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The Reproducing Kernel Hilbert Space method for solving a fractional order differential equations problem

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Abstract

In this article, we present a new approach based on the analytical method known as the Reproducing Kernel Hilbert Space Method (RKHSM), which is used for approximating the solution of fractional order differential equations. The results obtained demonstrate that RKHSM exhibits rapid uniform convergence of the approximate solution, along with high precision and efficiency. In this method, the solution is expressed in series form. To illustrate the effectiveness of the method, we provide numerical results including the exact solution, the approximate solution, and the absolute error.

Keywords: Reproducing Kernel Hilbert Space (RKHS), Fractional differential equation, convergence, Numerical solution, approximate solution, absolute error.

1.Introduction

During the last two decades, an increasing attention has been directed towards the numerical simulation of linear and nonlinear differential equations of fractional order since they have central applications in a large number of fields such as fluid mechanics, viscoelasticity, biology, physics and engineering [1–5].

Fractional-order derivatives and integrals embed the description of the memory and hereditary properties of different substances. Accordingly, the field of fractional partial differential equations has attracted interest of researchers in several important phenomenons in chemistry, hydrology, fluid mechanic, physics, gas dynamics, and signal processing (see, for instance, [1–17] and the references therein). Usually, it is too complicated to solve exactly this class of equations for most cases because, generally, the solution cannot be exhibited in a closed form even when it exists. Therefore, the development of analytical and numerical methods for the solutions of fractional PIDEs is of current importance. So finding exact solutions in closed forms for most differential equations of fractional order is a difficult task. As a result, a number of methods have been proposed and applied successfully to approximate fractional differential equations, such as Adomian decomposition method [6, 7], variational iteration method [8, 9], generalized differential transform method [10, 11], nonstandard difference method [12], Adams–Bashforth–Moulton method [13, 14] and the homotopy analysis method [15–17]. Recently, the reproducing kernel space method has been used for obtaining approximate solutions of differential and integral equations.

The main purpose of the present paper, is to construct a computational reproducing kernel Hilbert space method (RKHSM) to solve nonlinear differential equations of fractional order. Historically, Reproducing kernels were used for the first time at the beginning of the twentieth century by Zaremba in his work on boundary value problems for harmonic and biharmonic functions [18,19]. The general theory of reproducing kernel Hilbert spaces was established simultaneously and independently by Aronszajn [20] and Bergman [21] in 1950. The introduction of the reproducing kernel Hilbert spaces W_2^m [a, b] in [22] and $W_2^{(m,n)}$ ([a, b]×[c, d]) in [23] led to an explosion in applications of reproducing kernel Hilbert space methods to many areas of mathematics: ordinary and partial differential equations [23,24,25,26,27], fractional differential equations [28,29], nonlinear oscillators with discontinuities, nonlinear two-point boundary value problems [30-32], difference equations [33], and integral equations [23,34].

The RKHSM is a numerical, as well as, analytical technique for solving a large variety of ordinary and partial differential equations associated to different kind of initial conditions, and usually provides the solutions in term of rapidly convergent series with components that can be elegantly computed. The main idea is to construct the direct sum of the RKHSs that satisfying the initial conditions of the given systems in order to determining their exact and their numerical solutions. The exact and the numerical solutions are represented in the form of series. The advantages of the utilized approach lie in the following main advantages; firstly, it can produce good globally smooth numerical solutions, and with ability to solve many differential systems with complex constraint conditions, which are difficult to solve; secondly, the numerical solutions and their derivatives are converge uniformly to the exact solutions and their derivatives, respectively; The organization of this paper is as follows. Section 2 introduces reproducing kernel Hilbert spaces and some of their properties which will be used in this work. Various definition lemmas, used in the proofs of the existence theorems, are presented in Sect. 3. Method of the Reproducing Kernel Hilbert space (RKHSM). Section 4. Solution of our problem in W_2^3 [0,1]. Section 5 Uniform convergence of the approximate solution. Section 6 illustration of the effectiveness of the method bythe presentation of numerical results (tables and figures). A conclusion for this article with some references is given in Sect. 7 and Sect 9.

2. Preliminary

In this section, we first give some basic definitions of the reproducing kernel and reproducing kernel Hilbert space, and subsequently, we introduce several reproducing kernel Hilbert spaces needed in this article.

Definition 2.1 The Riemann-Liouville integral operator of order $\alpha > 0$ with $a \geq 0$ is :

$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$ (1)

properties of the operator J α a can be found in **[1–5].**

Definition2.2 [35] The Caputo fractional derivative of $f(x)$ of order$\alpha > 0$ with $a \geq 0$ is :

$^{C}D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} \frac{d^m f(t)}{dt^m} dt, m-1 < \alpha < m, \ m \in \mathbb{N}.$ (2)

Definition 2.3 [36] The Caputo fractional derivative of $f(x)$ of order$\alpha > 0$ with $a \geq 0$ is :

$^{C}D_a^\alpha f(x) = J_a^{m-\alpha} f^{(m)}(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}} dt,$ (3)

for $m-1 < \alpha \leq m, \ m \in \mathbb{N}, \ x \geq a, \ f(x) \in C^m$.

Definition 2.4 [37] Let W be a real Hilbert space of functions on a set E. Denote by$\langle u, u \rangle_W$ the inner product and let $||u||_W = \sqrt{\langle u, u \rangle_W}$ be the norm in W, for $u, v \in W$. The real valued function $R : E \times E \to \mathbb{R}$ is called a reproducing kernel of Wif the followings are satisfied

1. For every x, $R_x(y) = R(x,y)$ as a function of y belongs to E.
2. The reproducing property:$u(x) = \langle u(.), R_x(.) \rangle_W$, for all $u \in W$, and all $x \in E$.

A Hilbert space W of functions on a set E is called a reproducing kernel Hilbert space if there exists a reproducing kernel R of W. If a Hilbert space W of functions on a set E possesses a reproducing kernel, then the reproducing kernel R is uniquely determined by the Hilbert space W.

Definition 2.5 [38]

$W_2^1[0,1]=$
$\left\{v:\ v\ are\ absolutely\ continuous\ functions\ on\ [0,1],\ \frac{dv}{dx}\in\mathbb{L}^2[0,1]\ ,\quad \forall x\in[0,1]\right\}$

The space is equipped with the following inner product and norm, respectively:

$\langle v,u\rangle_{W_2^1[0,1]}=v(0).u(0)+\int_0^1 v'(x).u'(x)\,dx,$ (4)

and $\|v\|_{W_2^1[0,1]}=\sqrt{\langle v,v\rangle_{W_2^1[0,1]}}$, $\forall v,u\in W_2^1[0,1].$

Definition 2.6 [39]

$W_2^3[0,1]=$
$\left\{v: v,\frac{dv}{dx},\frac{d^2v}{dx^2}: are\ absolutely\ continuous\ functions\ on\ [0.1],\ \frac{d^3v}{dx^3}\in\mathbb{L}^2[0,1],\ v(0)=\frac{dv(0)}{dx}=0\ ,\forall\ x\in[0,1]\right\}$

The space is equipped with the following inner product and norm, respectively:

$\langle v,u\rangle_{W_2^3[0,1]}=v(0).u(0)+v'(0).u'(0)+v^{(2)}(0).u^{(2)}(0)+\int_0^1 v^{(3)}(x).u^{(3)}(x)\,dx,$
(5)

and $\|v\|_{W_2^3[0,1]}=\sqrt{\langle v,v\rangle_{W_2^3[0,1]}}$, $\forall v,u\in W_2^3[0,1].$

Definition 2.7 Cauchy-Schwarz inequality:

$\forall(\mathrm{f},\mathrm{g})\in\mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega)$, we have:

$\int_\Omega|\mathrm{f(t)g(t)}|\,\mathrm{dt}\le(\int_\Omega(\mathrm{f(t)})^2\,\mathrm{dt})^{\frac{1}{2}}(\int_\Omega(\mathrm{g(t)})^2\,\mathrm{dt})^{\frac{1}{2}}$,

where

$$\|\mathrm{fg}\|_{\mathbb{L}^2(\Omega)}\le\|\mathrm{f}\|_{\mathrm{L}^2(\Omega)}\|\mathrm{g}\|_{\mathbb{L}^2(\Omega)}.$$

Definition 2.8 (Gram-Schmidt orthogonalization process)

Either $(a_1,\dots,a_n)$ a base of E. for i =1,...,n, we put:

$f_i=a_i-\sum_{j=1}^{i-1}\frac{\langle f_j,a_i\rangle}{\|f_j\|^2}f_j$ and $e_i=\frac{1}{\|f_i\|}f_i$.

Then the base $(f_1,\dots,f_n)$ is orthogonal and the base $(e_1,\dots,e_n)$ is orthonormal.

Definition 2.9 (Delta-Dirac function)

When *f(x)* is a well defined function at $x=x_0$,

$$\int_{-\infty}^{+\infty}\delta(x-x_0)f(x)dx=f(x_0)\int_{-\infty}^{+\infty}\delta(x-x_0)dx=f(x_0).$$

The Dirac-Delta function $\delta(x-x_0)$ has a sharp peak at $x=x_0$.

$\delta(x-x_0)=0$ if $x\neq x_0$ and $\delta(x-x_0)=+\infty$ if $x=x_0$ and $\int_{-\infty}^{+\infty}\delta(x-x_0)dx=1.$

3. RKHS method:

Consider the following problem:

$$\begin{cases} {}^{c}D^{\alpha}\mathrm{u(x)}=\mathrm{f}(\mathrm{x},\mathrm{u(x)}) \quad ,0\le \mathrm{x}\le 1 \quad , \quad 1\le\alpha\le 2 & (6)\\ \mathrm{u}(0)=\mathrm{c}_0, & (7)\\ \frac{\mathrm{du}(0)}{\mathrm{dx}}=\mathrm{c}_1, & (8)\end{cases}$$

where ${}_{c}D^{\alpha}$ is the fractional derivative in the sense of Caputo of order α, c_i, $i = 0,1$, are real constants and $f(x, u(x))$ is a linear or nonlinear function.
Since the conditions of problem (6)-(8) are non-homogeneous, we have introduced the following function:
$V(x) = -w(x) - c_0 + w(0)$, such that $w(x) = c_1\left(x - \frac{1}{2}\right)$.
And we define a new function: $v(x) = u(x) + V(x)$. Thus, the problem (6)-(8) can be reformulated as follows:

$$\begin{cases} {}_{c}D^{\alpha}v(x) = g(x, v(x)) \quad ,0 \leq x \leq 1 \quad , \quad 1 \leq \alpha \leq 2 & (9) \\ v(0) = 0, & (10) \\ \dfrac{dv(0)}{dx} = 0, & (11) \end{cases}$$

where: $g(x, v(x)) = f(x, u(x)) + {}_{c}D^{\alpha}V(x)$.
To apply the Reproducing Kernel Hilbert Space method, we must proceed through two essential steps: Constructing the RKHS and deriving the Reproducing Kernel function within the Hilbert space.
For problem (9)-(11), we require the RKHS: $W_2^3[0,1]$ and $W_2^1[0,1]$. (already defined in definitions 2.5 and 2.6).
Theorem 3.1: $S_z(x)$ is the RKHS function associated with the space $W_2^3[0,1]$, defined as follows:

$$S_z(x) = \begin{cases} \frac{x^2z^2}{4} + \frac{x^2z^3}{12} - \frac{xz^4}{24} + \frac{x^5}{120} & ,x \leq z, \\ \frac{z^2x^2}{4} + \frac{z^2x^3}{12} - \frac{zx^4}{24} + \frac{z^5}{120} & ,x > z, \end{cases} \quad (12)$$

Proof: According to definition 2.6, we have:

$$\langle v, S_z(x)\rangle_{W_2^3[0,1]} = v(0).S_z(0) + v'(0).S_z{}'(0) + v^{(2)}(0).S_z{}^{(2)}(0) + \int_0^1 v^{(3)}(x).S_z{}^{(3)}(x)\,dx \quad (13)$$

We use the integration by parts of (13) three times, and we obtain:

$$\begin{aligned} \langle v, S_z(x)\rangle_{W_2^3[0,1]} &= v(0).S_z(0) + v'(0).S_z{}'(0) + v^{(2)}(0).S_z{}^{(2)}(0) + v^{(2)}(1).S_z{}^{(3)}(1) \\ &- v^{(2)}(0).S_z{}^{(3)}(0) - v'(1).S_z{}^{(4)}(1) + v'(0).S_z{}^{(4)}(0) + v(1).S_z{}^{(5)}(1) \\ &- v(0).S_z{}^{(5)}(0) \\ &- \int_0^1 v(x).S_z{}^{(6)}(x)\,dx. \end{aligned} \quad (14)$$

Let us take:

$$\begin{cases} S_z(0) - S_z{}^{(5)}(0) = 0 \\ S_z{}'(0) + S_z{}^{(4)}(0) = 0 \\ S_z{}^{(2)}(0) - S_z{}^{(3)}(0) = 0 \\ S_z{}^{(3)}(1) = 0 \\ S_z{}^{(4)}(1) = 0 \\ S_z{}^{(5)}(1) = 0 \end{cases} \quad (15)$$

We will have:

$$\langle v, S_z(x)\rangle_{W_2^3[0,1]} = -\int_0^1 v(x).S_z{}^{(6)}(x)\,dx. \quad (16)$$

By applying the property of the reproducing kernel: $\langle v, S_z(x)\rangle_{W_2^3[0,1]} = v(z)$,
then we obtain:$-\int_0^1 v(x).S_z^{(6)}(x)\,dx = v(z)$.
According to the property of the Dirac delta function, we find that:

$$S_z^{(6)}(x) = -\delta(z-x), \tag{17}$$

where δ is the Dirac-Delta function. Forz ≠ x, (17) becomes:

$$S_z^{(6)}(x) = 0.$$

On the other hand, we obtain:

$$S_z(x) = \begin{cases} a_1(z) + a_2(z)x + a_3(z)x^2 + a_4(z)x^3 + a_5(z)x^4 + a_6(z)\,x^5 & ,x \le z, \\ b_1(z) + b_2(z)x + b_3(z)x^2 + b_4(z)x^3 + b_5(z)x^4 + b_6(z)\,x^5 & ,x > z, \end{cases} \tag{18}$$

where a_i(z)and b_i(z) , i=1,...6 are unknown coefficients to be determined. Using the property of the Delta-Dirac function, we find that:

$$\begin{cases} S_{z^+}(z) = S_{z^-}(z) & (19) \\ S_{z^+}{}'(z) = S_{z^-}{}'(z) & (20) \\ S_{z^+}{}^{(2)}(z) = S_{z^-}{}^{(2)}(z) & (21) \\ S_{z^+}{}^{(3)}(z) = S_{z^-}{}^{(3)}(z) & (22) \\ S_{z^+}{}^{(4)}(z) = S_{z^-}{}^{(4)}(z) & (23) \\ S_{z^+}{}^{(5)}(z) - S_{z^-}{}^{(5)}(z) = -1 & (24) \end{cases}$$

Using (15), (19)-(24) and expression (18) along with its derivatives, we obtain the unknown coefficients as follows:

$$a_1(z) = 1\ , \quad a_2(z) = z\,, \quad a_3(z) = \frac{z^2}{4}, \quad a_4(z) = \frac{z^2}{12}, \quad a_5(z) = -\frac{z}{24}, \quad a_6(z) = \frac{1}{120}$$

and

$$b_1(z) = \frac{z^5}{120}\ , \quad b_2(z) = -\frac{z^4}{24}\,, \quad b_3(z) = \frac{z^3}{12}, \quad b_4(z) = 0, \quad b_5(z) = 0, \quad b_6(z) = 0$$

Hence the Kernel Reproduction function in $W_2^3[0,1]$ is determined as follows:

$$S_z(x) = \begin{cases} \frac{x^2z^2}{4} + \frac{x^2z^3}{12} - \frac{xz^4}{24} + \frac{x^5}{120} & ,x \le z, \\ \frac{z^2x^2}{4} + \frac{z^2x^3}{12} - \frac{zx^4}{24} + \frac{z^5}{120} & ,x > z, \end{cases}$$

■

Now, we will show that: $\langle v, S_z(x)\rangle_{W_2^3[0,1]} = v(z)$.
Either:

$$\langle v, S_z(x)\rangle_{W_2^3[0,1]} = v(0).S_z(0) + v'(0).S_z{}'(0) + v^{(2)}(0).S_z{}^{(2)}(0) + \int_0^1 v^{(3)}(x).S_z{}^{(3)}(x)\,dx, \tag{25}$$

let us integrate by part (25) twice, we obtain:

$$\langle v, S_z(x)\rangle_{W_2^3[0,1]} = v(0).S_z(0) + v'(0)\left[S_z{}'(0) + S_z{}^{(4)}(0)\right] + v^{(2)}(0)\left[S_z{}^{(2)}(0) - S_z{}^{(3)}(0)\right] + v^{(2)}(1).S_z{}^{(3)}(1) - v'(1).S_z{}^{(4)}(1) + \int_0^1 v'(x).S_z{}^{(5)}(x)\,dx.$$

(26)
Let's take:

$$\begin{cases} S_z(0) = 0 \\ S_z{}'(0) = y \\ S_z{}^{(2)}(0) = \frac{y^2}{2} \\ S_z{}^{(3)}(0) = \frac{y^2}{2} \\ S_z{}^{(4)}(0) = -y \\ S_z{}^{(3)}(1) = 0 \\ S_z{}^{(4)}(1) = 0 \end{cases} \quad (27)$$

Using (27), we get:

$$\langle v, S_z(x)\rangle_{W_2^3[0,1]} = \int_0^z v'(x).S_z{}^{(5)}(x)\,dx + \int_z^1 v'(x).S_z{}^{(5)}(x)\,dx, \quad (28)$$

we know that: $S_z{}^{(5)}(1) = 0$, $S_z{}^{(5)}(0) = 1$,
therefore:

$$\langle v, S_z(x)\rangle_{W_2^3[0,1]} = v(z) - v(0), \quad (29)$$

since: $v(0) = 0$, we obtain:

$$\langle v, S_z(x)\rangle_{W_2^3[0,1]} = v(z). \quad (30)$$

We now move on to the second RKHS $W_2^1[0,1]$.

Theorem 3.2 [45] $B_z(x)$is the RKHS function associated with the space $W_2^1[0,1]$, given as follows:

$$B_z(x) = \begin{cases} 1+x & ,x \le z, \\ 1+z & ,x > z, \end{cases} \quad (31)$$

Proof: According to definition 2.5, we have

$$\langle v, B_z(x)\rangle_{W_2^1[0,1]} = v(0).B_z(0) + \int_0^1 v'(x).B_z{}'(x)\,dx \ , \quad (32)$$

let's integrate by parts in (32), and we obtain:

$$\langle v, B_z(x)\rangle_{W_2^1[0,1]} = v(0).B_z(0) + v(1).B_z{}'(1) - v(0).B_z{}'(0) - \int_0^1 v(x).B_z{}^{(2)}(x)\,dx \ , \quad (33)$$

if we have the following equations:

$$\begin{cases} B_z(0) - B_z{}'(0) = 0 \\ B_z{}'(1) = 0 \end{cases} \quad (34)$$

Then we'll have:

$$\langle v, B_z(x)\rangle_{W_2^1[0,1]} = -\int_0^1 v(x).B_z{}^{(2)}(x)\,dx.$$

By applying the Kernel property, we obtain:

$$-\int_0^1 v(x).B_z{}^{(2)}(x)\,dx = v(z), \quad (35)$$

by application of the Dirac-Delta function:

$$-B_z{}^{(2)}(x) = \delta(z-x), \quad (36)$$

therefore: $B_z{}^{(2)}(x) = 0$,when: $z \neq 0$.
On the other hand, we have:

$$B_z(x) = \begin{cases} a_1(z) + a_2(z)x & ,x \le z, \\ b_1(z) + b_2(z)x & ,x > z, \end{cases} \quad (37)$$

we notice that there are four unknown coefficients, so we need to add two more equations:

$$\begin{cases} B_{z^+}(z) = B_{z^-}(z) \\ B_{z^+}{}'(z) - B_{z^-}{}'(z) = -1. \end{cases} \quad (38)$$

Using (28), (32) and the expression (31) with its derivatives, we then obtain the unknown coefficients as follows:

$$\begin{cases} a_1(z) = 1 \\ a_2(z) = 1 \\ b_1(z) = 1 + z \\ b_2(z) = 0 \end{cases}$$

We therefore obtain the Kernel Reproduction function in $W_2^1[0,1]$ given as follows:

$$B_z(x) = \begin{cases} 1 + x & , x \leq z, \\ 1 + z & , x > z. \end{cases}$$

Now we have to show that: $\langle v, S_z(x) \rangle_{W_2^1[0,1]} = v(z)$.

We have: $\langle v, B_z(x) \rangle_{W_2^1[0,1]} = v(0). B_z(0) + \int_0^z v'(x). B_z'(x)\, dx + \int_z^1 v'(x). B_z'(x)\, dx$,(39)

we also have:$B_z'(0) = 1,\ B_z'(1) = 0$,

hence:

$\langle v, B_z(x) \rangle_{W_2^1[0,1]} = v(z)$. (40)

4.Solution of problem (9)-(11) in $\mathbf{W_2^3[0,1]}$:

Now, we will give the representation of analytical solution to (9)–(11) in $W_2^3[0,1]$. We assume that problem (9)–(11) has a unique solution which belongs to $W_2^3[0,1]$.

Define the linear operator L_1 from the reproducing kernel Hilbert space $W_2^3[0,1]$ to $W_2^1[0,1]$.

Therefore, problem (9)–(11) is turned into the following operator form:

$$\begin{cases} L_1 v(x) = g\big(x, v(x)\big), \quad 0 \leq x \leq 1, \quad 1 \leq \alpha \leq 2 & (41) \\ v(0) = 0, & (42) \\ v'(0) = 0, & (43) \end{cases}$$

Lemma 4.1: L_1 is a linearly bounded operator.

Proof: It suffices to show that:

$\exists\, j > 0\ tel\ que \|L_1 v\|^2_{W_2^1[0,1]} \leq j \|v\|^2_{W_2^3[0,1]}$.

We have:

$\|L_1 v\|^2_{W_2^1[0,1]} = \langle L_1 v, L_1 v \rangle_{W_2^1[0,1]} = [L_1 v(0)]^2 + \int_0^1 [L_1 v(x)]'^2\, dx$, (44)

by applying the property of the Kernel function, we have:

$$L_1 v(x) = \langle v, L_1 S_z(.) \rangle_{W_2^3[0,1]}.$$

Applying the Cauchy Schwartz inequality, we find:

$|L_1 v(x)| \leq j_1 \|v\|_{W_2^3[0,1]}$, (45)

where:

$j_1 = \|L_1 S_z\|_{W_2^1[0,1]}$,

hence:

$(L_1 v)^2 \leq {j_1}^2 \|v\|^2_{W_2^3[0,1]}$, (46)

on the other hand:

$\left|(L_1 v)'\right| \leq j_2 \|v\|_{W_2^3[0,1]}$,(47)

where:

$j_2 = \left\|(L_1 v)'\right\|_{W_2^1[0,1]}$,

hence:

$(L_1 v)'^2 \leq {j_2}^2 \|v\|^2_{W_2^3[0,1]}$, (48)

substituting (46) and (48) into (44), we obtain:

$\|L_1 v\|^2_{W_2^1[0,1]} \leq j. \|v\|^2_{W_2^3[0,1]}$, (49)

such that:

$j = {j_1}^2 + {j_2}^2$.

Which completes our proof ■

Our aim is to find v(x)so we select a dense, countable subset such that $\{x_1, x_2, ...\}$then assume :

$\sigma_i(x) = S_{x_i}(x)$and$\phi_i(x) = L_1{}^*\sigma_i(x)$,

such that $L_1{}^*$the conjugate operator of $L_1$is defined as follows: $L_1{}^*: W_2^1[0,1] \to W_2^3[0,1]$.

And let's present the following lemma:

Lemma: By applying the Gram-Schmidt orthogonalization procedure to the complete system $\{\phi_i(x)\}_{i=1}^{\infty}$ we obtain the orthonormal system $\{\overline{\phi}_i(x)\}_{i=1}^{\infty}$ of $W_2^3[0,1]$such that:

$$\overline{\phi}_i(x) = \sum_{k=1}^{i} \rho_{ik}\, \phi_k(x), \qquad i=1,2,... \qquad (50)$$

such that:

$$\rho_{ik} = \begin{cases} \dfrac{1}{\|\phi_1\|}\ , & i = j = 1 \\ \dfrac{1}{e_i}\ , & i = j \neq 1 \\ -\dfrac{1}{e_i}\sum_{k=j}^{i-1} c_{ik}\rho_{kj}\ , & i > j \end{cases} \qquad (51)$$

where:

$$e_i = \sqrt{\|\phi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}{}^2}\ , \qquad c_{ik} = \langle \phi_i, \overline{\phi_k} \rangle_{W_2^3[0,1]}. \quad (52)$$

Theorem: Let : $\{x_i\}_{i=1}^{\infty}$ dense in [0,1] and $\phi_i(x) = L_{1(z)}S_z(x)\big|_{Z=x_i}$, then $\{\phi_i(x)\}_{i=1}^{\infty}$ is a complete system in $W_2^3[0,1]$.

Proof:

We have: $(L_1{}^*\sigma_i)(x) = \langle (L_1{}^*\sigma_i)(x), S_z(x)\rangle_{W_2^3[0,1]}$

$$= \langle \sigma_i(x), S_z(x)\rangle_{W_2^3[0,1]} = L_1 S_z(x)|_{z=x_i}, \qquad (53)$$

and we have:$\phi_i(x) \in W_2^3[0,1]$and $v \in W_2^3[0,1]$, then:

$$\langle v(x), \phi_i(x)\rangle_{W_2^3[0,1]} = 0\ , \qquad i = 1,2,3 \dots.. \qquad (54)$$

we obtain:$\langle v(x), \phi_i(x)\rangle_{W_2^3[0,1]} = \langle v(x), L_1{}^*\sigma_i(x)\rangle_{W_2^3[0,1]}$

$$= \langle L_1 v(x), \sigma_i(x)\rangle_{W_2^3[0,1]}$$

$$= (L_1 v)(x_i),$$

hence:

$$(L_1 v)(x_i) = 0, \qquad (55)$$

we have: $\{x_i\}_{i=1}^{\infty}$ dense in [0,1] , hence:

$$x_i \xrightarrow[i\to\infty]{} x \Rightarrow (L_1 v)(x_i) \xrightarrow[x_i \xrightarrow[i\to\infty]{} x]{} (L_1 v)(x)\ , \qquad (56)$$

by applying (55) and (56) and the inverse of the operator L_1, we obtain:

$$v(x) = 0.$$

Which completes our proof.■

Theorem:

Suppose that $\{x_i\}_{i=1}^{\infty}$ dense in $W_2^3[0,1]$ then there is a unique solution $v(x) \in W_2^3[0,1]$ of the problem:

$$\begin{cases} L_1 v(x) = g\big(x, v(x)\big)\ , & 0 \le x \le 1, \quad 1 \le \alpha \le 2 \\ v(0) = 0, \\ v'(0) = 0\ , \end{cases}$$

which is given by:

$$v(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \rho_{ik} . g(x_k, v_k) . \bar{\phi}_i(x).$$

Proof: We have: $\bar{\phi}_1(x)$ a complete orthonormal basis in $W_2^3[0,1]$ so we can write:

$$v(x) = \sum_{i=1}^{\infty} \langle v(x), \bar{\phi}_i(x) \rangle_{W_2^3[0,1]} . \bar{\phi}_i(x)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \rho_{ik} . \langle v(x), \phi_k(x) \rangle_{W_2^3[0,1]} . \bar{\phi}_i(x)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \rho_{ik} . \langle v(x), L_1{}^* \sigma_k(x) \rangle_{W_2^3[0,1]} . \bar{\phi}_i(x)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \rho_{ik} . \langle L_1 v(x), \sigma_k(x) \rangle_{W_2^3[0,1]} . \bar{\phi}_i(x)$$

$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \rho_{ik} . \langle g(x,v), \sigma_k(x) \rangle_{W_2^3[0,1]} . \bar{\phi}_i(x),$ (57)

by applying the Kernel function property to (57), we obtain:

$v(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \rho_{ik} . g(x_k, v_k) . \bar{\phi}_i(x).$ (58)

This completes the proof of the theorem. ■

So we can write $v_n(x)$ the approximate solution as follows:

$v_n(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \rho_{ik} . g(x_k, v_k) . \bar{\phi}_i(x).$ (59)

Lemma: Let $\{x_i\}_{i=1}^{\infty}$ be dense in [0,1], then:

$L_1 v_n(x_i) = L_1 v(x_i) = g(x_i, v(x_i)),$ i=1,2,...,n.

Proof:

$$L_1 v_n(x_i) = \langle L_1 v_n(x), \sigma_i(x) \rangle_{W_2^1[0,1]}$$

$$= \langle v_n(x), L_1{}^* \sigma_i(x) \rangle_{W_2^3[0,1]}$$

$$= \langle v(x), L_1{}^* \sigma_i(x) \rangle_{W_2^3[0,1]}$$

$= \langle L_1 v(x), \sigma_i(x) \rangle_{W_2^1[0,1]},$

by applying the property of the Kernel function, we obtain:

$$L_1 v_n(x_i) = L_1 v(x_i) = g(x_i, v(x_i)), \quad i = 1,2, \dots n.$$

we have previously demonstrated that :

$$v_n(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \rho_{ik} . g(x_k, v_k) . \bar{\phi}_i(x),$$

we have already shown that :

$$v_n(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \rho_{ik} . g(x_k, v_k) . \sum_{s=1}^{i} \rho_{is} . \phi_s(x)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \sum_{s=1}^{n} \rho_{ik} . g(x_k, v_k) . \rho_{is} . \phi_s(x). \tag{60}$$

From this, we derive the following lemma.

Lemma: Let $\{x_i\}_{i=1}^{\infty}$ be dense in [0,1], then:

$v_n(x) = \sum_{i=1}^{n} \Lambda_i . \bar{\phi}_i(x)$, (61)

such that: $\Lambda_i = \sum_{k=1}^{i} \rho_{ik} \cdot g(x_k, v_k)$.
Proof:
We have:
$v_n(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \sum_{s=1}^{n} \rho_{ik} \cdot g(x_k, v_k) . \rho_{is} . \phi_s(x)$,(62)
from equation (45) we write : $\overline{\phi}_\iota(x) = \sum_{s=1}^{i} \rho_{is}\, \phi_s(x)$ then replacing it in (57), we obtain:
$v_n(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \rho_{ik} . g(x_k, v_k) . \overline{\phi}_\iota(x)$, (63)
let's define:$\Lambda_i = \sum_{k=1}^{i} \rho_{ik} \cdot g(x_k, v_k)$, where (58) becomes :
$v_n(x) = \sum_{i=1}^{n} \Lambda_i \cdot \overline{\phi}_\iota(x)$. (64)

■

5. Convergence in space $W_2^3[0,1]$:

Theorem 5.1 The approximate solution $v_n(x)$ and its derivatives: $v_n{}'(x)$, $v_n{}^{(2)}(x)$: converge uniformly.
Proof:

$$|\square_\square(\square) - \square(\square)| = \left|\langle \square_\square(\square) - \square(\square), \square_\square(\square)\rangle_{\square_2^3[0,1]}\right|$$

$\leq \square_0 . \|\square_\square(\square) - \square(\square)\|_{\square_2^3[0,1]}$, (65)
such as:
$\square_0 = \|\square_\square(\square)\|_{\square_2^3[0,1]}$, (66)

$$\left|\square_\square{}'(\square) - \square'(\square)\right| = \left|\langle \square_\square{}'(\square) - \square'(\square), \square_\square{}'(\square)\rangle_{\square_2^3[0,1]}\right|$$

$\leq \square_1 . \left\|\square_\square{}'(\square) - \square'(\square)\right\|_{\square_2^3[0,1]}$, (67)
such as:
$\square_1 = \left\|\square_\square{}'(\square)\right\|_{\square_2^3[0,1]}$. (68)
And finally, let's finish with:

$$\left|\square_\square{}^{(2)}(\square) - \square^{(2)}(\square)\right| = \left|\langle \square_\square{}^{(2)}(\square) - \square^{(2)}(\square), \square_\square{}^{(2)}(\square)\rangle_{\square_2^3[0,1]}\right|$$

$\leq \square_2 . \left\|\square_\square{}^{(2)}(\square) - \square^{(2)}(\square)\right\|_{\square_2^3[0,1]}$, (69)
such as:
$\square_2 = \left\|\square_\square{}^{(2)}(\square)\right\|_{\square_2^3[0,1]}$. (70)
So: $v_n(x), v_n{}'(x)$ and $v_n{}^{(2)}(x)$ converge uniformly to: $v(x), v'(x)$ and $v^{(2)}(x)$ respectively. ■

6. Numerical results:

In this section, a numerical experiment is presented to verify the efficiency and performance of the RKHSM. In process of computation, all the calculation are done by using Maple 15 software package.and the graphs with origin 2017. The results reveal that the algorithm is highly accurate, rapidly converge, and convenient to handle various physical problems in fractional calculus.

Example: Consider the following differential equation:

$$\square_{\square}^{\square}\square(\square)+\square\square\square(\square)+\square\square\square(\square)=1-\int_0^{\square}\square(\square)\square\square,$$

subject to the following conditions:

$$\square(0) = 0, \quad \square'(0) = 1.$$

Where: $0 \leq \square \leq 1$, $1 < \square \leq 2$. Here the exacte solution for $\square = 2$ is: $\square(\square) = \square\square\square(\square)$.

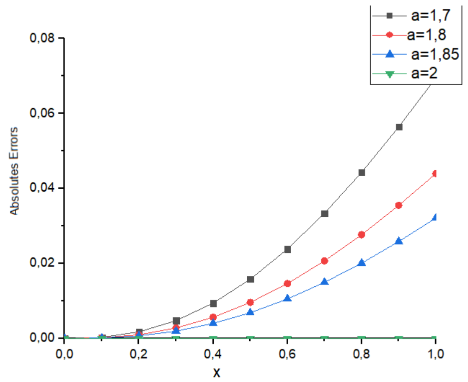
After homogenization of the initial conditions and by using the**RKHSM** , we take: $□_□ = \frac{□}{□}$, $□ = 1,2, \dots, □$, and $□ = 10$, and the absolute errors of approximate solutions are given in Table1 and Figure1 for different values of □. On the other hand the numerical results of the approximate solutions are given in Table2. The graphs of theses approximates for different values of □ are traced in Figure2.
We remark from the numerical results in Table 2 that the approximate solutions are in agreement with the exacte solution for $□ = 2$.

Table1 Absolute errors

x	$□ = 1.7$	$□ = 1.8$	$□ = 1.85$	$□ = 2$
0	0	0	0	0
0.1	3.2511665×10^{-4}	1.8091665×10^{-4}	1.2511665×10^{-4}	4.91665×10^{-6}
0.2	1.7583308×10^{-3}	1.0063308×10^{-3}	6.993308×10^{-4}	4.6692×10^{-6}
0.3	4.7702067×10^{-3}	2.8072067×10^{-3}	1.9802067×10^{-3}	2.2067×10^{-6}
0.4	9.4393423×10^{-3}	5.6533423×10^{-3}	4.0213423×10^{-3}	6.577×10^{-7}
0.5	$1.58255386 \times 10^{-2}$	9.6105386×10^{-3}	6.8715386×10^{-3}	4.614×10^{-7}
0.6	$2.38564734 \times 10^{-2}$	$1.46484734 \times 10^{-2}$	$1.05464734 \times 10^{-2}$	1.5266×10^{-6}
0.7	$3.34156872 \times 10^{-2}$	$2.07126872 \times 10^{-2}$	$1.49816872 \times 10^{-2}$	4.6872×10^{-6}
0.8	$4.43280909 \times 10^{-2}$	$2.76910909 \times 10^{-2}$	$2.01040909 \times 10^{-2}$	8.9091×10^{-6}
0.9	$5.64429096 \times 10^{-2}$	$3.55099096 \times 10^{-2}$	$2.58729096 \times 10^{-2}$	1.19096×10^{-5}
1	$6.95179848 \times 10^{-2}$	$4.39899848 \times 10^{-2}$	$3.21419848 \times 10^{-2}$	1.52×10^{-8}

Table2 Approximates and exacte solutions

x	$□ = 1.7$	$□ = 1.8$	$□ = 1.85$	$□ = 2$	Exacte solution
0	0	0	0	0	0
0.1	0.09951	0.09965	0.0997083	0.0998280	0.09983341665
0.2	0.19692	0.19771	0.1979	0.198674	0.1986693308
0.3	0.2908	0.29271	0.29344	0.295516	0.2955202067
0.4	0.37998	0.38381	0.385287	0.389417	0.3894183423
0.5	0.4641	0.46981	0.47261	0.479425	0.4794255386
0.6	0.5408	0.5499	0.554211	0.56464	0.5646424734
0.7	0.6108	0.62351	0.62924	0.644213	0.6442176872
0.8	0.67303	0.68970	0.69731	0.717360	0.7173560909
0.9	0.7269	0.74782	0.7575	0.783316	0.7833269096
1	0.7721	0.7975	0.81012	0.84147	0.8414709848



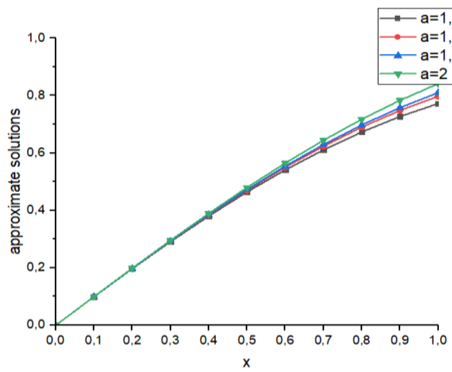


Figure1. Absolute errors Figure2. Approximate solutions

Based on the results from Table 1 and the curves obtained in Figure 1, we can see that as the values of α increase, the error decreases, leading to an improvement in the accuracy of the obtained solution.

7. Conclusion

In this article, we have presented the Reproducing Kernel Hilbert Space Method (RKHSM) for solving fractional-order differential equations. Our study reveals that RKHSM is highly accurate and efficient in determining a uniformly convergent approximate solution and its derivatives. we observe that the computed approximate solutions coincide and are nearly identical to the exact solution for □ $= 2$, and to reduce the error, it is sufficient to increase the values of α. Comparative studies based on the direction of the absolute error function indicate that RKHSM provides more acceptable approximate values in terms of accuracy and stability. Finally, we can see that the kernel reproduction method demonstrates potential for solving significantly more complex problems involving fractional linear and nonlinear equations.

8. Conflicts of Interest

The authors declare no conflicts of interest.

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