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Some New Optical Waves Interactions Via Modified Jacobi Elliptic Functions Method

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Abstract

This article explains how long and short waves influence each other within a highly complex nonlinear structure when there is a detuning factor involved. The explanation is based on the nonlinear complex Benney system, and we first consider this system before systematically deriving its solutions using Jacobian elliptic functions. We illustrate that one specific ellipticity modulus is on the verge of occurring. The findings from this study can contribute to the understanding of previous research on theinteraction between long and short waves in highly complex nonlinear system. Additionally, we utilize Jacobi's elliptic functions to define specific solutions, especially when the ellipticity modulus approaches either unity or zero. These solutions correspond to particular periodic wave solitons, which have been previously discussed in the literature.

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Introduction

Optical domain have a major role in the industry and daily life, scientists and researchers are treated as one of themost notable studies in this era.Many techniques for partial differential equations PDEs have been developed by mathematicians in their efforts to explain many physical interpretations of the optical olitons waves.

Solitons are the primary structure of wave transmission technology, data transmission, transoceanic ..etc.Thiskind of waves comprises of a special type of optical field, which does not alter through multiplication. Nonlinear long-short waves interactions system NLSWIS processes do model nonlinear dynamical interaction betweenlow-frequency long waves,and high frequency short waves [3].Highly motivating is uncovering basic physicalinteractions leading to further study and investigation of various nonlinear interactions the general solution structure including analytical [7] and approximate solutions [14].We motion some important and recent works thattreated this model by exponential expansion methods [2], [4], [13].The exact travelling wave solutions of thenonlinear partial differential equations NLPDEs were obtained on the hypothesis that the exact solution can beexpressed as finite expansion of a function which is the solution of a simpler equation. We use this methodology to transform NLPDEs to nonlinear ordinary differential equation NLODE. There are range of methods which employ this methodology with small variants which are: $tanh$method[1], $sine-cosine$method[7], Jacobielliptic function Expansion method [12], auxiliary equation method [11], sub-equation method [15]...etc, but generally speaking, all of the above methods have their own advantages and short comings, respectively.

Various methods have been proposed to construct analytical solutions, among which a straightforward but veryeffective method has drawn a lot of interest. This is the so-called Jacobian elliptic functions

JEFs method [5] and ithas been used in a rather wide range of equations [9]. We propose a modified JEFs method as an auxiliary method that makes it possible to obtain new and original explicit travelling waves solutions of NLSWIS[3].

The aim of this paper, in the first, is to perform a second order nonlinear ordinary differential equation NODE with sixth-degree nonlinear term which is, in nature, an extension of a type of elliptic equation, into a modifiedauxiliary equation to seek exact solution to a NLSWIS 1D [3]. Secondly, the objective is to construct solutions to the NLSWIS interms of Jacobi's elliptic functions. In the limiting case of the modulus of ellipticity, optical solitons and other solutions will emerge.The remainder of the paper is devoted to the application of a modified auxiliaryequation method to reveal solitons and other solutions that are in terms of JEFs with also yield solitons-solutions with the appropriate limiting value of the modulus of ellipticity.

The rest of this research paper is organized as follows: In section 2, we give the headlines of the modified auxiliary method (modified JEFs method) to get the solitary solutions of NLSWIS. In section3, we discuss the obtained results and their novelty with the previous methods. In section 4, we represent the conclusion of this study.

Mathematicalresults

Here, we use the modified auxiliary equation method, which is the most general analytical method.This model NLSWIS[3]isspecifiedby:

$$\begin{cases} i\Phi_t + \Phi_{xx} - \Phi\Theta = 0, \\ \Theta_t + \Theta_x + (|\Phi|^2)_x = 0, \end{cases} \qquad (1)$$

Where$\Phi(x,t)$and$\Theta(x,t)$reflect the slowly changin g envelope of the short transverse and the long longitudinal waves, respectively.x is the positional harmonize and t is the time.

We propose an appropriate travelling wave transformation as follows,

$$\begin{cases} \Phi(x,t) = exp^{i\eta} u(\xi), \\ \Theta(x,t) = v(\xi), \end{cases} \qquad (2)$$

were

$$\begin{cases} \eta = \rho x + ct, \\ \xi = ax + bt. \end{cases} \qquad (3)$$

Whereb, c are the frequencies of the travelling waves and $\rho, a$are the numbers of the waves. We reduce NLPDEs(1) into one dimensional ordinary differential equation ODE; if we take the necessary of (2)for(1),we get the following expressions:

$$\begin{cases} \Phi_x = exp^{i\eta}\rho u + aexp^{i\eta}u', \Phi_t = iexp^{i\eta}cu + bexp^{i\eta}u',\ \Phi_{xx} = -\rho^2 exp^{i\eta}u + 2ipaexp^{i\eta}u' + a^2 exp^{i\eta}u'', \\ \Theta_t = bv',\ \Theta_x = av', (|\Phi|^2)_x = a(u^2)', \dots \end{cases} \qquad (4)$$

Byusing(2),(3)and(4)in(1),wegetthefollowingsystem:

$$\begin{cases} (b + 2a\rho)iu' - (\rho^2 + c)u + a^2u'' - uv = 0, \\ \xi = ax + bt(a + b)v' + a(u^2)' = 0. \end{cases} \qquad (5)$$

When the complex part equals zero,we get:

$$b = -2a\rho. \qquad (6)$$

The second equation of the system(5) after integrationbecomes:

$v = \frac{-1}{1-2\rho} u^2,$ (7)

Where the integration constant is zero for simplicity.
Substituting(6)and(7)inthefirstequationofsystem(5)yields:

$a^2 u'' - (\rho^2 + c)u + \frac{1}{1-2\rho} u^3 = 0.$ (8)

Thebalanceruledetailedin[10]gives$N = 1$.
Solitonsemergesfromthelimitingprocessarepresentinginthenextsection.

Optical solutionsofNLSWIS

ApplyingthemodifiedauxiliaryequationmethodtotheNLSWIS[3]andusingthebalanceruleof[10](when $N = 1$),wegettowritethesolutionof(8)asfollow:

$u(\xi) = \sum_{i=0}^{N=1} a_i F^i(\xi) = a_0 + a_1 F(\xi).$ (9)

wherea_0, a_1 arearbitraryconstantssuchthat$a_1 \neq$ 0and$F(\xi)$isaJacobianellipticfunction[9]when$F(\xi)$satisfyingthefollowing:

$(F'(\xi))^2 = A_2 F^2(\xi) + A_4 F^4(\xi) + A_6 F^6(\xi),$(10)

whereA_2, A_4andA_6arearbitraryconstantsdeterminedbyJacobiellipticfunctionsJEFsmethod[9].Substituting(9)andthederivativeof(10)in(8)andcollectingalltermswiththesamepowerandsettingthemtozero,wehavethefollowingofalgebraicequations:

$$\begin{cases} F^5: 3a^2 a_1 A_6 = 0, \\ F^3: 2a^2 A_4 + \frac{a_1^2}{1-2\rho} = 0, \\ F^2: \frac{3a_0 a_1^2}{1-2\rho} = 0, \\ F^1: a^2 A_2 - (\rho^2 + c) + \frac{3a_0^2}{1-2\rho} = 0, \\ F^0: -a_0(\rho^2 + c) + \frac{3a_0^3}{1-2\rho} = 0. \end{cases} \quad (11)$$

Solving algebraic system (11) by using any computer software (Matlab, Maple, Wolfram, Mathematica,...) yieldstwo types of solutions as follow:
Optical solutions of NLSWIS:

Family I :

$a_0 = 0, a_1^2 = 2a^2 A_4(2\rho - 1), c = a^2 A_2 - \rho^2$ and $A_6 = 0$ with $A_4(2\rho - 1) > 0$.

Case 1 : When $A_2 = -(1 + k_1^2)$ and $A_4 = k_1^2 > 0 \Rightarrow 2\rho - 1 > 0 \Rightarrow \rho > \frac{1}{2} (\rho \in \mathbb{N}^*)$, we can obtain the following new complex Jacobi sine function solution for system (1):

$\Phi_1(x, t, k_1) = \pm\sqrt{2a^2 k_1^2 (2\rho - 1)} e^{i(\rho x - a^2(1+k_1^2)t - \rho^2 t)} sn(ax - 2a\rho t, k_1).$(12)

$\theta_1(x, t, k_1) = \frac{-1}{1-2\rho} 2a^2 k_1^2 (2\rho - 1) sn^2(ax - 2a\rho t, k_1).$(13)

Case 2 : When $A_2 = 2k_1^2 - 1$ and $A_4 = -k_1^2 < 0 \Rightarrow 2\rho - 1 < 0 \Rightarrow \rho < \frac{1}{2} (\rho = 0)$, we can obtain the following new complex Jacobi cosine function solution for system (1):

$\Phi_2(x, t, k_1) = \pm\sqrt{2a^2 k_1^2} e^{ia^2(2k_1^2-1)t} cn(ax, k_1).$(14)

$\theta_2(x,k_1) = -2a^2k_1^2cn^2(ax,k_1)$.(15)

Case 3: When $A_2 = 2 - k_1^2$ and $A_4 = -1 < 0 \Rightarrow 2\rho - 1 < 0 \Rightarrow \rho < \frac{1}{2}(\rho = 0)$, we can obtain the following new complex Jacobi function solution of the third kind for system (1):

$\Phi_3(x,t,k_1) = \pm\sqrt{2a^2}e^{ia^2(2-k_1^2)t}dn(ax,k_1)$.(16)

$\theta_3(x,k_1) = -2a^2dn^2(ax,k_1)$.(17)

Family II
${a_0}^2 = c = \frac{-a^2A_2}{2}, a_1^2 = -2a^2A_4, \rho = 0$ and $A_6 = 0$.

When $A_2 = 2k_1^2 - 1 < 0$ and $A_4 = -k_1^2 < 0$, we can obtain the following new complex Jacobi cosine function solution for system (1):

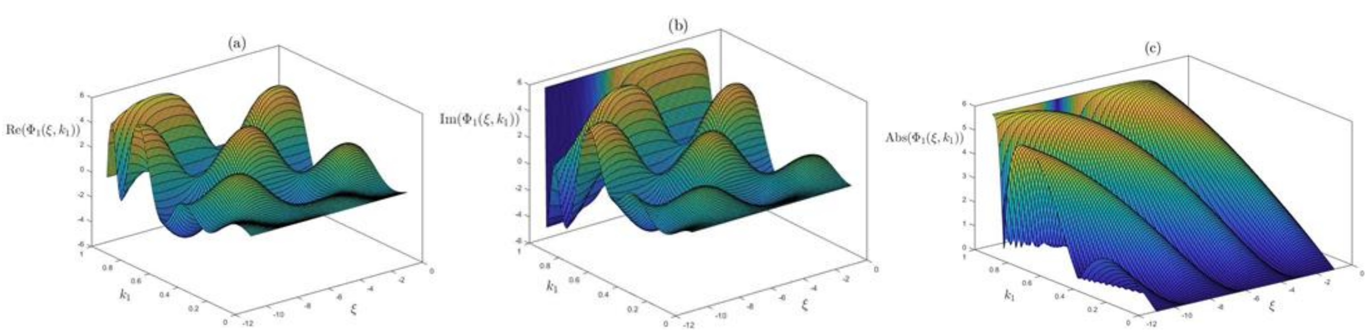
$\Phi_4(x,t,k_1) = e^{i\frac{a^2(1-2k_1^2)t}{2}}\left[\sqrt{\frac{a^2(1-2k_1^2)}{2}} + \sqrt{2a^2k_1^2}cn(ax,k_1)\right]$.(18)

$\theta_4(x,k_1) = -\frac{a^2(1-2k_1^2)}{2} - 2a^2k_1^2cn^2(ax,k_1) - \sqrt{a^4(1-2k_1^2)k_1^2}cn(ax,k_1)$.(19)

Particular cases:
When $k_1 \to 0$, the JEFs (12)-(19) degenerate to the triangular functions, that is,
$sn\xi \to sin\xi, cn\xi \to cos\xi, dn\xi \to 1$.(20)
When $k_1 \to 1$, the JEFs (12)-(19) degenerate to the hyperbolic functions, that is,
$sn\xi \to tanh\xi, cn\xi \to sech\xi, dn\xi \to sech\xi$.(21)
The particular solutions in (20)-(21) detailed in [13].

Concluding results

It would be very nice if we had true figures which illustrate graphically some of the obtained new solutions of (1),corresponding to a Family1 (Fig.1), Family2 (Fig.2) and Family3 (Fig.3) in a special cases of the constants when fixing the variablesa, bandρ.



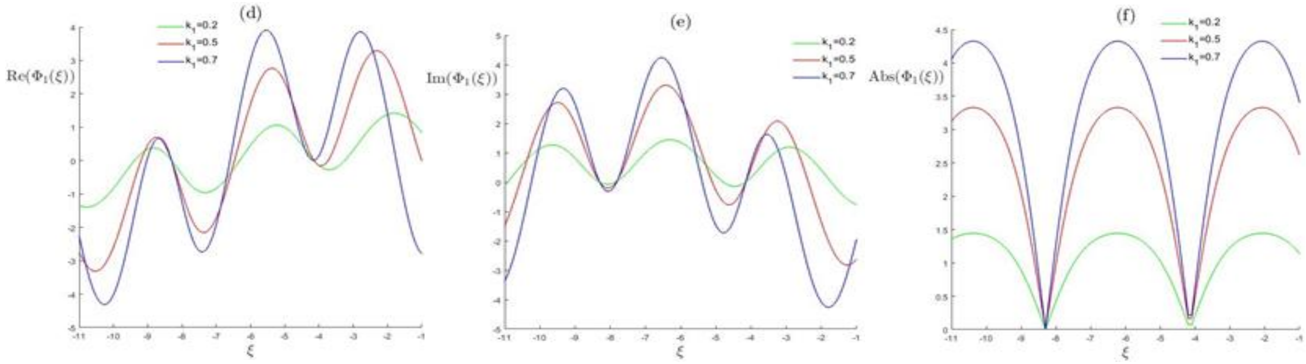
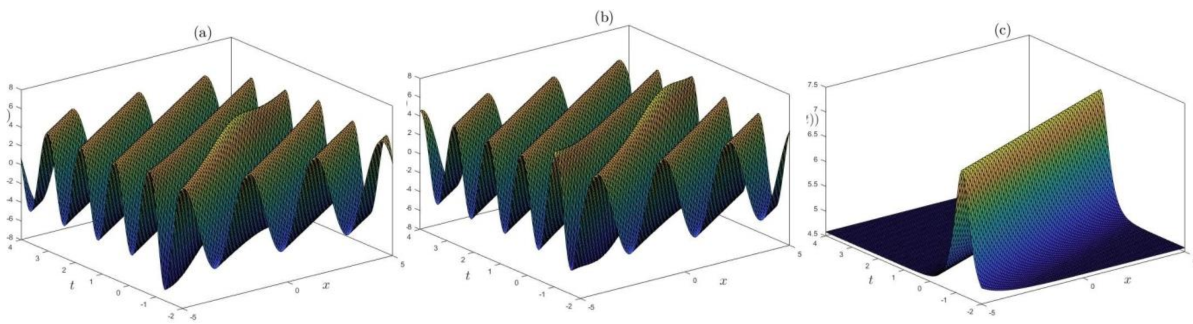


Figure1:[(a)and(d)]Real,[(b)and(e)]imaginaryand[(c)and(f)]absoluteplotsin3Dand 2D sketches of equation(12),respectively,when$a = 1, b = -4$and$\rho = 2$.

In this paper, important properties of modified JEFs method have been used to give physical meaning of some complex and trigonometric, hyperbolic, Jacobi elliptic functions solutions obtained for system (1).Modified JEFs method has some advantages over the classical methods (like $tanh$ method, $\sin - cos$method, simplest equationmethod [6] and expansion method) that modified JEFs method is more general because we can find other different style analytical solutions which cannot be obtained by using only expansion method([2],[4]and[13]). Therefore, this procedure of (9)-(10) will contribute to obtain more analytical solutions and for better understanding of engineering and physical problemsalong with new physical predictions.

In section 2, we demonstrate that JEFs solutions of (1) have only one modulus$k_1(0 \leq k_1 \leq 1)$. To the best of ourcurrent state of knowledge, we think that complex solutions (12)-(19) of (1) may have been obtained here for the firsttime,intheliterature.

Then, as its name suggest, hyperbolic functions are circular functions [see wikipedia].Moreover, they arise in many problems of mathematics and mathematical physics. For instance, the hyperbolic tangenta rises in the calculation of and rapidity of special relativity. The hyperbolic secant a rises in the profile of alaminarjet. It is estimated that all these analytical solutions obtained in this paper are related to such physical realizations. More general andcomprehensive information about longitudinal waves such as in sound, pressure waves and numerical instruments waves, has been treated in[8].



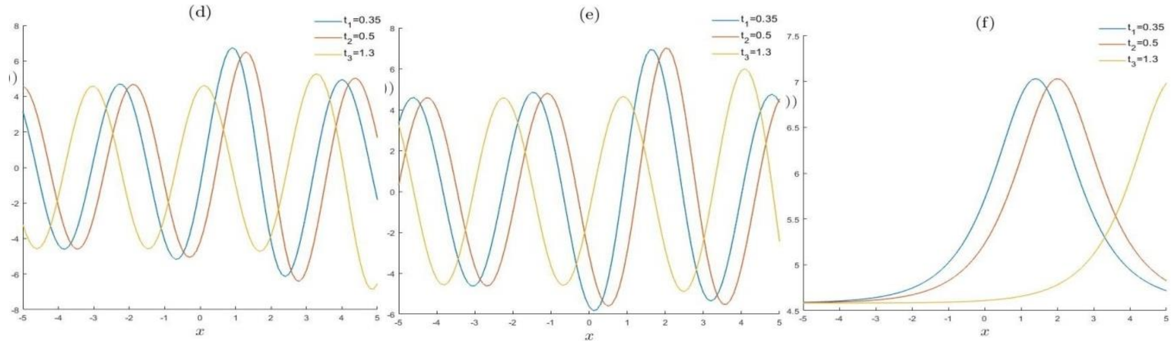


Figure2:[(a)and(d)]Real,[(b)and(e)]imaginaryand[(c)and(f)]absoluteplotsin3Dand 2D sketches of equation(14),respectively,when$a$=1,b=−4andρ=2.

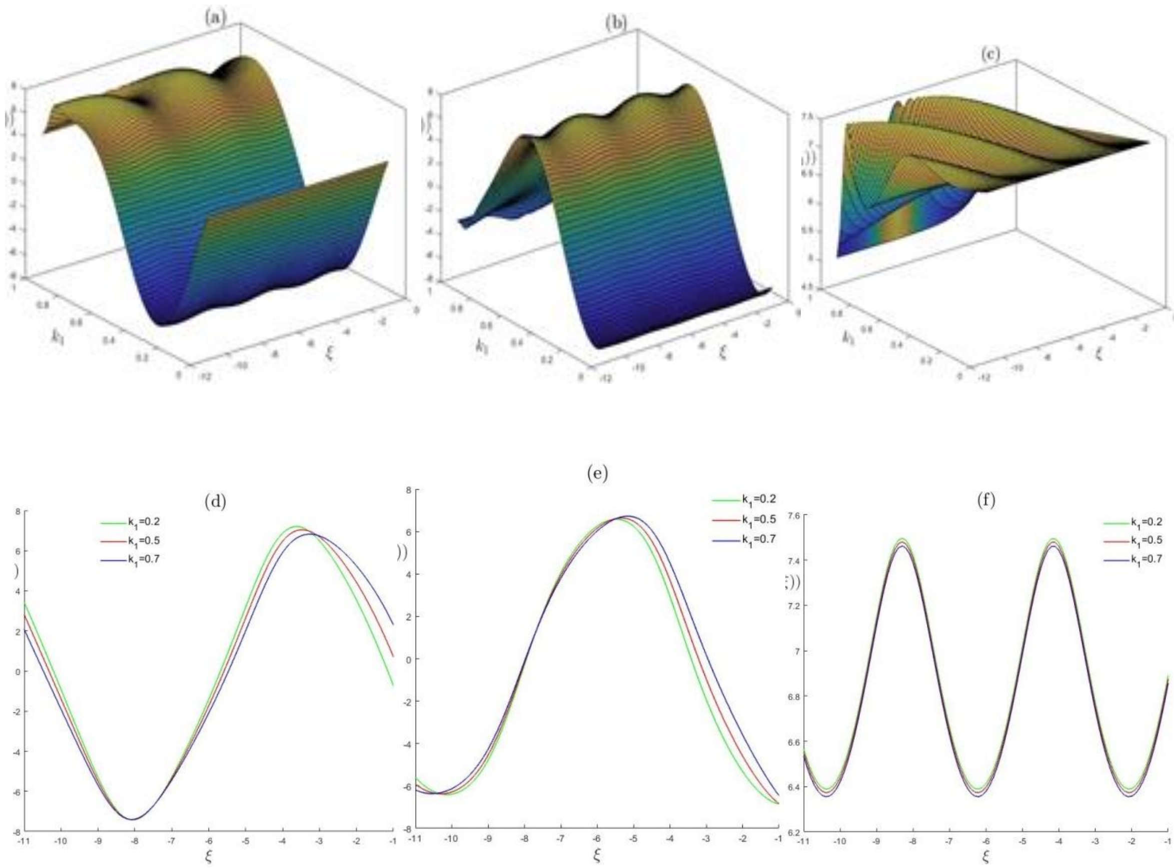


Figure3:[(a)and(d)]Real,[(b)and(e)]imaginaryand[(c)and(f)]absoluteplotsin3Dand 2D sketches of equation(16),respectively,when$a$=1,b=−4andρ=2.

Finally, when it comes to figures, surfaces have been plotted by considering the suitable values for the parameters. When we check all analytical solutions obtained in this paper by modified JEFs method, we observe that both two dimensional surfaces and three-dimensional surfaces in figures 1, 2 and 3 are gotten. Therefore, it can be said thatthey are physically reasonable, because almost all figures show us similar wave behaviours of the suitable values ofparameters.

The modified JEFs method has been applied to the NLSWIS (1).This newly improved scheme has given some newcomplex trigonometric, hyperbolic and Jacobian elliptic functions solutions such as equations (12)-(19) for NLSWIS(1). To the best of our knowledge, the application of modified JEFs method to system(1) has not been submittedto literature, beforehand. This modified JEFs method led to a new set of results (solutions have one modulus)which are being reported for the first time in this work. We observe that modified method has been a powerful tool for obtaining new analytical solutions to NLSWIS. We think that this newly modified method can also be applied to similar and more complex partial differential equations and systems with strong non-linearities.

References

[1] M.J. Ablowitz, P.A. Clarkson, Solitons Nonlinear Evolution Equations and Inverse Scattering,Cambridge Uni-versityPress,Cambridge,UK,1991.
[2] H.M.Baskonus, H.Bulut and F.BM.Belgacem, *Analytical solutions for nonlinear long-short wave interactionsystemswithhighlycomplexstructure*,J.Comp.Appl.Math312(2017)257-266.
[3] D.J.Benny,*Ageneraltheoryforinteractionsbetweenshortandlongwaves.*,J.Stud.Appl.Math,56,81-94,1977.
[4] Ahmet Bekir, Esin Aksoy and OzkanGuner, *Optical soliton solutions of the long-short wave interaction system.*,JournalofNonlinearOpticalPhysicsandMaterials,Vol.22,No.2(2013)1350015(11pages).
[5] C.DaiandJ.Zhang,Jacobianellipticfunctionmethodfornonlineardifferential-differenceequations.Chaos,solitonsandfractals.Vol.27,no.4,pp.1042-1047,2006.
[6] N.A.Kudryashov,Simplestequationmethodtolookforexactsolutionsofnonlineardifferentialequations.RussianFederation(2004).
[7] W.X.Ma,T.Huang,Y.Zhang,Y.Chen,AmultipleExp-functionmethodfornonlineardifferentialequationsanditsapplications,Phys.Scr.82(2010)065003.
[8] DavidMorin,Longitudinalwaves,Chapter5,2011.
[9] ZeidI.A.Al-Muhiameed and EmadA.-B.Abdel-Salam, Generalized Jacobielliptic function solution to a class of Nonlinear Schrdinger-type equations. Mathematical Problems in Engineering.Vol2011,ArticleID575679,11pages.
[10] Pinar. Z., Ozis. T., Observations on the class of ”Balancing Principle” for nonlinear PDEs that can treated bytheauxiliaryequationmethod.NonlinearAnal.RealWorldAppl.(2015)23,9-16.
[11] Sirendaoreji, Sun Jiong, Auxiliary equation method for solving nonlinear partial differential equations, (2003)aid:12234.
[12] X. Yong, X. Zeng, Z. Zhang, Y. Chen, Symbolic computation of Jacobi function solutions to nonlineardifferential-differenceequations,Comput.Math.Appl.57(2009)1107-1114.
[13] C.Yue, A.Elmoasry et al.,*On complex wave structures related to the nonlinear long-short wave interactionsystem:Analyticalandnumericaltechniques.*,AIPAdvances10,045212(2020).
[14] Haiyong Qin, Mostafa M. A. Khater, Raghda A. M. Attia and Dianchen Lu, *Approximate simulations for thenon-linearlong-Shortwaveinteractionsystem*,FrontiersinPhysics(2019).
[15] H. Zhang, A note on some sub-equation methods and new types of exact travelling wave solutions for twononlinearpartialdifferentialequations,ActaAppl.Math.106(2009)241-249.