)+(\eta_1(t),v)_{V'\times V}+\int_{\Gamma_3} h(\mathrm{S}_t(\dot{u}_\eta),\varphi_\eta(t)-\varphi_F)\,j^0(\dot{u}_\eta(t);v)da$$
$$\geq (f(t),v)_{V'\times V},\forall v\in V,\ \ a.e.\ \ t\in(0,T), \tag{64}$$

$$a_\beta(\varphi_\eta(t),\psi)-(\eta_2(t),\psi)_{W'\times W}+\int_{\Gamma_3} h_e(S_t(\dot{u}_\eta))j_e^0(\varphi_\eta(t)-\varphi_F;\psi)\,da$$
$$\geq (q(t),\psi)_{W'\times W},\forall\psi\in W,a.e.\ \ t\in(0,T). \tag{65}$$

$$u_\eta(0)=u_0. \tag{66}$$

Let $w_\eta=\dot{u}_\eta$ denote the velocity field. Then, by using the initial condition (24), it follows that

$$u_\eta(t)=\int_0^t w_\eta(s)ds+u_0 forallt\in[0,T]. \tag{67}$$

Let $f_{\eta_1}:[0,T]\to V'$ be the function defined by

$$(f_{\eta_1}(t),v)_{V'\times V}=(f(t),v)_{V'\times V}-(\eta_1(t),v)_{V'\times V}\quad \forall v\in Vt\in[0,T]. \tag{68}$$

Then, (36) and the regularity $\eta_1\in W^{1,2}(0,T;V')$ imply that

$$f_{\eta_1}\in W^{1,2}(0,T;V'). \tag{69}$$

Let $q_{\eta_2}:[0,T]\to W'$ be the function defined by

$$(q_{\eta_2}(t),\psi)_{W'\times W}=(q(t),\psi)_{W'\times W}+(\eta_2(t),\psi)_{W'\times W}\quad \forall\psi\in Vt\in[0,T]. \tag{70}$$

Then, (37) and the regularity $\eta_2\in W^{1,2}(0,T;W')$ imply that

$$q_{\eta_2}\in W^{1,2}(0,T;W'). \tag{71}$$

The Problem $\mathcal{P}_\eta^{\mathcal{V}}$ is equivalent to the following inclusion:

$$\begin{cases} Findy_\eta(t)\in Ysuchthat \\ Ay_\eta(t)+\gamma^*\partial J_t(\gamma y_\eta(t))\ni(f_\eta(t),q_\eta(t))\quad a.e.t\in(0,T). \end{cases} \tag{72}$$

Where $y_\eta(t)=(w_\eta(t),\varphi_\eta(t))\in Y$, a.e. $t\in(0,T)$ and for all $z=(v,\psi)\in Y$, we define

$$((f_\eta(t),q_\eta(t)),z)_{Y'\times Y}=(f_{\eta_1}(t),v)_{V'\times V}+(q_{\eta_2}(t),\psi)_{W'\times W}. \tag{73}$$

By the formulate (60), we fix $\varphi_{\eta_1}(t)\in W$, a.e. $t\in(0,T)$and we transform the above problem (64) to the following inclusion

$$\begin{cases} Findw_{\eta_1}(t)\in Vsuchthat \\ A_\theta w_{\eta_1}(\mathrm{t})+\gamma^*\partial_{\gamma w_{\eta_1}}J_t^1(\gamma y_{\eta_1}(t))\ni f_{\eta_1}(t)\quad a.e.t\in(0,T). \end{cases} \tag{74}$$

Where $A_\theta:V\to V'$ define by

$$(A_\theta w_{\eta_1}(t),v)_{V'\times V}=a_\theta(w_{\eta_1}(t),v)\quad \forall w_{\eta_1}(t),v\in Va.e.t\in(0,T),$$

and

$$J_t^1(x,r_1,r_2)=\int_{\Gamma_3} h_1(x,r_1,r_2)\,j_1(x,r_1,r_2)\ daa.e.t\in(0,T).$$

Now, we fix $w_{\eta_2}(t)\in V$, a.e. $\mathrm{t}\in(0,T)$ and we transform the above problem (65) to the following inclusion

$$\begin{cases} Find\varphi_{\eta_2}(t) \in Wsuchthat \\ A_\beta \varphi_{\eta_2}(t) + \gamma^* \partial_{\gamma\varphi_{\eta_2}} J_t^2(\gamma y_{\eta_2}(t)) \ni q_{\eta_2}(t) \quad a.e.t \in (0,T). \end{cases} \tag{75}$$

Where the operator $A_\beta: W \to W'$ define by

$$(A_\beta \varphi_{\eta_2}(t), \psi)_{W' \times W} = a_\beta(\varphi_{\eta_2}(t), \psi) \quad \forall \varphi_{\eta_2}(t), \psi \in Wa.e.t \in (0,T),$$

with

$$J_t^2(x, r_1, r_2) = \int_{\Gamma_3} h_2(\mathrm{x}, r_1, r_2)\, j_2(x, r_1, r_2)\, daa.e.t \in (0,T).$$

It is clear, that A_θ and A_β are linear, continuous and coercive.

The functional $J_t^{\gamma\varphi_{\eta_1}}: L^2(\Gamma_3; \mathbb{R}) \to \mathbb{R}$ is defined by

$$\partial J_t^{\gamma\varphi_{\eta_1}}(\gamma w_{\eta_1}(t)) = \int_{\Gamma_3} h(S_t(\gamma w_{\eta_1}), \gamma\varphi_{\eta_1}(t) - \varphi_F)\, \partial j(\gamma w_{\eta_1}(t))\, daa.e.t \in (0,T),$$

For $\gamma\varphi_{\eta_1}(t) \in L^2(\Gamma_3; \mathbb{R})$ fixed, a.e$(0,T)$, such that

$\partial J_t^{r_2}(r_1) = \partial_{r_1} J_t^1(x, r_1, r_2) = \int_{\Gamma_3} h_1(x, r_1, r_2)\, \partial_{r_1} j_1(x, r_1, r_2)\, da.$

The functional $J_t^{\gamma w_{\eta_2}}: L^2(\Gamma_3; \mathbb{R}) \to \mathbb{R}$ is defined by

$$J_t^{\gamma w_{\eta_2}}(\gamma\varphi_{\eta_2}(t)) = \int_{\Gamma_3} h_e(S_t(\gamma w_{\eta_2}))\, j_e(\gamma\varphi_{\eta_2}(t) - \varphi_F)\, daa.e.t \in (0,T),$$

for $\gamma w_{\eta_2}(t) \in L^2(\Gamma_3; \mathbb{R})$ fixed, a.e. $(0,T)$, such that

$\partial J_t^{r_1}(r_2) = \partial_{r_2} J_t^2(x, r_1, r_2) = \int_{\Gamma_3} h_2(x, r_1, r_2)\, \partial_{r_2} j_2(x, r_1, r_2)\, da.$

Under assumptions **H**(j),**H**(h) and **H**(φ_F), the functional $J_t^{\varphi_{\eta_1}}$ satisfies

(**a**)$J_t^{\varphi_{\eta_1}}$ is locally Lipschitz on bounded subsets $L^2(\Gamma_3; \mathbb{R}^2)$.

(**b**) For $\overline{\varphi} \in L^2((\Gamma_3); \mathbb{R})$ fixed,

$\| \partial_{\overline{w}} J_t(\overline{w}, \overline{\varphi}) \| \leq \tilde{c}$, for all $\overline{w} \in L^2((\Gamma_3); \mathbb{R})$ with $\tilde{c} > 0$.

(**c**) either $J_t^{\varphi_{\eta_1}}$ or $-J_t^{\varphi_{\eta_1}}$ is regular.

(**d**) For $\overline{\varphi} \in L^2((\Gamma_3); \mathbb{R})$ fixed, we define

$J_t^{0\overline{\varphi}}(\overline{w}; \overline{v}) = \int_{\Gamma_3} h(S_t(\overline{w}), \overline{\varphi}(t) - \varphi_F)\, j^0(\overline{w}(t); \overline{v})\, da \forall (\overline{w}, \overline{v}) \in L^2((\Gamma_3); \mathbb{R}^2).$

Under assumptions **H**(j_e),**H**(h_e) and **H**(φ_F), the functional $J_t^{w_{\eta_2}}$ satisfies

(**a**)$J_t^{w_{\eta_2}}$ is locally Lipschitz on bounded subsets $L^2(\Gamma_3; \mathbb{R}^2)$.

(**b**) For $\overline{w} \in L^2((\Gamma_3); \mathbb{R})$ fixed,

$\| \partial_{\overline{\varphi}} J_t(\overline{w}, \overline{\varphi}) \| \leq \tilde{\tilde{c}}$, for all $\overline{\varphi} \in L^2((\Gamma_3); \mathbb{R})$ with $\tilde{\tilde{c}} > 0$.

(**c**) either $J_t^{w_{\eta_2}}$ or $-J_t^{w_{\eta_2}}$ is regular.

(**d**) For $\overline{w} \in L^2((\Gamma_3); \mathbb{R})$ fixed, we define

$J_t^{0\overline{w}}(\overline{\varphi}; \overline{\psi}) = \int_{\Gamma_3} h(S_t(\overline{w}))\, j_e^0(\overline{\varphi}(t) - \varphi_F; \overline{\psi})\, da \forall (\overline{\varphi}, \overline{\psi}) \in L^2((\Gamma_3); \mathbb{R}^2).$

5 Existence and uniqueness results

In this section, we will focus the study on the existence, uniqueness and continuous dependence results of the above problem (74)-(75).

Proposition 5.1 Assume that **H**(c)(a,c),(69), (71) **H**(j),**H**(h),**H**(j_e),**H**(h_e), **H**(φ_F), **H**(0), (47) and (48) hold. Then (**i**)For all $\eta_1 \in L^2(0,T; V')$. The unique solution w_{η_1} of problem (74) satisfies $w_{\eta_1} \in L^2(0,T; V)$. Moreover, if $\varphi_{\eta_1}(t), \varphi_{\zeta_1}(t) \in W$, a.e. $t \in (0,T)$ and w_{η_1}, w_{ζ_1} denote the solutions to problem (74), for $\varphi =$

φ_{η_1} and $\varphi = \varphi_{\zeta_1}$, respectively, then there exists $c > 0$ such that

$$\| w_{\eta_1}(t) - w_{\zeta_1}(t) \|_V \leq c\{\| \eta_1(t) - \zeta_1(t) \|_{V'} + \| w_{\eta_1} - w_{\zeta_1} \|_{L^2(0,t;V)}$$
$$+ \| \varphi_{\eta_1}(t) - \varphi_{\zeta_1}(t) \|_W\} \text{ a.e. } t \in (0,T). \qquad (76)$$

(**ii**) For all $\eta_2 \in L^2(0,T;W')$. The unique solution φ_{η_2} of problem (75) satisfies $\varphi_{\eta_2} \in L^2(0,T;W)$. Moreover, if $w_{\eta_2}(t), w_{\zeta_2}(t) \in V$ and $\varphi_{\eta_2}, \varphi_{\zeta_2}$ denote the solutions to problem (75), for $w = w_{\eta_2}$ and $w = w_{\zeta_2}$, respectively, then there exists $c > 0$ such that

$$\| \varphi_{\eta_2}(t) - \varphi_{\zeta_2}(t) \|_W \leq c\{\| \eta_2(t) - \zeta_2(t) \|_{W'} + \| w_{\eta_2} - w_{\zeta_2} \|_{L^2(0,t;V)}\} \ a.e.\ t \in (0,T). \qquad (77)$$

Proof. (**i**) We show that $w_{\eta_1} \in L^2(0,T;V)$. Indeed, using the inclusion (74) and the coercivity of the operator A_θ to find

$$\theta^* \| w_{\eta_1}(t) \|_V^2 + (b_{\eta_1}(t), \gamma w_{\eta_1}(t))_{L^2(\Gamma_3)} \leq |(f(t), w_{\eta_1}(t))_{V' \times V}| + |(\eta_1(t), w_{\eta_1}(t))_{V' \times V}|$$
$$\leq \{\| f(t) \|_{V'} + \| \eta_1(t) \|_{V'}\} \| w_{\eta_1}(t) \|_V \qquad (78)$$

such that $b_{\eta_1}(t) \in \partial_{w_{\eta_1}} J_t(\gamma y_{\eta_1}(t))$, a.e. $t \in (0,T)$. By the assumption (b) of functional $J_t^{\gamma\varphi_{\eta_1}}$, (78) becomes

$$\theta^* \| w_{\eta_1}(t) \|_V^2 \leq \{\tilde{c} \| \gamma \| + \| f(t) \|_{V'} + \| \eta_1(t) \|_{V'}\} \| w_{\eta_1}(t) \|_V$$

which implies

$$\| w_{\eta_1}(t) \|_V^2 \leq c\{1 + \| f(t) \|_{V'} + \| \eta_1(t) \|_{V'}\} \quad \text{a.}e.t \in (0,T). \qquad (79)$$

Since $\eta_1 \in L^2(0,T;V')$ and by (36), we deduce that $w_{\eta_1}(t) \in L^2(0,T;V)$. By theorem 4.1 we conclude the proof's existence part.

We turn now to the uniqueness part. Let w_{η_1}, w_{ζ_1} be solutions to (74), then, there exist $\gamma^* b_{\eta_1}^{\varphi_{\eta_1}}, \gamma^* b_{\zeta_1}^{\varphi_{\eta_1}} \in \mathcal{Z}'$ such that

$$A_\theta w_{\eta_1}(t) + \gamma^* b_{\eta_1}^{\varphi_{\eta_1}}(t) = \tilde{f}(t), \quad A_\theta w_{\zeta_1}(t) + \gamma^* b_{\zeta_1}^{\varphi_{\eta_1}}(t) = \tilde{f}(t) \quad a.e.t \in (0,T).$$

Subtracting the two equations, we get

$$A_\theta(w_{\eta_1}(t) - w_{\zeta_1}(t)) + \gamma^* b_{\eta_1}^{\varphi_{\eta_1}}(t) - \gamma^* b_{\zeta_1}^{\varphi_{\eta_1}}(\mathrm{t}) = 0 \quad a.e.t \in (0,T)$$

with $b_{\eta_1}^{\varphi_{\eta_1}}(t) \in \partial J_t^{\gamma\varphi_{\eta_1}}(\gamma w_{\eta_1}(t))$ and $b_{\zeta_1}^{\varphi_{\eta_1}}(t) \in \partial J_t^{\gamma\varphi_{\eta_1}}(\gamma w_{\zeta_1}(t))$. Multiplying the above equation by $w_{\eta_1}(t) - w_{\zeta_1}(t)$ and using the coercivity of A_θ. Then

$$\theta^* \| w_{\eta_1}(t) - w_{\zeta_1}(t) \|_V^2 + (b_{\eta_1}^{\varphi_{\eta_1}}(t) - b_{\zeta_1}^{\varphi_{\eta_1}}(t), \gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t))_{L^2(\Gamma_3)} \leq 0 \qquad (80)$$

a.e. $t \in (0,T)$. In the other hand, we have

$$\partial J_t^{\varphi_{\eta_1}}(\gamma w_{\eta_1}(t)) \subset \int_{\Gamma_3} h(S_t(\gamma w_{\eta_1}), \gamma\varphi_{\eta_1}(t) - \varphi_F)\, \partial j(\gamma w_{\eta_1}(t))\ da,$$
$$\partial J_t^{\varphi_{\eta_1}}(\gamma w_{\zeta_1}(t)) \subset \int_{\Gamma_3} h(S_t(\gamma w_{\zeta_1}), \gamma\varphi_{\eta_1}(t) - \varphi_F)\, \partial j(\gamma w_{\zeta_1}(t))\ da.$$

We find

$$s_{\eta_1}(x,t) = h(S_t(\gamma w_{\eta_1}), \gamma\varphi_{\eta_1}(t) - \varphi_F)\, p_{\eta_1}(x,t), \quad p_{\eta_1}(x,t) \in \partial j(x, \gamma w_{\eta_1}(x,t))$$
$$s_{\zeta_1}(x,t) = h(S_t(\gamma w_{\zeta_1}), \gamma\varphi_{\eta_1}(t) - \varphi_F)\, p_{\zeta_1}(x,t), \quad p_{\zeta_1}(x,t) \in \partial j(x, \gamma w_{\zeta_1}(x,t))$$

a.e. $(x,t) \in \Sigma_3$ and $s_{\eta_1}, s_{\zeta_1} \in L^2(0,T;Ł^2(\Gamma_3))$

$$(b_{\eta_1}^{\varphi_{\eta_1}}(t), \tilde{v})_{L^2(\Gamma_3)} = \int_{\Gamma_3} s_{\eta_1}(t)\tilde{v}da, \quad (b_{\zeta_1}^{\varphi_{\eta_1}}(t), \tilde{v})_{L^2(\Gamma_3)} = \int_{\Gamma_3} s_{\zeta_1}(t)\tilde{v}da \quad \forall \tilde{v} \in L^2(\Gamma_3)$$

a.e. $t \in (0,\mathrm{T})$. Using $\boldsymbol{H}(h)(c), \boldsymbol{H}(j)(c,f)$

$$(s_{\eta_1}(x,t) - s_{\zeta_1}(x,t))(\gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t)) \geq -L_h(c_0 + c_1|\gamma w_{\zeta_1}(t)|) \times$$
$$(|S_t(\gamma w_{\eta_1}(t)) - S_t(\gamma w_{\zeta_1}(t))||\gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t)|) - m\overline{h}\ \ |\gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t)|^2$$

we integrate on Γ_3, we obtain

$$(b_{\eta_1}^{\varphi_{\eta_1}}(t) - b_{\zeta_1}^{\varphi_{\eta_1}}(t), \gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t))_{L^2(\Gamma_3)} \geq -L_h(c_0 + c_1|\gamma w_{\zeta_1}(t)|) \times$$
$$\int_{\Gamma_3} (|S_t(\gamma w_{\eta_1}(t)) - S_t(\gamma w_{\zeta_1}(t))||\gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t)|)\ da - m\bar{h} \int_{\Gamma_3} |\gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t)|^2 da$$
$$\geq -L_h(c_0 + c_1|\gamma w_{\zeta_1}(t)|) \parallel S_t(\gamma w_{\eta_1}(t)) - S_t(\gamma w_{\zeta_1}(t)) \parallel_{L^2(\Gamma_3)} \parallel \gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t) \parallel_{L^2(\Gamma_3)}$$
$$-m\bar{h} \parallel \gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t) \parallel_{L^2(\Gamma_3)}^2$$
$$\geq -L_h(c_0 + c_1|\gamma w_{\zeta_1}(t)|)\sqrt{T} \parallel \gamma \parallel^2 \parallel w_{\eta_1} - w_{\zeta_1} \parallel_{L^2(0,t:V)} \parallel w_{\eta_1}(t) - w_{\zeta_1}(t) \parallel_V - m\bar{h} \parallel \gamma \parallel^2$$
$$\parallel w_{\eta_1}(t) - w_{\zeta_1}(t) \parallel_V^2.$$

L'inequality (80) becomes

$$(\theta^* - m\bar{h} \parallel \gamma \parallel^2)|w_{\eta_1}(t) - w_{\zeta_1}(t) \parallel_V^2 \leq L_h(c_0 + c_1|\gamma w_{\zeta_1}(t)|)\sqrt{T} \parallel \gamma \parallel^2 \parallel w_{\eta_1} - w_{\zeta_1} \parallel_{L^2(0,t:V)}$$
$$\parallel w_{\eta_1}(t) - w_{\zeta_1}(t) \parallel_V$$

Since $\gamma w_{\zeta_1}(t) \in L^2(\Gamma_3)$, then, there exist $c_2 > 0$ such that $c_0 + c_1|\gamma w_{\zeta_1}(t)| < c_2$. Using the inequality $xy \leq \frac{mx^2}{2} + \frac{y^2}{2m}$, for all $x, y, m > 0$, and (47), then, we have

$$\frac{(\theta^* - m\bar{h}\|\gamma\|^2)}{2} \parallel w_{\eta_1}(t) - w_{\zeta_1}(t) \parallel_V^2 \leq \frac{(L_h^2 c_2^2 T\|\gamma\|^4)}{2(\theta^* - m\bar{h}\|\gamma\|^2)} \parallel w_{\eta_1} - w_{\zeta_1} \parallel_{L^2(0,t:V)}^2.$$

Now, using the Gronwell's inequality to obtain

$$\parallel w_{\eta_1}(t) - w_{\zeta_1}(t) \parallel_V^2 \leq 0 \quad \text{a.}e.t \in (0,T).$$

We conclude that $w_{\eta_1}(t) = w_{\zeta_1}(t)$, a.e. $t \in (0,T)$. The uniqueness part of (**i**) is complete.

Next, we prove the inequality (76). Indeed, let η_1 and ζ_1 be two elements of $L^2(0,T;V')$ and denote by w_{η_1} and w_ζ the corresponding solutions of problem (74), for $\varphi = \varphi_{\eta_1}$ and $\varphi = \varphi_{\zeta_1}$, then, there exist $\gamma^* b_{\eta_1}^{\varphi_{\eta_1}}, \gamma^* b_{\zeta_1}^{\varphi_{\zeta_1}} \in \mathcal{Z}'$ such that

$$A_\theta w_{\eta_1}(t) + \gamma^* b_{\eta_1}^{\varphi_{\eta_1}}(t) = f_{\eta_1}(t), \quad A_\theta w_{\zeta_1}(t) + \gamma^* b_{\zeta_1}^{\varphi_{\zeta_1}}(t) = f_{\zeta_1}(t) \quad a.e.t \in (0,T)$$

with $b_{\eta_1}^{\varphi_{\eta_1}}(t) \in \partial J_t^{\gamma\varphi_{\eta_1}}(\gamma w_{\eta_1}(t))$ and $b_{\zeta_1}^{\varphi_{\zeta_1}}(t) \in \partial J_t^{\gamma\varphi_{\zeta_1}}(\gamma w_{\zeta_1}(t))$. Subtracting the two equations, we have

$$A_\theta(w_{\eta_1}(t) - w_{\zeta_1}(t)) + \gamma^* b_{\eta_1}^{\varphi_{\eta_1}}(t) - \gamma^* b_{\zeta_1}^{\varphi_{\zeta_1}}(t) = f_{\eta_1}(t) - f_{\zeta_1}(t) \quad a.e.t \in (0,T) \qquad (81)$$

Using the inclusions

$$\partial J_t^{\varphi_{\eta_1}}(\gamma w_{\eta_1}(t)) \subset \int_{\Gamma_3} h(S_t(\gamma w_{\eta_1}), \gamma\varphi_{\eta_1}(t) - \varphi_F)\, \partial j(\gamma w_{\eta_1}(t))\ da,$$
$$\partial J_t^{\varphi_{\zeta_1}}(\gamma w_{\zeta_1}(t)) \subset \int_{\Gamma_3} h(S_t(\gamma w_{\zeta_1}), \gamma\varphi_{\zeta_1}(t) - \varphi_F)\, \partial j(\gamma w_{\zeta_1}(t))\ da,$$

such that

$$s_{\eta_1}(x,t) = h(S_t(\gamma w_{\eta_1}), \gamma\varphi_{\eta_1}(t) - \varphi_F)\, p_{\eta_1}(x,t), \quad p_{\eta_1}(x,t) \in \partial j(x, \gamma w_{\eta_1}(x,t))$$
$$s_{\zeta_1}(x,t) = h(S_t(\gamma w_{\zeta_1}), \gamma\varphi_{\zeta_1}(t) - \varphi_F)\, p_{\zeta_1}(x,t), \quad p_{\zeta_1}(x,t) \in \partial j(\mathrm{x}, \gamma w_{\zeta_1}(x,t))$$

a.e. $(x,t) \in \Sigma_3$ and $s_{\eta_1}, s_{\zeta_1} \in L^2(0,T;Ł^2(\Gamma_3))$.
On the other hand, using $\boldsymbol{H}(h)(b,c), \boldsymbol{H}(j)(c,f)$ and the Hölder inequality, we obtain

$$(s_{\eta_1}(t) - s_{\zeta_1}(t))(\gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t)) \geq -L_h(c_0 + c_1|\gamma w_{\zeta_1}(t)|)(|S_t(\gamma w_{\eta_1}) - S_t(\gamma w_{\zeta_1})| +$$
$$|\gamma\varphi_{\eta_1}(t) - \gamma\varphi_{\zeta_1}(t)|)\ |\gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t)|\ - m\bar{h}\ |\gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t)|^2$$

and

$$(b_{\eta_1}^{\varphi_{\eta_1}}(t) - b_{\zeta_1}^{\varphi_{\zeta_1}}(t), \gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t))_{L^2(\Gamma_3)} \geq -L_h\sqrt{T} \parallel \gamma \parallel^2\ (c_0 + c_1|\gamma w_{\zeta_1}(t)|)$$
$$\parallel w_{\eta_1} - w_{\zeta_1} \parallel_{L^2(0,t,V)} \parallel w_{\eta_1}(t) - w_{\zeta_1}(t) \parallel_V - L_h \parallel \gamma \parallel^2\ (c_0 + c_1|\gamma w_{\zeta_1}(t)|)$$
$$\parallel \varphi_{\eta_1}(t) - \varphi_{\zeta_1}(t) \parallel_W \parallel w_{\eta_1}(t) - w_{\zeta_1}(t) \parallel_V - m\bar{h} \parallel \gamma \parallel^2\ \parallel w_{\eta_1}(\mathrm{t}) - w_{\zeta_1}(t) \parallel_V^2$$

On the other hand, by multiplying the equation (81) by $w_{\eta_1}(t) - w_{\zeta_1}(t)$ and using the coercivity of A_θ, we obtain

$\theta^* \parallel w_{\eta_1}(t) - w_{\zeta_1}(t) \parallel_V^2 \leq -(b_{\eta_1}^{\varphi_{\eta_1}}(t) - b_{\zeta_1}^{\varphi_{\eta_1}}(t), \gamma w_{\eta_1}(t) - \gamma w_{\zeta_1}(t))_{L^2(\Gamma_3)}$
$+(\eta_1(t) - \zeta_1(t), w_{\eta_1}(t) - w_{\zeta_1}(t))_{V' \times V}(\theta^* - m\overline{h} \parallel \gamma \parallel^2) \parallel w_{\eta_1}(t) - w_{\zeta_1}(t) \parallel_V^2$
$\leq \{\parallel \eta_1(t) - \zeta_1(t) \parallel_{V'} + L_h\sqrt{T} \parallel \gamma \parallel^2 (c_0 + c_1|\gamma w_{\zeta_1}(t)|) \parallel w_{\eta_1} - w_{\zeta_1} \parallel_{L^2(0,t,V)} +$
$L_h \parallel \gamma \parallel^2 (c_0 + c_1|\gamma w_{\zeta_1}(t)|) \parallel \varphi_{\eta_1}(t) - \varphi_{\zeta_1}(t) \parallel_W\} \parallel w_\eta(t) - w_\zeta(t) \parallel_V$ (82)

Using the inequality $xy \leq \frac{mx^2}{2} + \frac{y^2}{2m}$, for all $x, y, m > 0$ and (47), l'inequality (82) becomes

$$\parallel w_{\eta_1}(t) - w_{\zeta_1}(t) \parallel_V \leq c\{\parallel \eta_1(t) - \zeta_1(t) \parallel_{V'} + \parallel w_{\eta_1} - w_{\zeta_1} \parallel_{L^2(0,t;V)} + \parallel \varphi_{\eta_1}(t) - \varphi_{\zeta_1}(t) \parallel_W\}$$

a.e. $t \in (0, T)$, which proves (76).
By Gronwell inequality, (76) becomes

$$\parallel w_{\eta_1}(t) - w_{\zeta_1}(t) \parallel_V^2 \leq c\{\parallel \eta_1(t) - \zeta_1(t) \parallel_{V'}^2 + \parallel w_{\eta_1} - w_{\zeta_1} \parallel_{L^2(0,t;V)}^2 + \parallel \varphi_{\eta_1}(t) - \varphi_{\zeta_1}(t) \parallel_W^2\}$$
$$\leq c\{\parallel \eta_1(t) - \zeta_1(t) \parallel_{V'}^2 + \parallel \varphi_{\eta_1}(t) - \varphi_{\zeta_1}(t) \parallel_W^2\} \quad (83)$$

(**ii**) We do the same steps of the previous proof (**i**) with some modifications. First step, we show that $\varphi_{\eta_2}(t) \in L^2(0, T; W)$. To this end, using the inclusion (75), the coercivity of the operator A_β and the assumption (b) of functional $J_t^{\gamma w_{\eta_2}}$, we find

$$\parallel \varphi_{\eta_2}(t) \parallel_W^2 \leq c\{1 + \parallel q(t) \parallel_{V'} + \parallel \eta_2(t) \parallel_{W'}\} \quad (84)$$

Since $\eta_2 \in L^2(0, T; W')$ and by (37), we deduce that $\varphi_{\eta_2} \in L^2(0, T; W)$, which concludes the existence part of the proof.
Let's turn on to the uniqueness part of solutions $\varphi_{\eta_2}(t) \in W$ to problem (75). Let φ_1, φ_2 be solutions to (75) for $\eta_2, \zeta_2 \in L^2(0, T; W')$, then, there exist $\gamma^* d_{\eta_2}^{w_{\eta_2}}, \gamma^* d_{\zeta_2}^{w_{\eta_2}} \in \mathcal{Z}'$ such that

$$A_\beta \varphi_1(t) + \gamma^* d_{\eta_2}^{w_{\eta_2}}(t) = \tilde{q}(t), \quad A_\beta \varphi_2(t) + \gamma^* d_{\zeta_2}^{w_{\eta_2}}(t) = \tilde{q}(t)$$

a.e. $t \in (0, T)$, $d_{\eta_2}^{w_{\eta_2}} \in \partial J_t^{\gamma w_{\eta_2}}(\gamma \varphi_{\eta_2}(t))$ and $d_{\zeta_2}^{w_{\eta_2}} \in \partial J_t^{\gamma w_{\eta_2}}(\gamma \varphi_{\zeta_2}(t))$. Subtracting the two equations and multiplying the equation obtained by $\varphi_{\eta_2}(t) - \varphi_{\zeta_2}(t)$, then, using the coercivity of A_θ to obtain

$$\beta^* \parallel \varphi_{\eta_2}(t) - \varphi_{\zeta_2}(t) \parallel_W^2 + (d_{\eta_2}^{w_{\eta_2}}(t) - d_{\zeta_2}^{w_{\eta_2}}(t), \gamma \varphi_{\eta_2}(t) - \gamma \varphi_{\zeta_2}(t))_{L^2(\Gamma_3)} \leq 0 \quad (85)$$

a.e. $t \in (0, T)$. In the other hand, we have

$$\partial J_t^{w_{\eta_2}}(\gamma \varphi_{\eta_2}(t)) \subset \int_{\Gamma_3} h_e(S_t(\gamma w_{\eta_2}))\, \partial j_e(\gamma \varphi_{\eta_2}(t) - \varphi_F)\ da,$$
$$\partial J_t^{w_{\eta_2}}(\gamma \varphi_{\zeta_2}(t)) \subset \int_{\Gamma_3} h_e(S_t(\gamma w_{\eta_2}))\, \partial j_e(\gamma \varphi_{\zeta_2}(t) - \varphi_F)\ da,$$

with

$$s_{\eta_2}(x, t) = h_e(S_t(\gamma w_{\eta_2}))\, l_{\eta_2}(x, t), \quad l_{\eta_2}(x, t) \in \partial j_e(x, \gamma \varphi_{\eta_2}(x, t) - \varphi_F(x))$$
$$s_{\zeta_2}(x, t) = h_e(S_t(\gamma w_{\eta_2}))\, l_{\zeta_2}(x, t), \quad l_{\zeta_2}(x, t) \in \partial j_e(x, \gamma \varphi_{\zeta_2}(x, t) - \varphi_F(x))$$

a.e. $(x, t) \in \Sigma_3$ and $s_{\eta_2}, s_{\zeta_2} \in L^2(0, T; L^2(\Gamma_3))$ such that

$$(d_{\eta_2}^{w_{\eta_2}}(t), \tilde{\psi})_{L^2(\Gamma_3)} = \int_{\Gamma_3} s_{\eta_2}(t) \tilde{v} da, \qquad (d_{\zeta_2}^{w_{\eta_2}}(t), \tilde{\psi})_{L^2(\Gamma_3)} = \int_{\Gamma_3} s_{\zeta_2}(t) \tilde{\psi} da, \forall \tilde{\psi} \in L^2(\Gamma_3)$$

Using $\boldsymbol{H}(h_e)(c), \boldsymbol{H}(j_e)(f)$ and integrating on Γ_3 to obtain

$$(d_{\eta_2}^{w_{\eta_2}}(t) - d_{\zeta_2}^{w_{\eta_2}}(t), \gamma \varphi_{\eta_2}(t) - \gamma \varphi_{\zeta_2}(t))_{L^2(\Gamma_3)} \geq -\overline{h}_e m_e \parallel \gamma \parallel^2 \parallel \varphi_{\eta_2}(t) - \varphi_{\zeta_2}(t) \parallel_W^2$$

L'inequality (85) becomes

$$(\beta^* - \overline{h}_e m_e \parallel \gamma \parallel^2) \parallel \varphi_{\eta_2}(t) - \varphi_{\zeta_2}(t) \parallel_W^2 \leq 0 \quad a.e.\, t \in (0, T).$$

We use (48) to conclude that $\varphi_{\eta_2}(t) = \varphi_{\zeta_2}(t)$, a.e. $t \in (0, \mathrm{T})$. We have completed the uniqueness part.
Now, we prove the inequality (77). Using the inclusions

$$\partial J_t^{w_{\eta_2}}(\gamma\varphi_{\eta_2}(t)) \subset \int_{\Gamma_3} h_e(S_t(\gamma w_{\eta_2}))\,\partial j_e(\gamma\varphi_{\zeta_2}(t)-\varphi_F)\ da,$$
$$\partial J_t^{w_{\zeta_2}}(\gamma\varphi_{\zeta_2}(t)) \subset \int_{\Gamma_3} h_e(S_t(\gamma w_{\zeta_2}))\,\partial j_e(\gamma\varphi_{\zeta_2}(t)-\varphi_F)\ da,$$

with

$$s_{\eta_2}(x,t) = h_e(S_t(\gamma w_{\eta_2}))\,l_{\eta_2}(x,t),\quad l_{\eta_2}(x,t) \in \partial j_e(x,\gamma\varphi_{\eta_2}(x,t)-\varphi_F(x))$$
$$s_{\zeta_2}(x,t) = h_e(S_t(\gamma w_{\zeta_2}))\,l_{\zeta_2}(x,t),\quad l_{\zeta_2}(x,t) \in \partial j_e(x,\gamma\varphi_{\zeta_2}(x,t)-\varphi_F(x))$$
$$(d_{\eta_2}^{w_{\eta_2}}(t),\tilde{\psi})_{L^2(\Gamma_3)} = \int_{\Gamma_3} s_{\eta_2}(t)\tilde{v}da, (d_{\zeta_2}^{w_{\zeta_2}}(t),\tilde{\psi})_{L^2(\Gamma_3)} = \int_{\Gamma_3} s_{\zeta_2}(t)\tilde{\psi}da, \forall\tilde{\psi} \in L^2(\Gamma_3).$$

On the other hand, using $\boldsymbol{H}(h_e)(a,c), \boldsymbol{H}(j_e)(c,f)$ to obtain

$$(d_{\eta_2}^{w_{\eta_2}}(t) - d_{\zeta_2}^{w_{\zeta_2}}(t), \gamma\varphi_{\eta_2}(t) - \gamma\varphi_{\zeta_2}(t))_{L^2(\Gamma_3)} \geq -L_{h_e}\sqrt{T}\,\|\gamma\|^2\,(c_{0_e} + c_{1_e}|\gamma\varphi_{\zeta_2}(t) - \varphi_F|)$$
$$\times\|w_{\eta_2} - w_{\zeta_2}\|_{L^2(0,t,V)}\|\varphi_{\eta_2}(t) - \varphi_{\zeta_2}(t)\|_W - m_e\overline{h}_e\,\|\gamma\|^2\,\|\varphi_{\eta_2}(t) - \varphi_{\zeta_2}(t)\|_W^2$$

Multiplying the following equation

$$A_\beta(\varphi_{\eta_2}(t) - \varphi_{\zeta_2}(t)) + \gamma^* d_{\eta_2}^{w_{\eta_2}}(t) - \gamma^* d_{\zeta_2}^{w_{\zeta_2}}(t) = q_{\eta_2}(t) - q_{\zeta_2}(t) \quad a.e.\ t \in (0,T)$$

by $\varphi_{\eta_2}(t) - \varphi_{\zeta_2}(t)$ and using the coercivity of A_β, we obtain

$$(\beta^* - m_e\overline{h}_e\,\|\gamma\|^2)\,\|\varphi_{\eta_2}(t) - \varphi_{\zeta_2}(t)\|_W^2 \leq \{\|\eta_2(t) - \zeta_2(t)\|_{W'} + L_{h_e}\sqrt{T}\,\|\gamma\|^2$$
$$(c_{0_e} + c_{1_e}|\gamma\varphi_{\eta_2}(t) - \varphi_F|)\,\|w_{\eta_2} - w_{\zeta_2}\|_{L^2(0,t,V)}\}\,\|\varphi_{\eta_2}(t) - \varphi_{\zeta_2}(t)\|_W \qquad (86)$$

By the inequality $xy \leq \frac{mx^2}{2} + \frac{y^2}{2m}$, for all $x, y, m > 0$, (86) becomes

$$\|\varphi_{\eta_2}(t) - \varphi_{\zeta_2}(t)\|_W \leq c\{\|\eta_2(t) - \zeta_2(t)\|_{W'} + \|w_{\eta_2} - w_{\zeta_2}\|_{L^2(0,t;V)}\}\ a.e.\ t \in (0,T).$$

Which proves (77) and completes the proof of proposition 5.1.□

6 Solvability of problem $\mathcal{P}_{\boldsymbol{\eta}}^{\boldsymbol{\nu}}$

We have the following existence and uniqueness result.

Theorem 6.1 Assume that **H**(c)(a,c), (69),(71), **H**(j),**H**(j_e),**H**(h), **H**(h_e), **H**(φ_F), **H**(0), (36), (37), (47) and (48) hold. Then, the problem (72) has a unique solution, which satisfies

$y_\eta \in L^2(0,T;Y)$. (87)

For all $\zeta = (\zeta_1,\zeta_2) \in L^2(0,T;Y')$ and $\eta = (\eta_1,\eta_2) \in L^2(0,T;Y')$. If $y_\eta = (\dot{u}_\eta,\varphi_\eta) \in L^2(0,T;Y)$ and $y_\zeta = (\dot{u}_\zeta,\varphi_\zeta) \in L^2(0,T;Y)$ denote the solutions to problem (72), respectively, then there exists $c > 0$ such that

$$\|\dot{u}_\eta(t) - \dot{u}_\zeta(t)\|_V^2 \leq c\{\|u_\eta(t) - u_\zeta(t)\|_V^2 + \|u_\eta - u_\zeta\|_{L^2(0,t;V)}^2\}. \qquad (88)$$

And

$$\|\varphi_\eta(t) - \varphi_\zeta(t)\|_W^2 \leq c\{\|u_\eta(t) - u_\zeta(t)\|_V^2 + \|u_\eta - u_\zeta\|_{L^2(0,t;V)}^2\}. \qquad (89)$$

Proof. (**i**) Existence and uniqueness part.

For every $\eta \in L^2(0,T;Y')$ we consider the operator $\Lambda: L^2(0,T;Y') \to L^2(0,T;Y')$ defined by

$$(\Lambda\eta(t),z)_{Y'\times Y} = (a_\mu(\int_0^t w_\eta(s)ds + u_0, v) + a_e^*(\varphi_\eta(t),v), a_e(\int_0^t w_\eta(s)ds + u_0,\psi)) \qquad (90)$$

for all $z = (v,\psi) \in Y$, a.e. $t \in (0,T)$.

We prove that Λ has a unique fixed point η^*. Indeed, let $\eta = (\eta_1,\eta_2)$ and $\zeta = (\zeta_1,\zeta_2)$ be two elements of $L^2(0,T;Y')$ and denote by (w_η,φ_η) and (w_ζ,φ_ζ) the corresponding solutions of inclusion (72), we find

$$(Ay_\eta(t) - Ay_\zeta(t),z)_{Y'\times Y} + (\gamma^*(\xi_\eta(t),\kappa_\eta(t)) - \gamma^*(\xi_\zeta(t),\kappa_\zeta(t)),z)_{Z'\times Z} =$$
$$= (Ay_\eta(t) - Ay_\zeta(t),z)_{Y'\times Y} + ((\xi_\eta(t),\kappa_\eta(t)) - (\xi_\zeta(t),\kappa_\zeta(t)),\gamma z)_{L^2(\Gamma_3;\mathbb{R}^2)}$$
$$= ((\zeta_1(t) - \eta_1(t),\eta_2(t) - \zeta_2(t)),z)_{Y'\times Y}$$

for all $z \in Y$, a.e. $t \in (0,T)$, $(\xi_\eta(t),\kappa_\eta(t)) \in \partial J_t(\gamma y_\eta(t))$ and $(\xi_\zeta(t),\kappa_\zeta(t)) \in \partial J_t(\gamma y_\zeta(t))$. If we take $z = y_\eta - y_\zeta$, we obtain

$$(Ay_\eta(t) - Ay_\zeta(t), y_\eta(t) - y_\zeta(t))_{Y' \times Y} + ((\xi_\eta(t),\kappa_\eta(t)) - (\xi_\zeta(t),\kappa_\zeta(t)), \gamma y_\eta(t) - \gamma y_\zeta(t))_{L^2(\Gamma_3;\mathbb{R}^2)} = ((\zeta_1(t) - \eta_1(t), \eta_2(t) - \zeta_2(t)), y_\eta(t) - y_\zeta(t))_{Y' \times Y} \tag{91}$$

From (58), we find

$$\begin{aligned} min\{\theta^*,\beta^*\} \parallel y_\eta(t) - y_\zeta(t) \parallel_Y^2 + ((\xi_\eta(t),\kappa_\eta(t)) - (\xi_\zeta(t),\kappa_\zeta(t)), \gamma y_\eta(t) - \gamma y_\zeta(t))_{L^2(\Gamma_3;\mathbb{R}^2)} \\ \le \parallel (\zeta_1(t) - \eta_1(t), \eta_2(t) - \zeta_2(t)) \parallel_{Y'} \parallel y_\eta(t) - y_\zeta(t) \parallel_Y \\ \le \parallel (\eta(t) - \zeta(t) \parallel_{Y'} \parallel y_\eta(t) - y_\zeta(t) \parallel_Y \end{aligned}$$

hence

$$min\{\theta^*,\beta^*\} \parallel y_\eta(t) - y_\zeta(t) \parallel_Y^2 \le \parallel (\eta(t) - \zeta(t) \parallel_{Y'} \parallel y_\eta(t) - y_\zeta(t) \parallel_Y + ((\xi_\zeta(t),\kappa_\zeta(t)) - (\xi_\eta(t),\kappa_\eta(t)), \gamma y_\eta(t) - \gamma y_\zeta(t))_{L^2(\Gamma_3;\mathbb{R}^2)} \tag{92}$$

On the other hand, we have

$$((\xi_\zeta(t),\kappa_\zeta(t)) - (\xi_\eta(t),\kappa_\eta(t)), \gamma y_\eta(t) - \gamma y_\zeta(t))_{L^2(\Gamma_3;\mathbb{R}^2)} = \int_{\Gamma_3} ((\xi_\zeta(t) - \xi_\eta(t), \kappa_\zeta(t) - \kappa_\eta(t))).(\gamma y_\eta(t) - \gamma y_\zeta(t)) \; da.$$

From (60), we have the following inclusion

$$\partial J_t(x,\gamma y_\eta(t)) \subset \partial_{\gamma w_\eta(t)} J_t(x,\gamma y_\eta(t)) \times \partial_{\gamma \varphi_\eta(t)} J_t(x,\gamma y_\eta(t)) \quad a.e\;.t \in (0,T).$$
$$\partial J_t(x,\gamma y_\zeta(t)) \subset \partial_{\gamma w_\zeta(t)} J_t(x,\gamma y_\zeta(t)) \times \partial_{\gamma \varphi_\zeta(t)} J_t(x,\gamma y_\zeta(t)) \quad a.e\;.t \in (0,T).$$

and

$$\xi_\zeta(t) \in \partial_{\gamma w_\zeta(t)} J_t(x,\gamma y_\zeta(t)), \quad \xi_\eta(t) \in \partial_{\gamma w_\eta(t)} J_t(x,\gamma y_\eta(t)).$$
$$\kappa_\zeta(t) \in \partial_{\gamma \varphi_\zeta(t)} J_t(x,\gamma y_\zeta(t)), \quad \kappa_\eta(t) \in \partial_{\gamma \varphi_\eta(t)} J_t(x,\gamma y_\eta(t)).$$

There exists $s_\eta, s_\zeta \in L^2(0,T;L^2(\Gamma_3))$ and $d_\eta, d_\zeta \in L^2(0,T;L^2(\Gamma_3))$ such that

$$s_\eta(x,t) = h(S_t(\gamma w_\eta), \gamma\varphi_\eta(t) - \varphi_F)\, p_\eta(x,t), \quad d_\eta(x,t) = h_e(S_t(\gamma w_\eta))\, l_\eta(x,t)$$

with $p_\eta(x,t) \in \partial j(\gamma w_\eta(t))$ and $l_\eta(x,t) \in \partial j_e(\gamma\varphi_\eta(t) - \varphi_F)$, a.e. $(x,t) \in \Sigma_3$.

$$s_\zeta(x,t) = h(S_t(\gamma w_\zeta), \gamma\varphi_\zeta(t) - \varphi_F)\, p_\zeta(x,t), \quad d_\zeta(x,t) = h_e(S_t(\gamma w_\zeta))\, l_\zeta(x,t)$$

with $p_\zeta(x,t) \in \partial j(\gamma w_\zeta(t))$ and $l_\zeta(x,t) \in \partial j_e(\gamma\varphi_\zeta(t) - \varphi_F)$, a.e. $(x,t) \in \Sigma_3$. Therefore

$$((\xi_\zeta(t),\kappa_\zeta(t)) - (\xi_\eta(t),\kappa_\eta(t)), \gamma y_\eta(t) - \gamma y_\zeta(t))_{L^2(\Gamma_3;\mathbb{R}^2)} = \int_{\Gamma_3} (s_\zeta(x,t) - s_\eta(x,t), d_\zeta(x,t) - d_\eta(x,t)).(\gamma y_\eta(t) - \gamma y_\zeta(t)) \; da =$$
$$\int_{\Gamma_3} (s_\zeta(x,t) - s_\eta(x,t))(\gamma w_\eta(t) - \gamma w_\zeta(t))\, da + \int_{\Gamma_3} (d_\zeta(x,t) - d_\eta(x,t))(\gamma\varphi_\eta(t) - \gamma\varphi_\zeta(t))\, da$$

with

$$\int_{\Gamma_3} (s_\zeta(x,t) - s_\eta(x,t))(\gamma w_\eta(t) - \gamma w_\zeta(t))\, da \le L_h\sqrt{T} \parallel \gamma \parallel^2 (c_0 + c_1|\gamma w_\zeta(t)|) \parallel w_\eta - w_\zeta \parallel_{L^2(0,t,V)} \parallel w_\eta(t) - w_\zeta(t) \parallel_V + L_h \parallel \gamma \parallel^2 (c_0 + c_1|\gamma w_\zeta(t)|) \parallel \varphi_\eta(t) - \varphi_\zeta(t) \parallel_W \parallel w_\eta(t) - w_\zeta(t) \parallel_V + m\overline{h} \parallel \gamma \parallel^2 \parallel w_\eta(t) - w_\zeta(t) \parallel_V^2$$

and

$$\int_{\Gamma_3} (d_\zeta(x,t) - d_\eta(x,t))(\gamma\varphi_\eta(t) - \gamma\varphi_\zeta(t))\, da \le L_{h_e}\sqrt{T} \parallel \gamma \parallel^2 (c_{0_e} + c_{1_e}|\gamma\varphi_\zeta(t) - \varphi_F|) \parallel w_\eta - w_\zeta \parallel_{L^2(0,t;V)} \parallel \varphi_\eta(t) - \varphi_\zeta(t) \parallel_W + m_e\overline{h}_e \parallel \gamma \parallel^2 \parallel \varphi_\eta(t) - \varphi_\zeta(t) \parallel_W^2.$$

Then, we have

$$((\xi_\zeta(t),\kappa_\zeta(t)) - (\xi_\eta(t),\kappa_\eta(t)), \gamma y_\eta(t) - \gamma y_\zeta(t))_{L^2(\Gamma_3;\mathbb{R}^2)} \le \sqrt{T} \parallel \gamma \parallel^2 max\{L_h(c_0 + c_1$$

$$|\gamma w_\zeta(t)|), L_{h_e}(c_{0_e} + c_{1_e}|\gamma\varphi_\zeta(t) - \varphi_F|)\}\{\| w_\eta(t) - w_\zeta(t) \|_V + \| \varphi_\eta(t) - \varphi_\zeta(t) \|_W\}$$
$$\| w_\eta - w_\zeta \|_{L^2(0,t;V)} + L_h \| \gamma \|^2 (c_0 + c_1|\gamma w_\zeta(t)|) \| \varphi_\eta(t) - \varphi_\zeta(t) \|_W \| w_\eta(t) - w_\zeta(t) \|_V$$
$$+ max\{m\bar{h}, m_e\bar{h}_e\} \| \gamma \|^2 \{\| w_\eta(t) - w_\zeta(t) \|_V^2 + \| \varphi_\eta(t) - \varphi_\zeta(t) \|_W^2\}.$$

The inequality (92) becomes

$$min\{\theta^*, \beta^*\} \| y_\eta(t) - y_\zeta(t) \|_Y^2 \le \| (\eta(t) - \zeta(t) \|_{Y'} \| y_\eta(t) - y_\zeta(t) \|_Y + \sqrt{T} \| \gamma \|^2$$
$$max\{L_h(c_0 + c_1|\gamma w_\zeta(t)|), L_{h_e}(c_{0_e} + c_{1_e}|\gamma\varphi_\zeta(t) - \varphi_F|)\}\{\| w_\eta(t) - w_\zeta(t) \|_V +$$
$$\| \varphi_\eta(t) - \varphi_\zeta(t) \|_W\} \| w_\eta - w_\zeta \|_{L^2(0,t;V)} + L_h \| \gamma \|^2 (c_0 + c_1|\gamma w_\zeta(t)|) \| \varphi_\eta(t) - \varphi_\zeta(t) \|_W$$
$$\| w_\eta(t) - w_\zeta(t) \|_V + max\{m\bar{h}, m_e\bar{h}_e\} \| \gamma \|^2 \{\| w_\eta(t) - w_\zeta(t) \|_V^2 + \| \varphi_\eta(t) - \varphi_\zeta(t) \|_W^2\}.$$

$$\frac{1}{2}\left(min\{\theta^*, \beta^*\} - max\{m\bar{h}, m_e\bar{h}_e\} \| \gamma \|^2\right) \| y_\eta(t) - y_\zeta(t) \|_Y^2 \le \| (\eta(t) - \zeta(t) \|_{Y'}$$
$$\| y_\eta(t) - y_\zeta(t) \|_Y + c \| w_\eta - w_\zeta \|_{L^2(0,t;V)}^2 + \frac{1}{2}(L_h \| \gamma \|^2 (c_0 + c_1|\gamma w_\zeta(t)|))\{\| \varphi_\eta(t) - \varphi_\zeta(t) \|_W^2 +$$
$$\| w_\eta(t) - w_\zeta(t) \|_V^2\}.$$

$$\frac{1}{2}(min\{\theta^*, \beta^*\} - max\{m\bar{h}, m_e\bar{h}_e\} \| \gamma \|^2 - L_h \| \gamma \|^2 (c_0 + c_1|\gamma w_\zeta(t)|)) \| y_\eta(t) - y_\zeta(t) \|_Y^2 \le$$
$$\| (\eta(t) - \zeta(t) \|_{Y'} \| y_\eta(t) - y_\zeta(t) \|_Y + c \| w_\eta - w_\zeta \|_{L^2(0,t;V)}^2.$$

We obtain

$$\frac{1}{4}\left(min\{\theta^*, \beta^*\} - max\{m\bar{h}, m_e\bar{h}_e\} \| \gamma \|^2 - L_h\tilde{c}_2 \| \gamma \|^2\right) \| y_\eta(t) - y_\zeta(t) \|_Y^2 \le$$
$$\| (\eta(t) - \zeta(t) \|_{Y'}^2 + \| w_\eta - w_\zeta \|_{L^2(0,t;V)}^2\},$$

since $\gamma w_\zeta(t) \in L^2(\Gamma_3)$, a.e. $t \in (0,T)$, then, there exists $\tilde{c}_2 > 0$, such that $(c_0 + c_1|\gamma w_\zeta(t)|) < \tilde{c}_2$. Using (49) and Gronwall's inequality to obtain

$$\| y_\eta(t) - y_\zeta(t) \|_Y \le c \ \| (\eta(t) - \zeta(t) \|_{Y'}. \quad (93)$$

Since $\eta - \zeta \in L^2(0,T;Y')$, then we have the regularity (87).

Now, using the inequality

$$|a_\mu(\int_0^t (w_\eta(s) - w_\zeta(s))ds, v)| + |a_e^*(\varphi_\eta(t) - \varphi_\zeta(t), v)| \le c\{\int_0^t \| w_\eta(s) - w_\zeta(s) \|_V \, ds$$
$$+ \| \varphi_\eta(t) - \varphi_\zeta(t) \|_W\} \ \| v \|_V \quad (94)$$

for all $v \in V$, a.e. $t \in (0,T)$and

$$|a_e(\int_0^t (w_\eta(s) - w_\zeta(s))ds, \psi)| \le c\{\int_0^t \| w_\eta(s) - w_\zeta(s) \|_V \, ds\} \ \| \psi \|_W \quad (95)$$

for all $\psi \in W$, a.e. $t \in (0,T)$, with (90) and (93) to get

$$|(\Lambda\eta(t) - \Lambda\zeta(t), z)_{Y'\times Y}|^2 \le c \| z \|_Y^2 \{\sqrt{T}(\int_0^t \| w_\eta(s) - w_\zeta(s) \|_V^2 \, ds)^{\frac{1}{2}} +$$
$$\| \varphi_\eta(t) - \varphi_\zeta(t) \|_W\}^2$$
$$\le c\{\| w_\eta - w_\zeta \|_{L^2(0,t;V)}^2 + \| \varphi_\eta(t) - \varphi_\zeta(t) \|_W^2\} \ \| z \|_Y^2 \quad (96)$$

for all $z \in Y$, a.e. $t \in (0,T)$, so

$$\| (\Lambda\eta(t) - \Lambda\zeta(t) \|_{Y'}^2 \le c\{\| w_\eta - w_\zeta \|_{L^2(0,t;V)}^2 + \| \varphi_\eta(t) - \varphi_\zeta(t) \|_W^2\}. \quad (97)$$

On the other hand, using (93), we obtain

$$\| w_\eta(t) - w_\zeta(t) \|_V \le c \| \eta(t) - \zeta(t) \|_{Y'}. \quad (98)$$
$$\| \varphi_\eta(t) - \varphi_\zeta(t) \|_W \le \mathrm{c} \| \eta(t) - \zeta(t) \|_{Y'}. \quad (99)$$

Integrating (98) and (99), which implies

$$\| w_\eta - w_\zeta \|_{L^2(0,T;V)} \leq c \| \eta - \zeta \|_{L^2(0,T;Y')}. \quad (100)$$
$$\| \varphi_\eta - \varphi_\zeta \|_{L^2(0,T;W)} \leq c \| \eta - \zeta \|_{L^2(0,T;Y')}. \quad (101)$$
From (100) and (101), l'inequality (97) becomes
$$\| (\Lambda\eta - \Lambda\zeta \|^2_{L^2(0,T;Y')} \leq c \| \eta - \zeta \|^2_{L^2(0,T;Y')}. \quad (102)$$
Reiterating the previous inequality p times, we find that
$$\| \Lambda^p\eta - \Lambda^p\zeta \|_{L^2(0,T;Y')} \leq \sqrt{\frac{c^pT^p}{p!}} \| \eta - \zeta \|_{L^2(0,T;Y')}. \quad (103)$$
This last inequality shows that for a sufficiently large p, the operator Λ^p is a contraction on the Banach space $L^2(0,T;Y')$ and, therefore, there exists a unique element $\eta^* \in L^2(0,T;Y')$ such that $\Lambda\eta^* = \eta^*$, which shows that η^* is the unique fixed point of Λ.

(**ii**) Next, we show the continuous dependence result (88)-(89). For $\eta = \eta^*$, $\zeta = \zeta^*$ and using (67) to obtain
$$|(\eta_1(t) - \zeta_1(t), v)_{V'\times V}| \leq c\{\| u_\eta(t) - u_\zeta(t) \|_V + \| \varphi_\eta(t) - \varphi_\zeta(t) \|_W\} \| v \|_V$$
for all $v \in V$. a.e. $t \in (0,T)$, so
$$\| (\eta_1(t) - \zeta_1(t) \|_{V'} \leq c\{\| u_\eta(t) - u_\zeta(t) \|_V + \| \varphi_\eta(t) - \varphi_\zeta(t) \|_W\} \quad a.e. t \in (0,T). \quad (104)$$
Using (76) and (104) to find
$$\| \dot{u}_\eta(t) - \dot{u}_\zeta(t) \|_V \leq c\{\| u_\eta(t) - u_\zeta(t) \|_V + \| \dot{u}_\eta - \dot{u}_\zeta \|_{L^2(0,t;V)} + \| \varphi_\eta(t) - \varphi_\zeta(t) \|_W\} \quad (105)$$
Therefore
$$\| \dot{u}_\eta(t) - \dot{u}_\zeta(t) \|^2_V \leq c\{\| u_\eta(t) - u_\zeta(t) \|^2_V + \| \dot{u}_\eta - \dot{u}_\zeta \|^2_{L^2(0,t;V)} + \| \varphi_\eta(t) - \varphi_\zeta(t) \|^2_W\} \quad (106)$$
By Gronwell inequality, (106) becomes
$$\| \dot{u}_\eta(t) - \dot{u}_\zeta(t) \|^2_V \leq c\{\| u_\eta(t) - u_\zeta(t) \|^2_V + \| \varphi_\eta(t) - \varphi_\zeta(t) \|^2_W\} \quad (107)$$
On the other hand, we have
$$|(\eta_2(t) - \zeta_2(t), \psi)_{W'\times W}| \leq c \| u_\eta(t) - u_\zeta(t) \|_V \| \psi \|_W$$
for all $\psi \in W$. a.e. $t \in (0,T)$, so
$$\| (\eta_2(t) - \zeta_2(t) \|_{W'} \leq c \| u_\eta(t) - u_\zeta(t) \|_V \ a.e. t \in (0,T). \quad (108)$$
Now, using (77) and (108) to obtain
$$\| \varphi_\eta(t) - \varphi_\zeta(t) \|_W \leq c\{\| u_\eta(t) - u_\zeta(t) \|_V + \| \dot{u}_\eta - \dot{u}_\zeta \|_{L^2(0,t;V)}\} \ a.e. \ t \in (0,T). \quad (109)$$
From (107), we have
$$\| \dot{u}_\eta - \dot{u}_\zeta \|^2_{L^2(0,t;V)} \leq c\{\| u_\eta - u_\zeta \|^2_{L^2(0,t;V)} + \| \varphi_\eta - \varphi^2_\zeta \|_{L^2(0,t;W)}\}.$$
L' inequality (109) becomes
$$\| \varphi_\eta(t) - \varphi_\zeta(t) \|^2_W \leq c\{\| u_\eta(t) - u_\zeta(t) \|^2_V + \| u_\eta - u_\zeta \|^2_{L^2(0,t;V)} + \| \varphi_\eta - \varphi_\zeta \|^2_{L^2(0,t;W)}\}$$
a.e. $t \in (0,T)$. By Gronwell inequality, we have
$$\| \varphi_\eta(t) - \varphi_\zeta(t) \|^2_W \leq c\{\| u_\eta(t) - u_\zeta(t) \|^2_V + \| u_\eta - u_\zeta \|^2_{L^2(0,t;V)}\} \quad (110)$$
we deduce that (89) is verified. Using (110) and (107) to find
$$\| \dot{u}_\eta(t) - \dot{u}_\zeta(t) \|^2_V \leq c\{\| u_\eta(t) - u_\zeta(t) \|^2_V + \| u_\eta - u_\zeta \|^2_{L^2(0,t;V)}\} \quad (111)$$
we deduce (88). □

7 Proof of Theorem 3.1

We have now all the ingredients to provide the proof of Theorem 3.1.

Existence. Let $\eta^* = (\eta^*_1, \eta^*_2) \in L^2(0,T;Y')$ be the fixed point of the operator Λ obtained in Theorem 6.1 and let u_{η^*} be the function defined by
$$u_{\eta^*}(t) = \int_0^t w_{\eta*}(s)ds + u_0 forallt \in [0,T]. \quad (112)$$
It follows from (90) and (112) that
$$\eta^*_1(t) = A_\mu u_{\eta^*}(t) + A^*_e \varphi_{\eta^*}(t), \quad \eta^*_2(t) = A_e u_{\eta^*}(t). \quad (113)$$
a.e. $t \in (0,T)$, and, writing (72) for $\eta = \eta^*$ and using (113) to find that $u = u_{\eta^*}$ and $\varphi = \varphi_{\eta^*}$ is a solution of

problem (72). Since $w_{\eta*} \in L^2(0,T;V)$, we deduce that $u \in L^2(0,T;V)$ and we have $\dot{u}(t) = \dot{u}_{\eta*}(t) = w_{\eta^*}(t) = w_\eta(t)$, a.e. $t \in (0,T)$, so $\dot{u} \in L^2(0,T;V)$.
Regarding the regularity of solution $\varphi \in L^2(0,T;W)$ it follow from Theorem 6.1.The passage from problem (72) to problem (64)–(66) found when we choose $z = (v,0) \in Y$ in (72) and using (55) with (56), we obtain the inequality (64), and when we choose $z = (0,\psi) \in Y$ in (72) we obtain the inequality (65). About the condition initial (65) it follow from (112).

It remains to show the regularity $\dot{\varphi} \in L^2(0,T;W)$. To this end, for any $t_1, t_2 \in [0,T]$, we use (45) and the arguments used in the proof of Proposition 5.1 to obtain

$$A_\beta(\varphi(t_1) - \varphi(t_2)) - A_e(u(t_1) - u(t_2)) + \gamma^* d(t_1) - \gamma^* d(t_2) = q(t_1) - q(t_2).$$

Multiplying by $\varphi(t_1) - \varphi(t_2)$, we get

$$a_\beta(\varphi(t_1) - \varphi(t_2), \varphi(t_1) - \varphi(t_2)) - a_e(u(t_1) - u(t_2), \varphi(t_1) - \varphi(t_2))$$
$$+(d(t_1) - d(t_2), \gamma\varphi(t_1) - \gamma\varphi(t_2))_{L^2(\Gamma_3)} = (q(t_1) - q(t_2), \varphi(t_1) - \varphi(t_2))_{W' \times W}.$$

The coercivity of A_β we find

$$\beta^* \parallel \varphi(t_1) - \varphi(t_2) \parallel_W^2 \leq c\{\parallel u(t_1) - u(t_2) \parallel_V^2 + \parallel q(t_1) - q(t_2) \parallel_{W'}^2\} \parallel \varphi(t_1) - \varphi(t_2) \parallel_W$$
$$-(d(t_1) - d(t_2), \gamma\varphi(\mathrm{t}_1) - \gamma\varphi(t_2))_{L^2(\Gamma_3)}.$$

We use the same method for finding the inequality (86) in proposition 5.1, so that we have

$$(\beta^* - m_e\overline{h}_e \parallel \gamma \parallel^2) \parallel \varphi(t_1) - \varphi(t_2) \parallel_W^2 \leq \{\parallel u(t_1) - u(t_2) \parallel_V + \parallel q(t_1) - q(t_2) \parallel_{W'}$$
$$+L_{h_e} \parallel \gamma \parallel (c_{0_e} + c_{1_e}|\gamma\varphi(t_1) - \varphi_F|)$$
$$\parallel S_{t_1}(\dot{u}) - S_{t_2}(\dot{u}) \parallel_{L^2(\Gamma_3)}\} \parallel \varphi(t_1) - \varphi(t_2) \parallel_W \quad (114)$$

On the other hand, we have

$$S_{t_1}(\dot{u})(x) - S_{t_2}(\dot{u})(x) = \int_{t_2}^{t_1} |\gamma\dot{u}(s)(x)|\, ds \leq |\gamma u(t_1)(x) - \gamma u(t_2)(x)| \quad a.e.\, x \in \Gamma_3$$

so

$$\parallel S_{t_1}(\dot{u}) - S_{t_2}(\dot{u}) \parallel_{L^2(\Gamma_3)} \leq \parallel \gamma \parallel \parallel u(t_1) - u(t_2) \parallel_V \; a.e.\, x \in \Gamma_3$$

the inequality (114) becomes

$$(\beta^* - m_e\overline{h}_e \parallel \gamma \parallel^2) \parallel \varphi(t_1) - \varphi(t_2) \parallel_W^2 \leq c\{\parallel u(t_1) - u(t_2) \parallel_V + \parallel q(t_1) - q(t_2) \parallel_{W'}\}$$
$$\times \parallel \varphi(t_1) - \varphi(t_2) \parallel_W (115)$$

By Gronwell inequality and (48), l'inequality (114) becomes

$$\parallel \varphi(t_1) - \varphi(t_2) \parallel_W \leq c\{\parallel u(t_1) - u(t_2) \parallel_V + \parallel q(t_1) - q(t_2) \parallel_{W'}\} \quad a.e.\, t \in (0,T).$$

This inequality combined with the regularity $q \in W^{1,2}(0,T;W')$, then $q: [0,T] \to W'$ is an absolutely continuous function and $u \in L^2(0,T;V)$, shows that $u: [0,T] \to V$ shows that is an absolutely continuous function as well, and it satisfies

$$\parallel \dot{\varphi}(t) \parallel_W \leq c\{\parallel \dot{u}(t) \parallel_V^2 + \parallel \dot{q}(t) \parallel_{W'}^2\} \quad a.e.\, t \in (0,T). \quad (116)$$

Since $\dot{u} \in L^2(0,T;V)$ and $\dot{q} \in L^2(0,T;W')$, it follows from (116) that $\dot{\varphi} \in L^2(0,T;W)$, which concludes the regularity (50) and the existence part of proof.
Uniqueness. The uniqueness of the solution to Problem $\mathcal{P}^{\mathcal{V}}$ is a consequence of the uniqueness part in Proposition 5.1 and the uniqueness of the fixed point of Λ.

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