A Novel w-Transform of Bicomplex numbers and their properties

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ABSTRACT

In this paper, I have derived a new w-Transform similar to z-transform of Bicomplex number. Also, I have derived some useful properties in W-Transform of Bicomplex numbers. I have solved w-transform of some standard sequences. I hope these results highly useful in the field of Electrical and electronics engineering fields.

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1. Introduction

Corrado Segre (year1848) [1], [2] introduced the tessarines in a series of articles in the philosophical magazine, H. Hankel developed a general system of "complex numbers with several real generators" in 1967.

atessarine is a hypercomplex number of the form

$$w = a + ib + jc + kd \ , a, b, c, d \in \mathbb{R}$$

Where $i^2 = j^2 = -1 \ , ij = ji = k \ and \ k^2 = 1$

1.1 Basic definitions

We know [2],[3],[4],[5],[6],[7],[8] the set of complex numbers $\mathbb{C}$ defined as follows

$\mathbb{C} = \{a + ib \ : a, b \in \mathbb{R}\}$ where $i$is an imaginary number, by using the extension of one imaginary to three imaginary numbers $i \, , \, j \ and \ k$ such that $i^2 = j^2 = -1 \ , ij = ji = k \ and \ k^2 = 1$

Definition 1 : (Bicomplex number)

Bicomplexnumers defined as $\mathbb{T} = \{z_1 + jz_2 \ : z_1, z_2 \in \mathbb{C}\}$

Definition 2 : (Bicomplex conjugates) There are three possible conjugates. Let $w \in \mathbb{T}$ and $z_1, z_2 \in \mathbb{C}$ such that $w = z_1 + jz_2$

$$w^* = \overline{z_1} + j\,\overline{z_2}$$
$$w^\# = z_1 - jz_2$$
$$w' = \overline{z_1} - j\,\overline{z_2}$$

Here $\overline{z_1}$ and $\overline{z_2}$ are standard complex conjugates of complex numbers $\mathbb{C}$

Definition 3 : (Bicomplexmodulii) There are four modulii possible. Let $w \in \mathbb{T}$ and $z_1, z_2 \in \mathbb{C}$ such that $w = z_1 + jz_2$

$|w|_i^{\,2} = w\,w^*$

$|w|_j^{\,2} = w\,w^\#$

$$|w|_k^{\,2} = w\,w'$$

Euclidean $\mathbb{R}^4 -$ norm defined as

$$\|w\| = \sqrt{|z_1|^2 + |z_2|^2}$$

Definition 4 : (Bicomplex multiplicative inverse)

$w \in T$ and $z_{1\,,}z_2 \in \mathbb{C}$ such that $w = z_1 + jz_2$ is invertible if

$$w^{-1} = \frac{z_1}{|z_1|^2 + |z_2|^2} + j\frac{z_2}{|z_1|^2 + |z_2|^2}$$

Definition 5: (Idempotent basis) A Bicomplexnumber$w = z_1 + jz_2$ has the following unique Idempotent representation

$w = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2$ where $z_{1\,,}z_2 \in \mathbb{C}$

$e_1 = \frac{1+k}{2} e_1 = \frac{1-k}{2}$, $ij = k$, $k^2 = 1$

Or $w = \big((a+d) + (c-d)\big)e_1 + \big((a-d) + (c+d)\big)e_1$

Definition 6: (Projection) Two projection operators P_1 and P_2 has the following unique Idempotent representation defined from $\mathbb{T} \to \mathbb{C}$

as $P_1(z_1 + jz_2) = z_1 - iz_2$

$$P_2(z_1 + jz_2) = z_1 + iz_2$$

Then$w = z_1 + jz_2 = P_1(w)e_1 + P_2(w)e_2$

$$= P_1(z_1 + jz_2)e_1 + P_2(z_1 + jz_2)e_2$$

Definition 7: (Trigonometric representation of Bicomplex numbers)

If $w = z_1 + jz_2 = \sqrt{{z_1}^2 + {z_1}^2}\left(\frac{z_1}{\sqrt{{z_1}^2+{z_2}^2}} + \frac{z_2}{\sqrt{{z_1}^2+{z}^{\ 2}}}\right)$

$$= r_c(cos\,\Theta_c + \mathrm{j}\sin\Theta_c)$$

$$= r_c e^{j\Theta_c}$$

Where $r_c = \sqrt{{z_1}^2 + {z_1}^2} = P_1(w)P_2(w)$,

$cos\,\Theta_c = \frac{z_1}{\sqrt{{z_1}^2+{z_2}^2}}$andsin $\Theta_c = \frac{z_2}{\sqrt{{z_1}^2+{z_2}^2}}$________________(1)

Also from (1)$\frac{z_1}{z_2} = \frac{cos\,\Theta_c}{\sin\Theta_c} \Rightarrow \frac{z_1 - iz_2}{z_1 + iz_2} = \frac{(cos\,\Theta_c - \mathrm{i}\sin\Theta_c)}{(cos\,\Theta_c + \mathrm{i}\sin\Theta_c)}$

$\Rightarrow \frac{P_1(w)}{P_2(w)} = \frac{(cos\,\Theta_c - \mathrm{i}\sin\Theta_c)}{(cos\,\Theta_c + \mathrm{i}\sin\Theta_c)}$

$$\Rightarrow \frac{P_1(w)}{P_2(w)} = \frac{e^{-i\Theta_c}}{e^{i\Theta_c}}$$

$$\Rightarrow \frac{P_1(w)}{P_2(w)} = e^{-i2\Theta_c}$$

Take ln on both sides

$$\Rightarrow \Theta_c = i\ln\sqrt{\frac{P_1(w)}{P_2(w)}}$$

Z-Transforms:

Definition :[9] , [10] Given a sequence x_n then Z-transform defined as

$X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$, z is a complex number
This transformation produces a new representation of x_n denoted by $X(z)$.
From $X(z)$ we can get original sequence x_n (inverse z-Transform) by finding nth power of the coefficient of z^{-1}.

2. w-Transform:

Definition : Given a sequence x_n then w-Transform defined as
$X(w) = \sum_{n=0}^{\infty} x_n w^{-n}$, $w = z_1 + jz_2$ is a Bicomplex number
This transformation produces a new representation of x_n denoted by $X(w)$.
From $X(w)$ we can get original sequence x_n (inverse w-Transform) by finding nth power of the coefficient of w^{-1}.

$$X(w) = \sum_{n=0}^{\infty} x_n w^{-n}$$

or

$$\Rightarrow X(w) = \sum_{n=0}^{\infty} x_n (u^{-n} e_1 + v^{-n} e_2)$$

$u = z_1 - iz_2 \, v = z_1 + iz_2$, $z_1, z_2 \in \mathbb{C}$

$$\Rightarrow X(w) = e_1 \sum_{n=0}^{\infty} x_n u^{-n} + e_2 \sum_{n=0}^{\infty} x_n v^{-n}$$

Example 1 : $x_n = \alpha^n$
Solution: $X(w) = \sum_{n=0}^{\infty} x_n w^{-n}$

$$\Rightarrow X(w) = \sum_{n=0}^{\infty} \alpha^n w^{-n}$$

$$\Rightarrow X(w) = \frac{w}{w - \alpha}$$

$$\Rightarrow X(w) = \frac{1}{1 - \alpha w^{-1}}$$

or

$\Rightarrow X(w) = \sum_{n=0}^{\infty} \alpha^n (u^{-n} e_1 + v^{-n} e_2)$, $u, v \in \mathbb{C}$

$$\Rightarrow X(w) = e_1 \sum_{n=0}^{\infty} \alpha^n u^{-n} + e_2 \sum_{n=0}^{\infty} \alpha^n v^{-n}$$

$$\Rightarrow X(w) = e_1 \sum_{n=0}^{\infty} \alpha^n u^{-n} + e_2 \sum_{n=0}^{\infty} \alpha^n v^{-n}$$

$$\Rightarrow\ X(w) = e_1 \frac{u}{u-\alpha} + e_2 \frac{v}{v-\alpha}$$

$$\Rightarrow\ X(w) = e_1 \frac{1}{1-\alpha u^{-1}} + e_2 \frac{1}{1-\alpha v^{-1}}$$

$$\Rightarrow\ X(w) = e_1 u_1 + e_2 v_1 = w_1$$

Where$u_1 = \frac{1}{1-\alpha u^{-1}}$, $u = z_1 - iz_2$

$v_1 = \frac{1}{1-\alpha v^{-1}}$, $v = z_1 + iz_2$ and $w = z_1 + jz_2$

Note : If $\alpha = 1$ then $x_n = 1$

$$X(w) = \sum_{n=0}^{\infty} x_n\, w^{-n}$$

$$X(w) = \sum_{n=0}^{\infty} 1\, w^{-n}$$

$$\Rightarrow\ X(w) = \frac{1}{1-w^{-1}}$$

Or $X(w) = e_1 \frac{1}{1-u^{-1}} + e_2 \frac{1}{1-v^{-1}}$, $u, v \in \mathbb{C}$

$$\Rightarrow\ X(w) = e_1 u_2 + e_2 v_2 = w_2$$

Where $u_2 = \frac{1}{1-u^{-1}}$, $u = z_1 - iz_2$

$v_2 = \frac{1}{1-v^{-1}}$, $v = z_1 + iz_2$ and $w = z_1 + jz_2$

Example 2 : $x_n = n$

Solution: $X(w) = \sum_{n=0}^{\infty} x_n\, w^{-n}$

$$\Rightarrow\ X(w) = \sum_{n=0}^{\infty} n\, w^{-n}$$

$$\Rightarrow\ X(w) = \frac{w}{(w-1)^2}$$

$$\Rightarrow\ X(w) = \frac{1}{w(1-w^{-1})^2}$$

Or

$\Rightarrow\ X(w) = e_1 \frac{u}{(u-1)^2} + e_2 \frac{v}{(v-1)^2}$, $u, v \in \mathbb{C}$

Example 3.Dirac delta sequence: $x_n = \delta(n-n_0)$

Solution: $X(w) = \sum_{n=0}^{\infty} x_n\, w^{-n}$

$$\Rightarrow\ X(w) = \sum_{n=0}^{\infty} \delta(n-n_0)\, w^{-n}$$

$$\Rightarrow\ X(w) = w^{-n_0}$$

$$\Rightarrow\ X(w) = (w^{-1})^{n_0}$$

or

$$\Rightarrow\ X(w) = \left(\frac{w^*}{|w|^2{}_i}\right)^{n_0} = \left(\frac{w^\#}{|w|^2{}_j}\right)^{n_0} = \left(\frac{w^*}{|w|^2{}_k}\right)^{n_0}$$

Put $n_0 = 1$ we get the impulse response of the unit delay

$$X(w) = \sum_{n=0}^{\infty} \delta(n-1)\, w^{-n}$$

$$\Rightarrow\ X(w) = w^{-1}$$

or

$$\Longrightarrow X(w) = \frac{w^*}{|w|^2{}_i} = \frac{w^\#}{|w|^2{}_j} = \frac{w^*}{|w|^2{}_k}$$

Example 4. Exponential function : $x_n = e^{\alpha n}$, $\alpha \in \mathbb{R}$

Solution: $X(w) = \sum_{n=0}^{\infty} x_n\, w^{-n}$

$$\Rightarrow\ X(w) = \sum_{n=0}^{\infty} e^{\alpha n}\, w^{-n}$$

$$\Rightarrow\ X(w) = \sum_{n=0}^{\infty} (e^{\alpha} w^{-1})^n$$

$$\Rightarrow\ X(w) = (1 - e^{\alpha} w^{-1})^{-1}$$

$$\Rightarrow\ X(w) = \frac{w}{w - e^{\alpha}}$$

$$\Rightarrow\ X(w) = \frac{1}{1 - e^{\alpha} w^{-1}}$$

3.Properties of w-Transform:

1. Linearity property:

$$a_1 x_n + a_2 y_n \overset{w}{\longleftrightarrow} w a_1 X(w) + a_2 Y(w)$$

Proof: $X(w) = \sum_{n=0}^{\infty} (a_1 x_n + a_2 y_n)\, w^{-n}$

$$= \sum_{n=0}^{\infty} (a_1 x_n\, w^{-n} + a_2 y_n\, w^{-n})$$

$$= \sum_{n=0}^{\infty} a_1 x_n \, w^{-n} + \sum_{n=0}^{\infty} a_2 y_n \, w^{-n}$$

$$= a_1 \sum_{n=0}^{\infty} x_n \, w^{-n} + a_2 \sum_{n=0}^{\infty} y_n \, w^{-n}$$

$= a_1 X(w) + a_2 Y(w)$

2.The time delay property:

$$x_{n-n_0} \overset{w}{\longleftrightarrow} w^{-n_0} X(w)$$

and $x_{n-1} \overset{w}{\longleftrightarrow} w^{-1} X(w)$

proof: $X(w) = \sum_{n=0}^{\infty} x_{n-n_0} \, w^{-n}$

replace $n - n_0$ my m

$$X(w) = \sum_{m=0}^{\infty} x_m \; w^{-(m+n_0)}$$

$$X(w) = w^{-n_0} \sum_{m=0}^{\infty} x_m \; w^{-m}$$

$$X(w) = w^{-n_0} X(w)$$

Special case if $n_0 = 1$

$$X(w) = w^{-1} X(w)$$

Thus $x_{n-n_0} \overset{w}{\longleftrightarrow} w^{-n_0} X(w)$

and $x_{n-1} \overset{w}{\longleftrightarrow} w^{-1} X(w)$

3.convolution and w-Transform:

$$x_n * h_n \overset{w}{\longleftrightarrow} H(w) X(w)$$

Proof : Let $y_n = x_n * h_n$

Take w-transform on both sides

$$y_n = \sum_{l=0}^{\infty} h_l \; x_{n-l}$$

Take the w-transform on both sides using superposition and general delay property

$$Y(w) = \sum_{l=0}^{\infty} h_l w^{-l} X(w)$$

$$\Rightarrow Y(w) = \left(\sum_{l=0}^{\infty} h_l w^{-l}\right) X(w)$$

$$\Rightarrow Y(w) = H(w)X(w)$$

hence

$$x_n * h_n \overset{w}{\longleftrightarrow} H(w)X(w)$$

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