UtilitasMathematica
ISSN 0315-3681 Volume 120, 2023

Uniform decay in a wave equation and Numerical simulations

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Abstract

This research paper focuses on investigating the phenomenon of uniform decay in a wave equation. The study encompasses scenarios involving dynamic boundary conditions, localized memory terms, and fractional dampings. The main objective is to establish that the presence of a localized memory term, in conjunction with frictional dampings, holds significant strength. This combined effect, operating through a transmission process (u on the boundary Γ equals v), is demonstrated to ensure the asymptotic stability of the entire system.

Key words: Wave equation, Viscoelasticity, Memory term, Stabilization, Frictional damping

1.Introduction

We will explore the wave equation characterized by a non-local term in time and a discrete internal damping factor.

$$\left\{ u_{tt} + a(x)\,u_t = \Delta u - \int_0^t h(t-s)\Delta u(s)ds, \text{ in } \Omega \times R+ \right.$$

$u = 0$, *on* $\Gamma \times \mathbf{R}_+$

$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x)$, *in* Ω (1)

Here, Ω denotes a bounded region within $\mathbb{R}^n$, featuring a smooth boundary $\Gamma = \partial\Omega$. The functions $u_0(x)$ and $u_1(x)$ represent provided initial data, a stands as a nonnegative constant, and the precise definition of the relaxation function h(t) will be presented subsequently.

This issue simulates certain phenomena within the realm of viscoelasticity. It's worth mentioning that in prior research efforts, the subsequent constraint on the kernel has consistently been observed.

$h'(t) \leq -\gamma' h(t)$ (2)

was enforced.

We consider the kernel $h(t)$ to be a differentiable function of class C^1(R+,R+) that fulfills the following lemma:

lemma.1

If a function $h(0) > 0$ satisfies for $t \geq 0$,

$$h(t) \leq ae^{-\gamma' t} + b\int_0^t e^{-\gamma(t-s)}h(s)ds, \quad (3)$$

with $a,b,\gamma,\gamma' > 0$, for $\gamma > \gamma' + b$, $t > 0$,

$$\int_0^t e^{\gamma(s)}h(s)ds \leq ae^{(\gamma-\gamma')t}\left(\frac{\gamma-\gamma'}{\gamma-\gamma'-b}\right)$$

$$h(t) \leq a\left[1 + b\left(\frac{\gamma-\gamma'}{'}\right)\right]e^{-\gamma' t}. \quad (4)$$

then

(5)

$\gamma - \gamma - b$
and
$h'(t) \leq -\gamma' h(t)$ (6)

Proof. We set

$$r(t) = \int_0^t e^{\gamma(s)} h(s)ds,$$

and infer from (1) that

$'$

$$r'(t) = e^{\gamma t} h(t) \leq ae^{(\gamma - \acute{\gamma})t} + b\int_0^t e^{\gamma(s)} h(s)ds,$$

$r'(t) \leq ae^{(\gamma - \acute{\gamma})t} + b \times r(t)$, By Gronwall’s lemma and $r(0) = 0$, we find

$$r(t) \leq ae^{(\gamma-\gamma')t} + \int_0^t abe^{(\gamma-\gamma')s} e^{b\int_s^t d\tau} ds \quad \leq ae^{(\gamma-\gamma')t} + ab\int_0^t e^{(\gamma-\gamma')s} e^{b(t-s)} ds,$$

$$\leq ae^{(\gamma-\gamma')t} + ab \times e^{bt} \int_0^t e^{(\gamma-\gamma'-b)s} ds,$$

$$\leq ae^{(\gamma-\gamma')t} + \frac{ab}{(\gamma-\gamma'-b)} \times e^{bt}\left(e^{(\gamma-\gamma'-b)t} - 1\right),$$

$$\leq ae^{(\gamma-\gamma')t} + \frac{ab}{(\gamma-\gamma'-b)} \times \left(e^{(\gamma-\gamma')t} - e^{bt}\right),$$

$$\leq ae^{(\gamma-\gamma')t}\left(1 + \frac{b}{\gamma-\gamma'-b}\right) - \frac{ab}{\gamma-\gamma'-b} \times e^{bt},$$

$$\leq ae^{(\gamma-\gamma')t}\left(1 + \frac{b}{\gamma-\gamma'-b}\right),$$

$$\leq ae^{(\gamma-\gamma')t}\left(\frac{(\gamma-\gamma')}{\gamma-\gamma'-b}\right),$$

then

$$h(t) \leq ae^{-\gamma' t} + abe^{-\gamma' t}\left(\frac{\gamma-\gamma'}{\gamma-\gamma'-b}\right).$$

Hence

$$h(t) \leq a \times \left(\frac{1 + b(1-\xi^{-1})}{1 - b \times \xi^{-1}}\right) e^{-\gamma' t}, \quad \xi = \gamma - \gamma', \quad \eta = a \times \left(\frac{1 + b(1-\xi^{-1})}{1 - b \times \xi^{-1}}\right)$$

$$h(t) \leq \eta e^{-\gamma' t}.$$

$$h'(t) \leq -\gamma' \times \eta e^{-\gamma' t}.$$

.

then $h'(t) + \gamma' \times h(t) \leq 0$.
Hence $h'(t) \leq -\gamma' \times h(t)$.

lemma.2

If $1 > \gamma' > a$ and $0 < b < 1$ then

$$1-\int_{0}^{\infty} h(s)ds = l > 0$$
.

Proof. We now (7) and (8)

$$h(t) \leq a\left[1+b\left(\frac{\gamma-\gamma'}{\gamma-\gamma'-b}\right)\right]e^{-\gamma' t}.$$

$$\int_0^\infty h(s)ds \leq \frac{a}{\gamma'}\left[1+b\left(\frac{\gamma-\gamma'}{\gamma-\gamma'-b}\right)\right]$$

$$\gamma' > a \ \ and \ \ b < 1, \qquad (7)$$

we have

$$\gamma'(b-1)+a<0,\ \ then\ \ \gamma'-a > b(\gamma'),\ \ hence\ \ \frac{\gamma'}{a}\times\frac{\gamma'-a}{b(\gamma')} > 1$$

$$\frac{\gamma'}{a}\times\frac{\gamma'-a}{b(\gamma')} > \frac{\gamma-\gamma'-b}{\gamma-\gamma'},\ \ hence\ \ \frac{\gamma'-a}{\gamma'} > \frac{1}{\gamma'}ab\left(\frac{\gamma-\gamma'}{\gamma-\gamma'-b}\right)$$

$$hence\ \ 1 > \frac{a}{\gamma'}+\frac{1}{\gamma'}ab\left(\frac{\gamma-\gamma'}{\gamma-\gamma'-b}\right) \geq \int_0^\infty h(s)ds \qquad (8)$$

next then,

$$1-\int_0^\infty h(s)ds = l > 0$$
.

lemma.3

$e^{\zeta t}h(t) \in L^1(R+)$for $\gamma' > \zeta > 0$.

Easily understandable mathematical operations and abilities. In another section, we analyze the scenario where a(x) = 0, thus examining problem (1). In this particular case, we establish the presence and uniqueness of solutions, as previously demonstrated by Aries and Medjden.

The calculation of the stabilizations for wave equations is not required, as already addressed in the works of M. Medjden and N.-e. Tatar. Our approach boasts two distinct qualities: it is straightforward (without the need for complex methodologies) and it accommodates kernels that have not been addressed previously.

To address this, we introduce a novel 'Lyapunov functional.' We alter the system's energy by adding an extra term, thoughtfully selected to counteract undesired elements. Additionally, we delve into the case of a(x) = 0, signifying the absence of internal dissipation. In this scenario, we will establish that the integral term introduces a gentle damping effect, sufficient on its own to guide the system toward equilibrium, again in an exponential fashion

This paper will solely focus on addressing the question concerning the asymptotic behavior. In the context of the mathematical exposition at hand, our principal aim is to diligently channel the reader's attention towards the elucidated theory.

This theoretical construct is underpinned by meticulously constructed and exhaustive proofs, meticulously expounded upon in the cited sources, including references to Messaoudi, S.A. [12] and Medjden. M. and Tatar.N.E. [11], in conjunction with the Magister Memoir Aries. M. E.[1] The deliberate inclusion of these succinct citations serves the purpose of exemplifying the adept approach to seamlessly engage with the content delineated in references [1],[12], and [13].

Theorem 1.1. Assume that *h* is a continuous function and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.
In that case, a singular solution to problem (1.1) exists such that
$u \in L^\infty(0,\infty; H_0^1(\Omega)),\ u_t \in L^\infty(0,\infty; L^2(\Omega)),\ u_{tt} \in L^2(0,\infty; L^2(\Omega))$. (9)

Proof. Let *u* denote the exclusive global weak solution of issue (1). We proceed to deny the corresponding classical energy functional as follows:

$$E(t) = \frac{1}{2}\left\{\| u_t(t) \|_2^2 + \| \nabla u(t) \|_2^2\right\}. \qquad (10)$$

The energy function abides by the subsequent identity:

$$\frac{dE}{dt}(t) = -\| u_t(t) \|_2^2 + \int_\Omega a(x)\nabla u_t(t) \int_0^t h(t-s)\nabla u(s)dsdx. \qquad (11)$$

Setting,

$$(h\Box\nabla u)(t) = \int_\Omega \int_0^t h(t-s)\left|\nabla u(t) - \nabla u(s)\right|^2 dsdx, \qquad (12)$$

It becomes evident that

$$\frac{1}{2}\frac{d}{dt}(h\Box\nabla u)(t) = -\int_\Omega \nabla u_t \int_0^t h(t-s)\nabla u(s)dsdx + \frac{1}{2}(h'\Box\nabla u)(t)$$
$$+\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t h(s)ds\right)\int_\Omega |\nabla u|^2 dx\right\} - \frac{1}{2}h(t)\int_\Omega |\nabla u|^2 dx.$$

Subsequently, defining

$$e(t) := \frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\left(1 - \int_0^t h(s)ds\right)\int_\Omega |\nabla u|^2 dx + \frac{1}{2}(h\Box\nabla u)(t) \quad ,(14)$$

By employing (11) and (13), we derive

$$e'(t) = -\int_\Omega |u_t|^2 dx - \frac{1}{2}h(t)\int_\Omega |\nabla u|^2 dx + \frac{1}{2}(h'\Box\nabla u)(t) \quad . (15)$$

Notice that as per lemma 1, it follows that $e'(t) \le 0$ for $t \ge 0$. Additionally, considering the definitions of $e(t)$, $(h\Box\nabla u)(t)$, and lemma 2, we can establish the existence of a positive constant *M* such that:

$E(t) \le Me(t),\ t \ge 0$ (16)

Subsequently, we introduce a pair of functionals:

$$\Phi(t) := \int_\Omega u_t u dx \qquad (17)$$

and

$$\Psi(t) := \int_\Omega \int_0^t L_\alpha(t-s)\left|\nabla u(t) - \nabla u(s)\right|^2 dsdx =: (L_\alpha\Box\nabla u)(t) \qquad (18)$$

where

$$L_\alpha(t) := e^{-\alpha t}\int_t^{+\infty} l_\alpha(s)ds = e^{-\alpha t}\int_t^{+\infty} l(s)e^{\alpha s}ds \qquad (19)$$

where $\alpha = \zeta$ is as dened in lemma 3.

$V(t) := e(t) + \varepsilon\Phi(t) + \eta\Psi(t)$. Opting for $0 < \varepsilon < 1$, along with $\eta > 0$ (20)
and α chosen such that:

$$\frac{dV(t)}{dt} \le -\left[\frac{\varepsilon}{2} - \eta\left(\int_0^{+\infty} h(s)e^{\alpha s}ds\right) - 2(1-l)\mu\right]\int_\Omega |\nabla u|^2\,dx$$
$$-(1-\varepsilon)\int_\Omega |u_t|^2\,dx - \varepsilon\Phi(t) - \alpha\eta\Psi(t) - \mu(h\Box\nabla u)(t)$$
$$-\left(\eta - \frac{\varepsilon(1-l)}{2} - 2\mu\right)\int_\Omega\int_0^t h(t-s)\,|\nabla u(s)|^2\,dsdx. \qquad (21)$$

Lastly, we select μ to be sufficiently small to ensure the positivity of both coefficients within the brackets. Consequently, a positive constant β greater than 0 exists, such that:

$$\frac{dV(t)}{dt} \le -\beta\ V(t), \quad t \ge 0. \qquad (22)$$

As a result, we conclude that:

$$V(t) \le V(0)e^{-\beta t} \qquad t \ge 0. \qquad (23)$$

□

1. The undamped case $a(x) = 0$

If h satisfies an additional condition, namely $|h'(t)|\,e^{\sigma t} \in L^1(0,\infty)$ for some $\sigma > 0$, an exponential decay result can be established even in the absence of internal dissipation specifically when $a(x) = 0$. In this scenario, only the memory term remains, which is a less favorable situation compared to the initial case with the memory and internal dissipation combined.

However, it can be shown that this memory term by itself is adequate to create a modest dissipative effect, which leads the system to equilibrium exponentially. The method of proof is analogous to that used in the preceding theorem. The focus now shifts to how we offset the term $-\int_\Omega |u_t|^2 dx$in (11), which is forfeited due to the choice of

$a(x) = 0$. In this context, we consider the functional:

$$\Phi(t) := \int_\Omega u_t u dx$$

$$\Lambda(t) := -\int_\Omega u_t \int_0^t h(t-s)(u(t)-u(s))dsd \qquad (24)$$

$$L(t) := e(t) + \varepsilon\Phi(t) + \eta\Lambda(t)$$

And

$$\Psi(t) = \int_\Omega\int_0^t H_\alpha(t-s)\,|\nabla u(s)|^2\,dsdx \qquad (25)$$

With

$$H_\alpha(t) = e^{-\alpha t}\int_t^{+\infty} h(s)e^{\alpha s}ds \qquad (26)$$

For the verification of our second finding, the ensuing proposition is necessary for the proof:

Proposition 1.1.

Positive constants ρ1 and ρ2 exist such that

(27)

$\rho_1 e(t) \leq L(t) \leq \rho_2 e(t)$, for any $t \geq 0$

In order to demonstrate our result, it is necessary to display the subsequent auxiliary functionals:

$$W(t) = V(t) + \mu\Gamma(t) = e(t) + \varepsilon\Phi(t) \qquad (28)$$

And

$V(t) = e(t) + \varepsilon\Phi(t) + \eta\Lambda(t) + \delta\Psi(t)$

$$\Gamma(t) = \int_\Omega \int_0^t \tilde{H}_\sigma(t-s) \left|\nabla u(s)\right|^2 dsdx \qquad (29)$$

With

$$\tilde{H}_\sigma(t) = e^{-\sigma t} \int_t^{+\infty} \left|h'(s)\right| e^{\sigma s} ds. \qquad (30)$$

We apply our results.

Proposition 1.2.

Positive constants $\xi_1,.\xi_2..$ existsuch that

$$\xi_1 [E(t) + h\Box\nabla u(t)] \leq W(t) \leq [\xi_2 E(t) + h\Box\nabla u(t) + \Gamma(t)], \qquad (31)$$

for any $t \geq 0$

The classical energy $E(t)$ of equation (10) exhibits exponential decay to zero. We compute.

$$\frac{dW}{dt} = \frac{de}{dt} + \varepsilon\frac{d\Phi}{dt} + \delta\frac{d\Psi}{dt} + \eta\frac{d\Lambda}{dt} + \mu\frac{d\Gamma}{dt} \qquad (32)$$

we obtain

$$\begin{cases} \dfrac{dW}{dt} \leq -\left(\eta h_0 - \varepsilon - \eta\delta_3\right) \displaystyle\int_\Omega \left|u_t\right|^2 dx \\ -\left(\dfrac{\varepsilon}{2} - (1-l)\,\delta_1\eta - \mu l - \mu(2-l)\right) \displaystyle\int_\Omega \left|\nabla u\right|^2 dx \\ -\mu\sigma\Gamma(t) - \alpha\delta\Psi(t) - \left[\mu - \eta\dfrac{\overline{h}}{4\delta_1} - \eta\overline{h}\right] h\Box\nabla u(t) \\ -\left[\left(\delta - \dfrac{\varepsilon\overline{h}}{2} - (2-l)\,\mu\right) \displaystyle\int_\Omega \int_0^t h(t-s) \left|\nabla u(s)\right|^2 dsdx\right] \end{cases}$$

$$\delta \qquad (33)$$

–

Taking $\delta_1 = \frac{h}{4} \quad \eta = \frac{\mu}{4\overline{h}}, \quad \varepsilon = \frac{h_0}{4}, \ \delta_3 = \frac{h_0}{4\eta}, \quad \frac{hh_0}{2} + (2-l)\,\mu < \frac{}{2}$, we deduce for some positive constant. $C_1 = \min(h_0, \sigma, \alpha\delta, \mu) > 0$ such that

$$\begin{aligned} \frac{dW(t)}{dt} &\leq -C_1 \begin{pmatrix} \frac{1}{2}\int_\Omega \left|u_t\right|^2 dx + \frac{1}{2}\int_\Omega \left|\nabla u\right|^2 dx + \Gamma(t) \\ +\delta\Psi(t) + \frac{1}{2} h\Box\nabla u(t) \end{pmatrix} \\ &\leq -C_1 \left[E(t) + \frac{1}{2}(h\Box\nabla u)(t) + \Gamma(t)\right] \leq -\frac{C_1}{\xi_2} W(t) \end{aligned} \qquad (34)$$

Consequently, we infer that

$$W(t) \leq Ce^{-\frac{C_1}{\xi_2}t} \quad (35)$$

As for the other inequality,

$E\,(t) \leq E\,(0)e^{-\kappa t}$ *avec* $\kappa > 0$)

2. Numerical simulations

In this section, our objective is to analyze the impact of the supplementary term and demonstrate the scope of stability both preceeding to and following its incorporation into the system.

2.1 Numerical simulations of the one-dimensional form of the classical wave equation:

In order to derive a basic linear equation of motion, we make the assumption that both the relative displacement of the string and the gradient $\frac{\partial y}{\partial x}$ are of a small magnitude. We focus on a tiny segment $\triangle x$ of the string and observe that the variation in vertical tension between its ends gives rise to a restoring force, responsible for the vertical acceleration of this segment. Through the application of Newton's laws to this specific segment, we arrive at the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad (37)$$

We seek a solution $y(x,t)$ exclusively using distinct values of the independent variables x and t within a grid.

$$x = i\triangle x, \quad i = 1,\cdots,N_x, \quad t = j\triangle t, \quad j = 1,\cdots,N_t, \qquad (38)$$

$$y(x,t) = y(i\triangle x, j\triangle t) =^{def} y_{i,j}. \qquad (39)$$

The wave equation is discretized into a difference equation by applying the central-difference approximation. Initially, finite differences are used to express the second derivatives:

$$\frac{\partial^2 y}{\partial t^2} \simeq \frac{y_{i,j+1} + y_{i,j-1} - 2y_{i,j}}{(\triangle t)^2}$$

$$\frac{\partial^2 y}{\partial x^2} \simeq \frac{y_{i+1,j} + y_{i-1,j} - 2y_{i,j}}{(\triangle x)^2} \quad (40)$$

(41)

By inserting (40) and (41) into the wave equation (37), we arrive at the resulting difference equation

$$\frac{y_{i,j+1} + y_{i,j-1} - 2y_{i,j}}{(\triangle t)^2} = \frac{1}{c^2}\frac{y_{i+1,j} + y_{i-1,j} - 2y_{i,j}}{(\triangle x)^2} \quad (42)$$

Observe that this equation encompasses three distinct time values:

$j + 1$ = the future,
j = the present,
j-1= *the past.*

As a result, we reorganize it into a format that enables us to anticipate the forthcoming solution based on the existing and preceding solutions:

$$y_{i,j+1} = 2y_{i,j} - y_{i,j-1} + \frac{c^2}{c'^2}\left(y_{i+1,j} + y_{i-1,j} - 2y_{i,j}\right), \quad c'^2 =^{def} \frac{\triangle x}{\triangle t}.(43)$$

In this context, c' represents a composite of numerical parameters possessing velocity like dimensions, and its proportion to c governs the stability of the algorithm. For our calculations, we set the initial time as $j = 1$, signifying that $j = 0$ corresponds to $t = -\triangle t$.

At the outset,

$$y_{i,2} = 2y_{i,1} - y_{i,0} + \frac{c^2}{c'^2}\left(y_{i+1,1} + y_{i-1,1} - 2y_{i,1}\right), \quad (44)$$

with

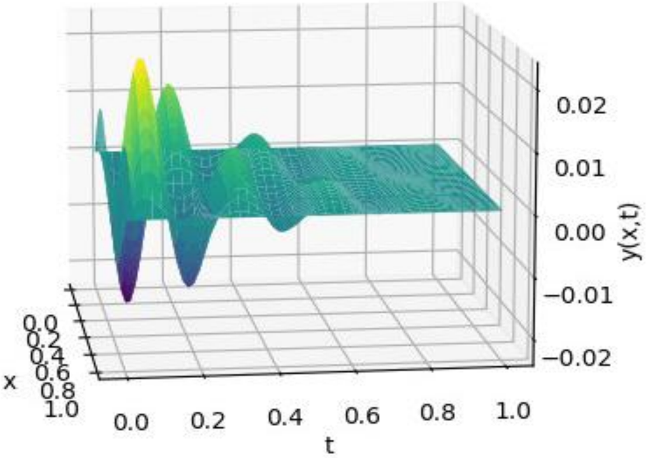
$$\frac{\partial y}{\partial t}(x,0) = 0 \Longrightarrow y_{i,2} = y_{i,0} \quad (45)$$

This signifies

$$y_{i,2} = y_{i,1} + \frac{c^2}{2c'^2}\left(y_{i+1,1} + y_{i-1,1} - 2y_{i,1}\right), \quad (for\ \ j = 2\ \ only). \quad (46)$$

Example

$y\ tt - \triangle y = 0$ *in* (0,1) × (0,1)
$y(x,0) = 0.001 sin(2\pi x)$ (47)
$y_t(x,0) = 0$



2.2 Numerical simulations of the two-dimensional form of the classical wave equation:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad (48)$$

This represents the two-dimensional counterpart of the wave equation (37) we previously examined in a single dimension. In this context c, the velocity of propagation, remains the square root of tension divided by density.

The process of devising an algorithm to solve the 2D wave equation (48) mirrors the approach employed for the 1D equation in Section 3.

We begin by translating the second derivatives into central differences:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u(x,y,t+\triangle t)}{(\triangle t)^2} \quad (49)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+\triangle x,y,t)}{(\triangle x)^2} \quad (50)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u(x,y+\triangle y,t)}{(\triangle y)^2} \quad (51)$$

Once we discretize the variables as: $u(x = i\triangle x, y = i\triangle y, t = k\triangle t) \equiv u^k_{i,j}$, we derive our time-stepping algorithm by calculating the future solution based on the existing and preceding solutions:

$$u^{k+1}_{i,j} = 2u^k_{i,j} - u^{k-1}_{i,j} + \frac{c^2(\triangle t)^2}{(\triangle x)^2}[u^k_{i+1,j} + u^k_{i-1,j} - 4u^k_{i,j} + u^k_{i,j+1} + u^k_{i,j-1}] \quad (52)$$

While the current solution at time (k) and the previous solution at time ($k - 1$) are established after the initial step, initiating the algorithm requires knowledge of the solution at $t = -\triangle t$, which corresponds to before the initial time. To determine this, we leverage the fact that the membrane starts from a state of rest:

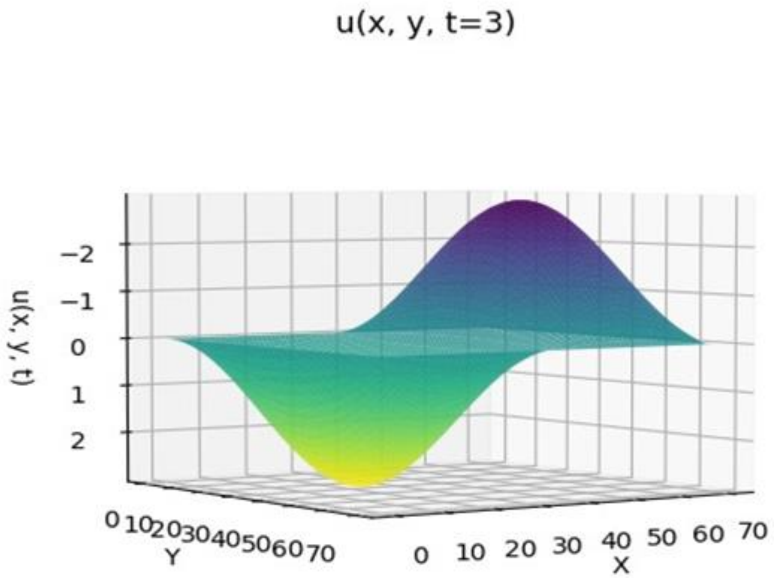
$$0=\frac{\partial u(x,y,0)}{\partial t}\approx\frac{u_{i,j}^{1}-u_{i,j}^{-1}}{2\triangle t}\Longrightarrow u_{i,j}^{1}=u_{i,j}^{-1}. \qquad (53)$$

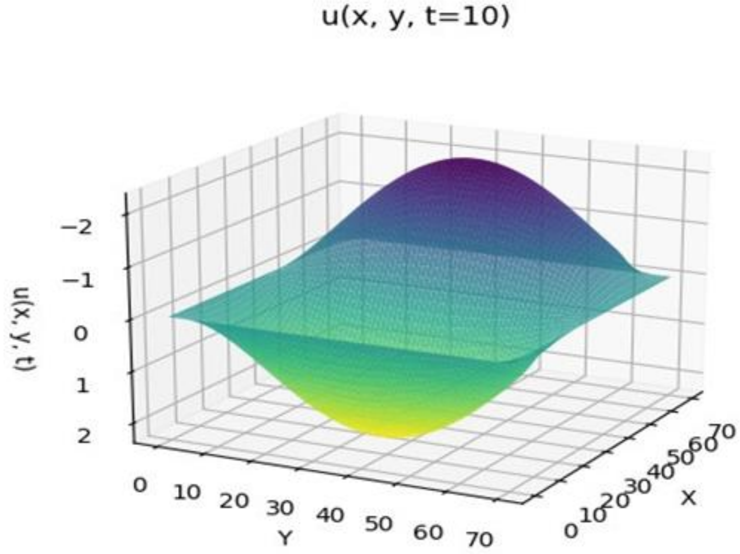
Upon inserting (53) into (52) and solving for u^1, , we derive the algorithm for the initial step:

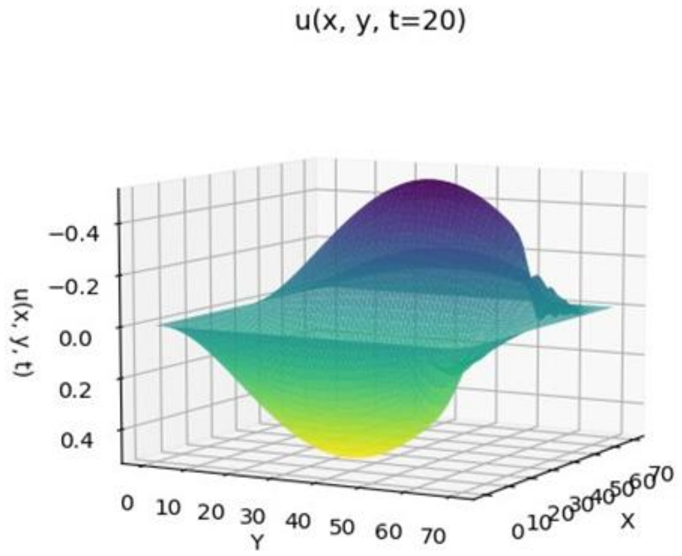
$$u_{i,j}^{1}=u_{i,j}^{0}-\frac{c^{2}(\triangle t)^{2}}{2(\triangle x)^{2}}[u_{i+1,j}^{0}+u_{i-1,j}^{0}-4u_{i,j}^{0}+u_{i,j+1}^{0}+u_{i,j-1}^{0}] .(54)$$

The displacement $u^0_{i,j}$ is already determined at time $t = 0$ (*with* $k = 0$). Example

$$u_{tt}(x,y,t)-\triangle u(x,y,t)=0 \quad in \quad (0,2)\times(0,2)\times(0,1)$$
$$u(x,y,0)=0.001cos(2\pi x)sin(2\pi y) \quad (55)$$
$$u_t(x,y,0)=0$$


u(x, y, t=3)


u(x, y, t=10)



2.3 Numerical simulation application for 2 DWaves

The standard method are used for the temporal part is the finite difference scheme. are solving the PDE by discretizing the spatial and temporal derivatives and using the finite difference method for numerical integration.

Here's a breakdown of the code's temporal integration process:

1. Time Discretization: You've defined a time grid t with Nt time points. The time step dt is calculated as the difference between consecutive time points.
2. Time Stepping Loop: The outer loop iterates over the time steps from 0 to $Nt-1$ (excluding the last time step).
3. Spatial Discretization Loop: The inner loop iterates over spatial grid points from 1 to $Nx-2$ (excluding boundary points) because you're using second-order finite difference that require neighbors for calculations.
4. Calculating Second Derivatives: You calculate the second derivatives of u with respect to space ($d2u/dx2$) and time ($d2u/dt2$) using finite difference approximations.
5. Integration Term: You've implemented numerical integration using the simps function from SciPy. The integration variable is the time t, and you're integrating $exp(-(t[n]-t)**2)*d2u/dx2$ over the time domain.
6. Updating u: The value of u at the next time step is updated based on the finite difference approximation for the second derivatives and the computed integral term. The coefficient in the update equation reflect the second-order accuracy of the finite difference scheme.

our approach is a combination of finite difference methods for the spatial derivatives and numerical integration for the time-dependent term. This approach works well for relatively simple cases and provides an approximate solution to the given PDE.

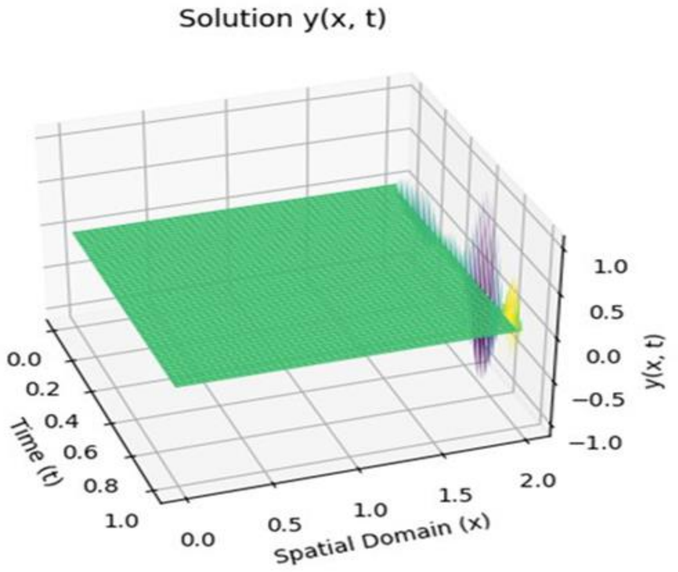
However, the overall stability and accuracy of the method can depend on the specific problem and the choice of numerical parameters (grid size, time step, etc.). For more complex problems or higher accuracy requirements, other numerical methods like the finite element method or spectral methods might be considered.

Example In the following subsection, we engage in numerical simulations to visually exemplify the theoretical finndings presented in the preceding sections (1D). The initial parameters for the system (37) have been selected as follows:

$y(x, 0) = 0.001\, sin(2\pi x)$
(56)
$y_t(x, 0) = 0$
with

$$y_{tt}(x,t) - \triangle y(x,t) + \int_0^t h(t-s)\triangle y(x,s)ds = 0 \qquad in \quad (0,1)\times(0,2)$$

with $h(t) = 2e^{-t2}$

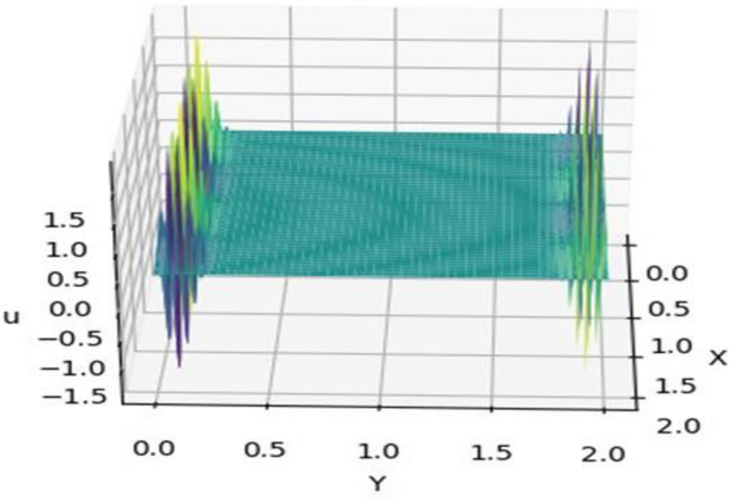


Solution y(x, t)

Example In the following subsection, we conduct numerical simulations to visually demonstrate the theoretical findings presented in the preceding sections (2D). The initial conditions for the system defined by equation (37) have been specially selected as follows:
u(x,y,0) = 0.001cos(2πx)sin(2πy)
(58)
ut(x,y,0) = 0

$$u_{tt}(x,y,t) - \triangle u(x,y,t) + \int_0^t h(t-s)\triangle u(x,y,s)ds = 0 \qquad in \quad (0,2)\times(0,2)\times(0,1)$$

with $h(t) = 2e^{-t2}$



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