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Estimation in nonparametric functional-on regression models with real responses

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Abstract

In this article, we are interested in the estimation of the conditional distribution cumulative function and its regression by the kernel method for functional random variables. We present the theory, the practice and the applications of this nonparametric method, which uses a weighting function to smooth the data and obtain an estimation of the conditional distribution function. We illustrate the method with simulated example, and we discuss its limitations and future research perspectives. We show that the kernel method is simple, flexible and robust, and that it can handle complex cases of functional regression

Keywords: Functional random variable, kernel estimator, nonparametric regression estimation, functional regression, conditional cumulative distribution function (c.d.f).

1. Introduction

Functionaldataanalysisisafieldofstatisticsthatdealswithobservationsthatarefunctionsorcurves,ratherthan scalars or vectors. Functional data appear in many areas of application, such as medicine, biology, finance,ecology, etc. It’s worth noting that the modelization of functional variables is gaining popularity, as evidenced by the monographs by James. O. Ramsay and Bernard. W. Silverman[8]. and Ferraty and Vieu [4]. On functional data analysis (FDA). One of the fundamental problems of functional data analysis is functional regression, which consists of studying the relationship between a functional response variable and one or more explanatory variables, which can also be functional or scalar. Functional regressionrelies on the estimation of the conditional cumulative distribution function of the response variable, given the explanatory variables, as discussed by Ahmed Ait Saidi and Mecheri Kheira [10]. The conditional distribution function contains all the information about the distribution of the response variable, and allows calculating quantities of interest, such as conditionalmean,the conditional variance, the conditional quantiles, etc., as discussed by Ali Gannoun, Jérôme Saracco, and Keming Yu [12].and Manteiga and Vieu [7]. The estimation of the conditionaldistribution function is therefore an essential step of functional regression. In this article, we are interested in the kernel method for the estimation of the conditional distribution function and its regression for functional variables. The kernel method is a nonparametric technique that uses a weighting function, called kernel, to smooth the data and obtain an estimation of the conditional distribution function, as detailed byGao and Gijbels [5]. and Ferraty and Vieu [4]. The kernel method has several advantages, such as its simplicity,its flexibility and its robustness. It also allows handling complex cases, such as multivariate functional regression, functional regression with censored data, or functional regression with heterogeneous data, as explained by Li and Hsing [6].We present in this article the theory, the practice and the applications of the kernel method for the estimation of the conditional distribution function and its regression for functional random variables (i.e. takes their values in a space of infinite dimension). We describe the principle of the method, the choices of the kernel and the bandwidth, the asymptotic properties, and the performance criteria. Citing that there are other suitable methods for estimating nonparametric functional regression, such as linear functional regression or spline regression. We illustrate the method with simulated data, and we discuss its limitations and future research perspectives. We show that the kernel method is simple, flexible, and robust, and that it can handle complex cases of functional regression. Consequently, the main contribution of this work is to validate

the theoretical results addressed by the researchers Ferraty et View [4]. and Delsol [3]. citing as an example, for the independent and identically distributed (i.i.d) case of the data, under certain conditions,
someasymptoticresultssuchasalmostcompletepointconvergenceandalmostcompleteuniformconvergence of the kernel estimator with their rates are established. Our interest mainly comes from the fact that many researchers have dealt with the theoretical aspect of the estimation of the conditional cumulative distribution function under the functional index regression model. For example, Ait Saidi and Kheira [10]. treated the theoretical framework of the conditional distribution function from an independent data sample with a scalar response variable and an explanatory variable valued in an infinite-dimensional semi-norm edvectors pace. On the one hand, Delsol[3] carried out a simulation for the estimation of the non-parametric conditional meanby the kernel method from a sample of independent and identically distributed functional data. On the other hand, Ait Saidi and Kheira[10]. treated the convergence of the estimator of the semi-parametric functional conditional mean from a sample of functional data i.i.d simulated. The works[1]. Proposes a local linear estimator of the conditional hazard function for index models incase of missingat random data. The method involves using a kernel-based approach to estimate the hazard function at each point in the domain and incorporating missing data using inverse probability weighting. The works[11]. Discusses nonparametric regression and classification with functional, categorical, and mixed covariates. The method involvesusing a generalized additive model (GAM) framework to model the relationship between the response and covariates, where the covariates can include functional data, categorical data, and mixed data. Therefore, the estimation of the conditional distribution function in the single index functional model for the case of independence. Of the simulated data should be important issues.

The remainder of the paper is structured as follows: In Section 2, we introduce our model and estimator, along with key concepts, assumptions, and notations. Section 3 is dedicated to a simulation study. Section 4 presents the results and discussion. Finally, we summarize our findings and present conclusions in the last section.

2. Model and estimator

2.1.Construct of the estimator

Let (X,Y) be a pairs of random variables taking its values in $E\times\mathbb{R}$, where (E,d) is a semi-metric space (i.e. $X$is a functional random variable (f.r.v) and d a semi-metric). We consider the sample $(X_i,Y_i)_{i=1,\dots,n}$of n independent pairs identically distributed (i.i.d) as the couple (X,Y). Let (x,y) be a fixed element of $(E,\mathbb{R})$, let $N_x\subset E$be a neighborhood of $x$and Sbe a fixed compact subset of$\mathbb{R}$. Given$x$, let us denote by $\hat{y}$a predicted value for the scalar response$y$. The regression (nonlinear) operator $r$of Yon $X$is defined by

$$\boldsymbol{r(x) = E(Y|X = x)} \qquad (2.1)$$

And the conditional cumulative distribution function (c.d.f.) of Y given X is defined by

$$\forall \boldsymbol{y} \in \mathbb{R}, \boldsymbol{F_Y^X(x,y) = P(Y\leq y|X = x)}. \qquad (2.2)$$

We consider the conditions introduced by Ferraty et al [4] on the regression operator. We define by $\hat{r}(x)$ the kernel estimator of $r(x)$ such that

$$\hat{\mathbf{r}}(\boldsymbol{x}) = \frac{\sum_{i=1}^{n} Y_i K(\frac{1}{\square}d(X_i,x))}{\sum_{i=1}^{n} K(\frac{1}{\square}d(X_i,x))} \qquad (2.3)$$

And by $\hat{F}_Y^X(x,y)$ the kernel estimator of$F_Y^X(x,y)$such that

$$\forall \boldsymbol{y} \in \mathbb{R}: \hat{\boldsymbol{F}}_{\boldsymbol{Y}}^{\boldsymbol{X}}(\boldsymbol{x},\boldsymbol{y}) = \frac{\sum_{i=1}^{n} G(\frac{1}{g}(y-Y_i))K(\frac{1}{\square}d(X_i,x))}{\sum_{i=1}^{n} K(\frac{1}{\square}d(X_i,x))} \qquad (2.4)$$

The function K is a kernel of type I or of type II and G is defined by$\forall t\in\mathbb{R}, G(t) = \int_{-\infty}^{t} K_0(v)dv$ where $K_0$is akernel of type 0 and $h = h(n)$ $(resp.g = g(n))$ is a sequence of positive real numbers which goes to zero as $n$tends to $+\infty$. This estimate extends, in different way, the works of Roussas [9]. in the real case and Ferraty and Vieu [4] in the functional case.

Recall that a function $K$from $\mathbb{R}$into $\mathbb{R}^+$such that $\int K = 1$is called kernel of type Iif there exist two real constants $c_1$and $c_2$satisfies

$0 < c_1 < c_2 < \infty$

such that

$c_2 1_{[0,1]} \leq K \leq c_2 1_{[0,1]}$

It's called kernel of type II if its support is [0,1] and if its derivative K' exists on [0,1] and satisfies for two real constants c_3 and c_4

$-\infty < c_3 < c_4 < 0$

such that

$c_3 \leq K' \leq c_4$

A function K_0 to $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K_0 = 1$ is called kernel of type 0 if its compact support is $[-1,1]$ such that

$$\forall t \in]0,1[: K_0(u) > 0 \qquad (2.5)$$

2.2. Point wise almost complete convergence

We will list some of the theoretical results mentioned at length in [4] with their proof, which we will rely on to conduct the theoretical side in this paper. Let $N_x \subset H$ be a neighborhood of x and S be a fixed compact subset of $\mathbb{R}$. In order to establish the almost complete convergence (a.co.) of our estimators we give some regularity conditions needed for stating this result introduced by ferraty [4].

(H_1) The model of the link function

$$Y = E(Y|X) + \epsilon, \textit{where } E(\epsilon|X) = 0. \qquad (2.6)$$

(H_2) The probability of the functional variable on a small ball is non null

$$\forall \epsilon > 0, P(X \in B(x,\epsilon)) = \varphi_x(\epsilon) > 0. \quad (2.7)$$

$$(H_3)\begin{cases} h \textit{ is a sequence of positive numbers such that} \\ \lim\limits_{n\to+\infty} h = 0 \textit{ and } \lim\limits_{n\to+\infty} \frac{1}{\varphi_x(h)} \cdot \frac{\log n}{n} = 0. \\ K \textit{ is a type } I \textit{ or type } II \textit{ Kernel and} \\ \exists C > 0, \exists \epsilon > 0, \forall \epsilon < \epsilon_0, \int_0^1 \quad \varepsilon\varphi_x(u)du > C\varepsilon\varphi_x(\varepsilon). \end{cases}$$

(H_4) The parameter g is a positive sequence such that, $\lim\limits_{n\to+\infty} h = 0$ and K_0 is a kernel of type 0.

(H_5) We will consider a scalar response variable Y such that

$\forall m \geq 2, E(|Y_m||X = x) < \sigma_m(x) < \infty \textit{ with } \sigma_m(x)$ continuous at x.

(H_6) The regression r (which is a nonlinear operator from E to $\mathbb{R}$) verify

$$r : E \to \mathbb{R}, \lim_{d(x_1,x_2)\to 0} r(x_1) = r(x_2) \qquad (2.8)$$

(H_7) The conditional cumulative distribution function F_Y^X (which is a nonlinear operator from $E \times \mathbb{R}$ to $\mathbb{R}$) verify

$$\begin{aligned} &F_Y^X : E \times \mathbb{R} \to \mathbb{R}, \forall x' \in \mathcal{N}_x, \lim_{d(x',x)\to 0} F_Y^X(x',y) = F_Y^X(x,y) \\ &\quad \textit{and } \forall y' \in \mathbb{R}, \lim_{|y'-y|\to 0} F_Y^X(x,y') = F(x,y) \end{aligned} \qquad (2.9)$$

Theorem 2.1 *Under hypothesis* (H_1), (H_2), (H_3), (H_5) and (H_6) *we have*

$$\lim_{n\to\infty} \hat{r}(x) = r(x), \textit{a.co.} \qquad (2.10)$$

Theorem 2.2 *Under hypothesis* (H_1), (H_2), (H_3), (H_4) and (H_7) *we have for any fixed real number y*

$$\lim_{n\to\infty} \hat{F}_Y^X(y) = F_Y^X(y), \textit{a.co.} \qquad (2.11)$$

Recall that we say that the sequence $(Z_n)_{n\in\mathbb{N}}$ of real-valued random variables is almost complete convergent (a.co.) to a real-valued random variable Z if only if

$$\forall \epsilon > 0, \sum_{n\geq 1} \quad P(|Z_n - Z| > \epsilon) < \infty \quad (2.12)$$

Next, the convergence rate of the almost complete convergence of $(T_n)_{n\in\mathbb{N}}$ to T *is* of v_n-order if and only if

$$\exists \epsilon_0 > 0, \sum_{n\geq 1} \quad P(|(T_n - T| > \epsilon_0 v_n) < \infty. \qquad (2.13)$$

And we write $T_n - T = O_{a.co}(v_n)$.

Proof of theorems (2.1) and (2.2). First all for $i = 1,\ldots,n$we recall that

$$\Omega_i(x) = \frac{K(\frac{d(X_i,x)}{\square})}{EK(\frac{d(X_i,x)}{\square})} \qquad (2.14\ )$$

The function Ω_i is defined if the expected value of $EK(\frac{d(X_i,x)}{h})$ is non-zero. Since let's show $EK(\frac{d(X,x)}{h}) > 0$

If $K$is a kernel of type$I$, we have

$$c_1\mathbf{1}_{[0,1]} \le K \le c_2\mathbf{1}_{[0,1]}\ where\ c_1, c_2 \in \mathbb{R}_+^* (2.15)$$

which implies

$$c_1\mathbf{1}_{B(x,h)}(X) \le K(\frac{d(X,x)}{h}) \le c_2\mathbf{1}_{B(x,h)}(X) \qquad (2.16)$$

which implies directly

$$c_1\varphi_x(h) \le EK(\frac{d(X,x)}{h}) \le c_2\varphi_x(h) \quad (2.17)$$

If$K$is a kernel of typeII, we have

$$EK(\frac{d(X,x)}{h}) = \int_0^1 K(\frac{d(X,x)}{h})dP(\frac{d(X,x)}{h} < 1)\ (2.18)$$

Since K'exists we can write

$$K(t) = K(0) + \int_0^t K'(u)du \qquad (2.19)$$

Which that implies

$$\begin{aligned} EK(\frac{d(X,x)}{h}) &= \int_0^1 K(0)dP(\frac{d(X,x)}{h} < 1) + \int_0^1 (\int_0^t K'(u)du)\, dP(\frac{d(X,x)}{h} < 1) \\ &= K(0)\varphi_x(h) + \int_0^1 (\int_0^1 K'(u)\mathbf{1}_{[u,1]}(t)du)\, dP(\frac{d(X,x)}{h} < 1) \end{aligned} \qquad (2.20)$$

By applying Fubini Theorem we obtained

$$EK(\frac{d(X,x)}{h}) = K(0)\varphi_x(h) + \int_0^1 K'(u)P(u \le \frac{d(X,x)}{h} \le 1)du (2.21)$$

Using the fact that $K(1) = 0$ we can write

$$EK(\frac{d_\lambda(X,x)}{\square}) = -\int_0^1 K'(v)\varphi_{x,\lambda}(\square v)dv \quad (2.22)$$

Using the hypothesis (H_5) with $h < \epsilon$we arrive to

$$EK(\frac{d(X,x)}{\square}) \ge c_4\varphi_x(\square) \qquad (2.23)$$

Since K is bounded and of support [0,1] using the result of equation (2.24) we get

$$EK(\frac{d(X,x)}{\square}) \le c_3\varphi_x(\square) \qquad (2.25)$$

To address the proof of the two mentioned theorems, we need decomposition (2.26)to prove the almost complete convergence of the regression and decomposition (2.27) for its conditional distribution function by combining the following lemmas.

$$\hat{r}(x) - r(x) = \frac{1}{\hat{r}_1(x)}[(\hat{r}_2(x) - E\hat{r}_2(x) - (r(x) - E\hat{r}_2(x))] - \frac{r(x)}{\hat{r}_1(x)}(\hat{r}_1(x) - 1)\ (2.28)$$

where

$$\hat{r}_1(x) = \frac{1}{n}\sum_{i=1}^{n} \Omega_i(x) \text{ and} \hat{r}_2(x) = \frac{1}{n}\sum_{i=1}^{n} Y_i\Omega_i(x).$$

$$\hat{F}_Y^X(x,y) - F_Y^X(x,y) = \frac{1}{\hat{r}_1(x)}[(\hat{r}_3(x,y) - E\hat{r}_3(x,y) - (F_Y^X(x,y) - E\hat{r}_3(x,y))] - \frac{F_Y^X(x,y)}{\hat{r}_1(x)}(\hat{r}_1(x) - 1)\ (2.29)$$

where

$$\hat{r}_1(x) = \frac{1}{n}\sum_{i=1}^{n} \Omega_i(x) \text{ and} \hat{r}_3(x,y) = \hat{r}_1(x)\hat{F}_Y^X(x,y) = \frac{1}{n}\sum_{i=1}^{n} \square_i(y)\Omega_i(x).\ (2.30)$$

$$\Omega_i(x) = \frac{K(\frac{X_i - x}{h})}{EK(\frac{X_i - x}{h})}\ and\ \Gamma_i(y) = G(\frac{Y_i - y}{g})(2.31)$$

The proof of Theorems 2.1 and 2.2 is a direct consequence of the following lemmas.

Lemma 2.3

1. Under assumptions of theorem, as n goes to infinity, we have

$$\hat{r}_1(x) - 1 = O_{a.ca.}\left(\sqrt{\frac{1}{\varphi_x(\square)} \cdot \frac{\log n}{n}}\right). \quad (2.32)$$

2. Under assumptions of theorem, as n goes to infinity, we have

$$E\hat{r}_2(x) - \hat{r}_2(x) = O_{a.co.}\left(\sqrt{\frac{1}{\varphi_x(\square)} \cdot \frac{\log n}{n}}\right). \qquad (2.33)$$

Lemma 2.4 *Under conditions* (H_3) *and* (H_6), *as n goes to infinity, we have*

$$\lim_{n\to\infty} E\hat{r}_2(x) = r(x), \text{a.co.} \qquad (2.34)$$

Lemma 2.5 *Under assumptions of theorem, as n goes to infinity, we have*

$$1. \lim_{n\to\infty} E\hat{r}_3(x,y) = F_Y^X(x,y), \text{a.co.} \qquad (2.32)$$

$$2. \lim_{n\to\infty} \hat{r}_3(x,y) - E\hat{r}_3(x,y) = 0, \text{a.co.} \qquad (2.3\ 3)$$

The denominator of decompositions (2.35)and (2.36) is treated by using the first part of Lemma 2.3 and the first part of Proposition $A.6$ (see [4]). Meanwhile the numerators of decomposition (2.37)are addressed by applying Lemmas 2.3 and 2.4 above, and the numerators of decomposition (2.38) are handled by utilizing the first part of Lemma 2.3 and Lemma 2.5. For further details of the proofs, one can refer to [4]. where the proofs of theorems (2.1) and (2.2) are elaborated upon with detailed explanations. That concludes the proof of both the theorems.

3. Simulation

In this section, our interest is the applied study of the theoretical study carried by [10] to estimate the conditional cumulative distribution function under the functional index model under the conditions of regularity cited in section 2, where we adopt the simulation data that used to estimate the functional semi parametric regression on the one hand, and then we adopt the simulation data proposed by [3] to estimate the functional non parametric regression on the other hand.

n	h	$r(x)$	$\hat{r}(x)$	MSE
10	0.001	0.9290	0.9206	0.0084
		0.8322	0.8238	
		1.3035	1.2951	
		0.9760	0.9676	
		1.0473	1.0389	
		1.2464	1.2380	
		1.0374	1.0290	
		1.1550	1.1466	
		0.8737	0.8653	
		1.3082	1.2998	

Table 1: Comparison table of regression and its estimator with MSE

To illustrate the concepts cited in section 2 by simulation, let's consider two examples. In the first example, we generate a sample of curves using data simulation techniques. We construct a set of 100 curves, each defined by the function One builds a sample of n = 100 curves as follows

$$x_i(t_j) = \cos(w_i + \pi(2t_j - 1)) \qquad (3.1)$$

Where $0 < t_1 < t_2 < \ldots < t_{100} = 1$ are equispaced points, $w_i$being independent observations uniformly distributed on$[0, \frac{\pi}{4}]$.

Fig. 1 gives an idea on their shape.

1. We take a link function r defined from $C^1(\mathbb{R})$ into $\mathbb{R}$by

$$\boldsymbol{r(f) = \int_{\frac{1}{4}}^{\frac{3}{4}} (f'(t))^2 dt} \quad (3.2)$$

2. Generate independently $\varepsilon_1, \ldots, \varepsilon_{100}$from a centered Gaussian of variance equal to 0.05 times.
3. Simulate the corresponding responses

$$\boldsymbol{Y_i = r(X_i) + \varepsilon_i, i = 1, \ldots, 100} \quad (3.3)$$

4. Simulate the corresponding estimators for the link function $r$and their corresponding conditional cumulative distribution function F_Y^X (c.d.f).

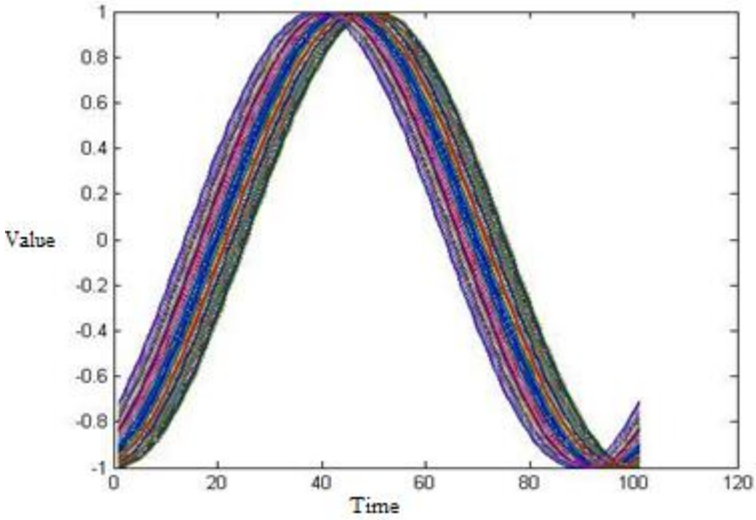


Figure 1: 100 curves of functional datasets

5. In addition, we used the mean squared error MSE values to assess the quality of any curve estimate for the regression.

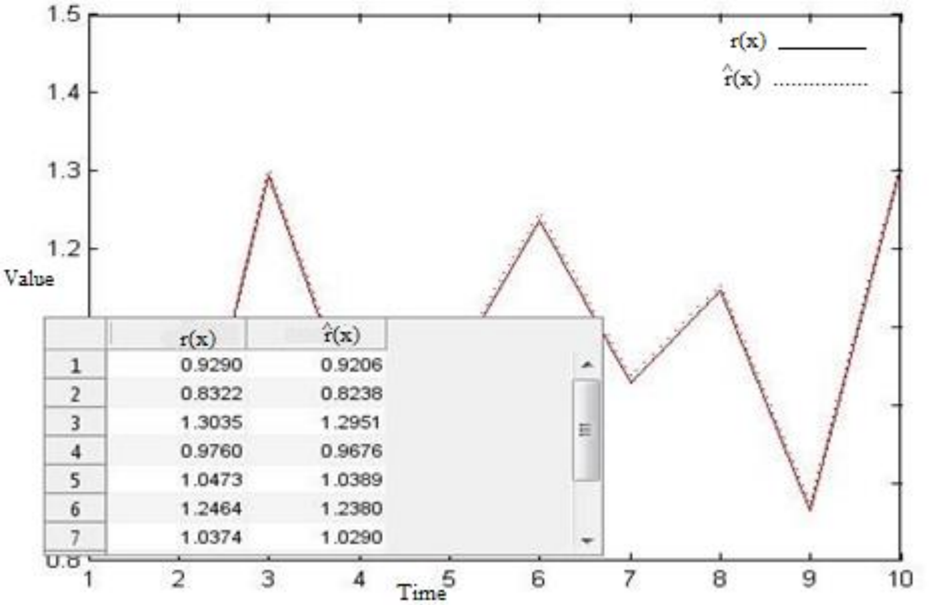
$$\boldsymbol{MSE = \frac{1}{100} \sum_{i=1}^{n} (\hat{r}(x_i) - r(x_i))^2} \quad (3.4)$$

In the second example we compare the corresponding estimator of conditional cumulative distribution to the true function while keeping the same data and regression model studied in the first example as follows

1. The true cumulative distribution is defined as a normal cumulative distribution function centered around 0.5 with a standard deviation of 0.1.
2. Simulate the corresponding estimator of the conditional cumulative distribution function c.d.f.
3. Simulate the mean and the standard deviation of the estimated c.d.f.

4. Results and Discussion

Westrivetovalidatetheconceptofalmostcompleteconvergenceinfunctionalregressionandsimulatethecorresponding conditional cumulative distribution function.Extensive research has been conducted in this realm, referencingthe works of [3]. for simulating nonparametric functional regression estimators and [2]. for semi parametric ones.Additionally, we leverage Ait'sresearch [10]. for simulating semi parametric functional regression estimators and drawing uponAit'stheoreticalstudiestosimulatethecumulativedistributionfunction.Ourcomprehensiveapproachaims to affirm the complete convergence of the conditional distribution function for functional random variables.By integrating empirical simulation techniques from [3]. and [2]. along with the theoretical framework provided by [10]. we seek a deeper understanding of the convergence properties and behavior of the cumulative distribution function. The challenge encountered during the simulation pertains to the absence of the theoretical conditional cumulative distribution function, crucial for comparing the simulated outcomes.



	$r(x)$	$\hat{r}(x)$
1	0.9290	0.9206
2	0.8322	0.8238
3	1.3035	1.2951
4	0.9760	0.9676
5	1.0473	1.0389
6	1.2464	1.2380
7	1.0374	1.0290

Figure 2: Regression and its Estimator for 10 curves

In addition to the difficulty associated with selecting the two smoothing parameters and the two kernels integral to the estimator, this theoretical gap represents a significant limitation.To address this constraint, we adopted a scanning approach to estimate the c.d.f.The simulation results are summarized in Figure 2 and 3 for example 1, where we examined the shape of the regression function under the considered regression model, as illustrated in Table 1. We found that this model is adaptable, as shown by the adjustments presented in Figure 4 for example 2. In this second part of the analysis, we conducted a comparison between the true cumulative function and its estimators using the kernel method. Thereby, we examined the average of the estimated cumulative distribution functions, emphasizing the need for further research to develop more robust and comprehensive techniques in this domain.

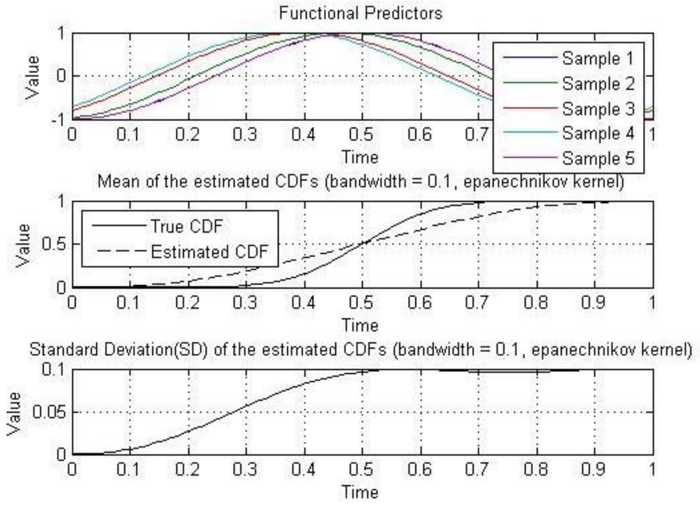


Figure 3: Mean and Standard Deviation of the Estimated CDFs.

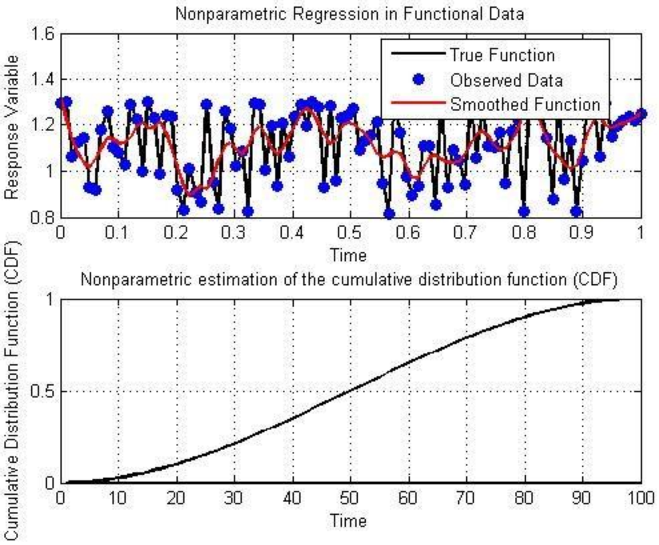


Figure 4: The Graph of the Estimated CDFs and Regression function.

5. Conclusion

In conclusion, our study focused on estimating the cumulative distribution function of functional data using the kernel method. For the selection of the window width, we chose scanning. Our approach consisted of validating our results by verifying the theoretical notion of almost complete convergence of the c.d.f estimator using the kernel method. Furthermore, we reinforced these asymptotic results during simulation execution, serving as a predictive tool and affirming the robustness and applicability of our approach. Furthermore, our results highlight the importance of judiciously selecting smoothing parameters in the kernel method. Although we used a sweep parameter selection approach, a method commonly used in the literature, it should be noted that alternative approaches, such as the Leave-One-Out method, could also be considered. The use of more sophisticated parameter selection techniques could improve the precision of our estimates and represents an interesting avenue for future research in this domain. Our results contribute to the growing body of research on functional data analysis and provide insight into best practices for estimating conditional cumulative distribution functions in this area, particularly in the context of predicting functional variables.

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