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NECESSARY CONDITIONS FOR THE EXISTENCE OF TRIPLE POSITIVE SOLUTIONS TO FRACTIONAL INTEGRODIFFERENTIAL BOUNDARY VALUE PROBLEMS AT RESONANCE

M. Latha Maheswari[a], K. Pandiyan[b]
Department of Mathematics, PSG College of Arts and Science,Coimbatore-641 014,
Tamil Nadu , India, lathamahespsg@gmail.com[a]
Department of Science and Humanities, Sri Krishna College of Engineering and Technology,
Coimbatore-641 008,
Tamil Nadu , India, pandiyan@skcet.ac.in[b]

Abstract
The fixed point index theory is used to determine whether there are set of solutions to boundary value problems with triple positives involving resonant fractional integrodifferential equations. To find the positive solutions, the spectrum theory and a few new height functions are also employed. An arbitrary fractional integral is used in the nonlinearity, which also allows for singularity.

Keywords: Fractional derivatives, Integrodifferential equation, Triple positive solution, At resonance, Singularity.
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1 Introduction

In this article, we consider the resonance-related problems with fractional integrodifferential equation boundary values:

$$D_{0+}^{\alpha}y(w) + f(w, y(w), \int_0^w k\left(w, p, y(p)\right)dp = 0, 0 < w < 1$$

$$y(0) = y'(0) = 0, D_{0+}^{\beta}y(1) = \sum_{i=1}^{m} \varepsilon_i D_{0+}^{\beta}y(\eta_i) \quad (1)$$

where $4 > \alpha > 3$, $\alpha - 3 > \beta > 0$, $\varepsilon_i > 0$, $0 < \eta_1 < \cdots < \eta_m < 1$ with $\sum_{i=1}^{m} \varepsilon_i \eta_i^{\alpha-\beta-1} = 1$, D_{0+}^{α} indicates derivatives of Riemann-Liouville, singularities at $w = 0,1$ and $y = x = 0$ are allowed by the nonlinearity $f(w, y, x)$. The equation $D_{0+}^{\alpha}y(w) + \lambda y(w) = 0$ is clearly solved by $\lambda = 0$ and $cw^{\alpha-1}$ with (1) as the boundary condition, FIBVP (1) is resonance.

Due to their proven uses in the scientific community, FBVPs have recently received a lot of attention in research ($[1 - 22]$). Non-local boundary value problems (BVPs) in particular have garnered a lot of interest ($[8 - 22]$). For instance, Zhong[11] and Zhang proved the following results of positive FBVP solutions using the Leggett-Williams theorem and Krasnosel'skii fixed point theorem:

$D_{0+}^{\alpha_1} z(w) + f\big(w, z(w)\big) = 0, 0 < w < 1, \qquad n-1 < \alpha_1 < n,$
$z(0) = z'(0) = \cdots = z^{(n-2)}(0) = 0,\ D_{0+}^{\beta_1} y(1) = \int_0^{\eta} a(w) D_{0+}^{\beta_1} z(w) dw.$

Zhang et al. [12] revealed existence findings of positive solutions to the FBVP utilising the fixed point index theory:

$$D_{0+}^{\alpha_2} y(w) + f\left(w, y(w), D_{0+}^{\beta_2} y(w)\right) > 0,\ w \in (0,1),$$
$$D_{0+}^{\beta_2} y(0) = 0,\ D_{0+}^{\beta_2} y(1) = \int_0^1 D_{0+}^{\beta_2} y(s) dB(s).$$

Many academics looked on ways to solve resonant BVPs using the coincidence degree theory [13-18]. It is important to highlight that few papers[16-20] have examined affirmative solutions for resonant BVPs. By utilising the Webb[19] technique, we were able to deduce certain sufficient and vital criteria to determine whether some resonant FBVPs in [21,22] have positive solutions. when $\alpha \in (1,2)$. As far as we are aware, for the resonant FBVP(1), three positive solutions have not been taken into account. We strive to fill this void, inspired by the masterpieces cited above.

2 Preliminaries

Let $\int_0^w K\big(w, p, y(p)\big) dp = x(w)$, It is clear from the approach in [12] that (1) is equivalent to

$$\begin{cases} D_0^{\alpha-\beta} x(w) + f(w, J_{0+}^{\beta} x(w), x(w) = 0, w \in (0,1), \\ J_{0+}^{\beta} x(0) = D_{0+}^{1-\beta} x(0) = 0, x(1) = \sum_{i=1}^{m} \varepsilon_i x(\eta_i) \end{cases} \tag{2}$$

The increasing function is known from [2].

$$h(\tau) := \sum_{l=0}^{+\infty} \frac{[(l+1)(\alpha-\beta)-3][(l+1)(\alpha-\beta)-4]\tau^l}{\Gamma\big((l+1)(\alpha-\beta)\big)}, \tau > 0$$

has a unique root $\tau^* > 0$, The resonant FBVP(2) is obviously the same as the FBVP:

$$\begin{cases} -D_0^{\alpha-\beta} x(w) + \tau x(w) = f\left(w, J_{0+}^{\beta} x(w), x(w)\right) + \tau x(w), w \in (0,1) \\ J_{0+}^{\beta} x(0) = D_{0+}^{1-\beta} x(0) = 0, x(1) = \sum_{i=1}^{m} \varepsilon_i x(\eta_i) \end{cases} \tag{3}$$

In this article, assume that $\tau \in (0, \tau^*]$ in order to keep things simple, we utilise the same notations:

$$U(w) = \sum_{l=0}^{+\infty} \frac{\tau^l w^{(l+1)(\alpha-\beta)-1}}{\Gamma\big((l+1)(\alpha-\beta)\big)},$$

$$E(w,q) = \frac{1}{U(1)} \begin{cases} U(w)U(1-q), & 0 \le w \le q \le 1, \\ U(w)U(1-q) - U(t-q)U(1), & 0 \le s \le w \le 1 \end{cases}$$

$$R(w,q) = E(w,q) + \frac{\sum_{i=1}^{k} \varepsilon_i E(q_i, q) U(w)}{U(1) - \sum_{i=1}^{k} \varepsilon_i U(\eta_i)}$$

We have the following Lemma, which is similar to Lemma 2.1 in (22).

Lemma 2.1. *Let's say $k \in L[0,1]$. Then the FIBVP for two terms.*

$$-D_0^{\alpha-\beta} x(w) + \tau x(w) = \mathrm{k(w)}, w \in (0,1)$$
$$J_{0+}^{\beta} x(0) = D_{0+}^{1-\beta} x(0) = 0, x(1) = \sum_{i=1}^{m} \varepsilon_i x(\eta_i) \tag{4}$$

has a unique solution

$$x(t) = \int_0^1 R(w,q) r(q) dq.$$

Lemma 2.2. *([3]) Let's say that $q^* \in (0,1)$ satisfies $q^* = (1-q^*)^{\alpha-\beta-2}$. $E(w,q)$ is satisfied then.*
(i) $E(w,q) \geq\geq \varrho_1 q(1-q)^{\alpha-\beta-1} t^{\alpha-\beta-1}, \ \forall\, w,q \ \in [0,1]$
(ii) $E(w,q) \leq \varrho_2 q(1-q)^{\alpha-\beta-1}$, $\forall w,q \in [0,1]$,
where $\rho_1 = \frac{1}{N(1)[\Gamma(\alpha-\beta)]^2}$, $\rho_2 = \frac{[N'(1)]^2}{N(1)q^*}$.
Lemma 3. *([25]) The following are the properties of the function $R(w,q)$:*
(i) $R(w,q) \geq \varrho_1 q(1-q)^{\alpha-\beta-1} t^{\alpha-\beta-1}, \ \forall\, w,q \ \in [0,1]$
(ii) $R(w,q) \leq \varrho_2 q(1-q)^{\alpha-\beta-1}$, $\forall w,q \in [0,1]$,
where $\varrho_1 = \frac{\rho_1\left(1-\sum_{i=1}^{k}\varepsilon_i\eta_i^{\alpha-\beta}\right)}{\Gamma(\alpha-\beta)\left[N(1)-\sum_{i=1}^{k}\varepsilon_i N(\eta_i)\right]}$, $\varrho_2 = \rho_2\left[1+\frac{N(1)\sum_{i=1}^{k}\varepsilon_i}{N(1)-\sum_{i=1}^{k}\varepsilon N(\eta_i)}\right]$.
Let $F = D[0,1]$ have the norm $||y|| = \max_{0\leq w\leq 1}|y(w)|$, θ be the zero element, and F be a Banach space.
$\psi_1(w) = \frac{\varrho_1\Gamma(\alpha-\beta)}{\varrho_2\Gamma(\alpha)} w^{\alpha-1}$, $\psi_2(t) = \frac{\varrho_1}{\varrho_2} w^{\alpha-\beta-1}$.
Describe a cone
$\mathbb{P} = \{y \in F: y(w) \geq \psi_2(w)\big||y|\big|, w \in [0,1]\}$
Let $0 < \alpha < 1$, indicate $\psi = \min_{w\in[\alpha,1]}\psi_2(w)$ and $v(y) = \min_{w\in[\alpha,1]} y(w)$, $y \in \mathbb{P}$,
$\forall S \geq s > 0$, set
$\mathbb{P}(v,s,S) = \{y \in \mathbb{P}: s \leq v(y), ||y|| \leq S\}$,
$\mathring{\mathbb{P}}(v,s,S) = \{x \in \mathbb{P}: s \leq v(y), ||y|| \leq S\}$,
$\mathbb{P}_s = \{y \in \mathbb{P}: ||y|| < s\}$.
Define height functions:

$$\phi(t,s,S) = max\{f(t,y,x) + \tau x: s\psi_1(t) \leq y \leq \frac{St^{\beta}}{\Gamma(\beta+1)}, s\psi_2(t) \leq x \leq s\}$$

$$\phi_1(t,s) = min\{f(t,y,x): s\psi_1(t) \leq y \leq \frac{st^{\beta}}{\Gamma(\beta+1)}, s\psi_2(t) \leq x \leq s\}$$

$$\phi_2(t,s,S) = min\{f(t,y,x) + \tau x: \frac{st^{\beta}}{\Gamma(\beta+1)} \leq y \leq \frac{St^{\beta}}{\Gamma(\beta+1)}, s \leq x \leq S\}$$

Lemma 2.4. *([23]) Let Ω be a bounded open set in F, cone $\mathbb{P}$ in Banach space F,and be a completely continuous operator where $\theta \in \Omega, \theta: \overleftarrow{\Omega} \cap \mathbb{P} \to \mathbb{P}$.*
(i) If $\exists y_0 \in \mathbb{P}/\{\theta\}$ such that $y - By \neq \lambda y_o$, $\forall\lambda \geq 0$, $y \in \partial\Omega \cap \mathbb{P}$, then $i(B, \Omega\cap\mathbb{P}, \mathbb{P}) = 0$
(ii) If $By \neq \lambda y$, $\forall\lambda \geq 1$,$y \in \partial\Omega \cap \mathbb{P}$, then $i(B, \Omega\cap\mathbb{P}, \mathbb{P}) = 1$.
Lemma 2.5. *([24]).Let $B: \overleftarrow{E}_{S_3} \to \mathbb{P}$ be completely continuous operator. If there exist a concave positive functional v with $v(y) \leq ||y||(y \in \mathbb{P})$ and numbers $s_3 \geq s_2 > s_1 > 0$ fulfills the following requirements.*
(i) $\mathring{\mathbb{P}}(v, s_1, s_2) \neq 0$, and $v(By) > s_1$ if $y \in \mathbb{P}((v, s_1, s_2)$;
(ii) $By \in \overleftarrow{E}_{S_3}$, if $y \in \mathbb{P}(v, s_1, s_3)$;
(iii) $v(By) > s_1 \forall y \in \mathbb{P}\big((v, s_1, s_3)\big)$, with $||By|| > s_2$.
Then $i\big(B, \mathring{\mathbb{P}}(v, s_1, s_3), \overleftarrow{E}_{s_3}\big) = 1$.

3 Main Results

Theorem 3.1. *Assume that there exist number* s_1, s_2, s_3, s_4, s_5 *with* $0 < s_1 < s_2 < s_3 < s_4 \leq s_5$ *and* $s_1 \leq s_4\psi^*$ *such that*

$(H_1) f$ *is continuous on* $(0,1) \times \left(0, \frac{s_5}{\Gamma(\beta+1)}\right) \times (0, s_5)$ *with* $f(w, y, x) \geq -\tau x$

$$(H_2) \phi(w, s_1, s_5) \in L[0,1]$$

$$(H_3) \phi_1(w, s_1) \geq 0$$

$$(H_4) \int_0^1 \phi\,(q, s_2, s_2) q(1-q)^{\alpha-\beta-1} dq < s_2 \rho_2^{-1}$$

$$(H_5) \int_b^1 \phi_2\,(q, s_3, s_4) q(1-q)^{\alpha-\beta-1} dq > s_3 [\psi^* \rho_2]^{-1}$$

$(H_6) \int_0^1 \phi\,(q, s_3, s_5) q(1-q)^{\alpha-\beta-1} dq \leq s_5 \rho_2^{-1}$.

Then there are at least three positive solutions for the resonant FIBVP(1).

Proof: Indicate two operators

$Qx(w) = \int_0^1 R\,(w, q) x(q) dq,$

$$Bx(w) = \int_0^1 R\,(w, q) \left[f\left(q, J_{0+}^{\beta} x(q), x(q)\right) + \tau x(q)\right] dq$$

It is clear that the linear operator $L: \mathbb{P} \to \mathbb{P}$ is completely continuous. We are aware that the spectrum radius of Q is $s(Q) = \tau^{-1}$, and $\phi(w) = w^{\alpha-\beta-1}$ is the appropriate eigenfunction, that is to say $Q\phi = \tau^{-1}\phi$. In any case $x \in \overleftarrow{\mathbb{P}}_{s_5}/\mathbb{P}_{s_1}$,there are $s_1\psi_2(w) \leq x(w) \leq s_5$ and $s_1\psi_1(w) \leq J_{0+}^{\beta} x(w) \leq \frac{s_5 w^{\beta}}{\Gamma(\beta+1}$. Then (H_1) and (H_2) yield that $B: \overleftarrow{\mathbb{P}}_{s_5}/\mathbb{P}_{s_1} \to \mathbb{P}$ is continuous. By applying the completely continuous operator extension theorem, B can be transformed into the completely continuous operator $\tilde{B}: \mathbb{P} \to \mathbb{P}$. We still spell it out as B for convenience. For $x \in \partial\mathbb{P}_{s_2}$, we have $s_5 \geq x(w) \geq s_3\psi_2(w)$ and $s_2\psi_1(w) \leq J_{0+}^{\beta} x(w) \leq \frac{s_2 w^{\beta}}{\Gamma(\beta+1)}$. By Lemma 2.3 and (H_4), we can get

$Bx(w) \leq \rho_2 \int_0^1 q\,(1-q)^{\alpha-\beta-1} \phi(q, s_2, s_2) dq < s_2,$

which implies that $Bx \neq \lambda x, \forall \lambda \geq 1$. So, based on Lemma 2.4, it follows that

$$i(B, \mathbb{P}_{s_2}, \mathbb{P}) = 1 \tag{5}$$

Similarly, for $x \in \partial\mathbb{P}_{s_5}$, by Lemma 2.3 and (H_6), we get

$$Bx(w) \leq \rho_2 \int_0^1 \phi\,(q, s_3, s_5) q(1-q)^{\alpha-\beta-1} dq < s_5$$

Therefore we have

$$i(B, \mathbb{P}_{s_5}, \mathbb{P}) = 1 \tag{6}$$

Following that, we will demonstrate

$i\left(B, \mathring{\mathbb{P}}(v, s_3, s_5), \overleftarrow{E}_{s_5}\right) = 1.$

(i) It is clear that $\mathring{\mathbb{P}}(v, s_3, s_4) \neq \phi$. For any $x \in \mathbb{P}(v, s_3, s_4)$, we have $s_4 \geq x(w) \geq s_3$ and $\frac{s_3 w^\beta}{\Gamma(\beta+1)} \leq J_{0+}^\beta x(w) \leq \frac{s_4 w^\beta}{\Gamma(\beta+1)}$, for $t \in [a, 1]$.

$$v(y) \geq \min_{w \in [a,1]} \rho_1 w^{\alpha-\beta-1} \int_0^1 q\,(1-q)^{\alpha-\beta-1} \left[f(q, J_{0+}^\beta x(q), x(q) + Jx(q)\right] dq$$

$\geq \rho_2 \psi^* \int_0^1 q\,(1-q)^{\alpha-\beta-1} \phi_2(t, s_3, s_4) ds > s_3.$

(ii) For $x \in \mathbb{P}(v, s_3, s_5)$, we have

$s_5 \geq x(w) \geq s_3 \psi_2(w)$ and $s_3 \psi_1(w) \leq J_{0+}^\beta x(w) \leq \frac{s_5 w^\beta}{\Gamma_{(\beta+1)}}$, for $w \in [0,1]$. Then

$$Bx \leq \rho_2 \int_0^1 q\,(1-q)^{\alpha-\beta-1} [f(q, J_0^+ x(q), x(q) + \tau x(q)] dq$$

$\leq \rho_2 \int_0^1 q\,(1-q)^{\alpha-\beta-1} \phi(w, s_3, s_5) dq \leq s_5.$

So $Bx \in \overleftarrow{E}_{s_5}$.

(iii) For $x \in \mathbb{P}(v, s_3, s_5)$ with $||\mathrm{Bx}|| > s_4$, nothing $s_3 \leq s_4 \psi^*$, we have

$v(Bx) = \min_{t \in [b,1]} \left(B_y\right)(t) \geq \psi^* s_4 \geq s_5.$

$$i(B, \mathring{\mathbb{P}}(v, s_3, s_5), \overleftarrow{E}_{s_5} = 1 \tag{7}$$

From (5) to (7) it is evident that

$$i\left(B, \mathbb{P}/\mathring{\mathbb{P}}(v, s_3, s_5) \cup \mathbb{P}_{s_5}\right), \overleftarrow{E}_{r_s} = -1 \tag{8}$$

As a result of (7) and (8), B has two fixed points.

$x_1 \in \mathring{\mathbb{P}}(v, s_3, s_5)$ and $x_2 \in \mathbb{P}_{s_5}/(\mathring{\mathbb{P}}(v, s_3, s_5) \cup \mathbb{P}_{s_2}$.

In the following section, we shall show that B has another positive fixed point on $\mathbb{P}_{s_2}$. We can assume that B does not have a fixed point on $\partial \mathbb{P}_{s_1}$. Then, we'll demonstrate

$$x - Ax \neq \lambda \chi, \forall \lambda > 0, x \in \partial \mathbb{P}_{s_1} \tag{9}$$

Otherwise , suppose there exist $\lambda_0 > 0$ and $x_1 \in \partial \mathbb{P}_{s_1}$ such that

$x - Bx = \lambda_0 \chi$

Thus $x_1 \geq \lambda_0 \chi$. Set $\lambda^* = sup\{\lambda : x_1 \geq \lambda_0 \chi\}$. Then $x_1 \geq \lambda^* \chi$. It follows from (H_3) that

$$Bx_1 = \int_0^1 R\,(w, q) \left[f\left(q, J_{0+}^\beta x_1(q), x_1(q)\right) + \tau x_1(q)\right] dq$$

$\geq \tau \int_0^1 R\,(w, q) x_1(q) dq = \tau Q x_1.$

Therefore,

$x_1 = Bx_1 + \lambda_0 \chi \geq \tau Q x_1 + \lambda_0 \chi \geq \tau Q(\lambda^* \chi) + \lambda_0 \chi = (\lambda^* + \lambda_0) \chi.$

With the definition of λ^* there is a contradiction. Therefore, Lemma 2.4 ensures that (9) holds.

$$i\left(B, C_{S_1} \cap W, W\right) = 0 \tag{10}$$

Thus, (5) and (10) indicate that B has a fixed point $x_3 \in \mathbb{P}_{s_2}/\mathbb{P}_{s_1}$. It is clear that $J_{0+}^\beta x_i(w), i = 1,2,3$ (1) has three positive solutions. The proof is now complete.

4 Example

Consider the resonant FIBVP:

$$\begin{cases} D_{0+}^{2.1}y(t) + f(t, y(t), J_0+^{0.2}y(t)) = 0, & t \in (0,1) \\ y(0) = y'(0) = 0, D_{0+}^{0.2}y(1) = 0.7^{-1.2}D_{0+}^{0.2}y(0.7), \end{cases} \quad (11)$$

with

$$f(t,y,x) = \begin{cases} \frac{t^8 y^{\frac{-1}{3}}}{59\sqrt{(1-t)}} + \frac{x^{\frac{-1}{3}}}{148} + \frac{(1-t)^8 x^{\frac{-1}{3}}}{59\sqrt{t}} - \frac{x}{4}, & (t,y,x) \in (0,1) \times (0,+\infty) \times (0,1], \\ \frac{(1-t)^8}{59\sqrt{t}} + \frac{t^8 y^{\frac{-1}{3}}}{59\sqrt{(1-t)}} + \frac{x^5}{149} - \frac{x}{4}, & (t,y,x) \in (0,1) \times (0,+\infty) \times (0,5], \\ \frac{(1-t)^8}{59\sqrt{t}} + \frac{t^8 y^{\frac{-1}{3}}}{59\sqrt{(1-t)}} + \frac{x^{\frac{1}{2}} + 5^4 - \sqrt{5}}{149} - \frac{x}{4}, & (t,y,x) \in (0,1) \times (0,+\infty) \times (5,+\infty). \end{cases}$$ By

$\frac{33.56\tau^3}{\Gamma(7.4)} \le k(\tau) - \left[\frac{-0.08}{\Gamma(2.1)} + \frac{1.64\tau}{\Gamma(3.2)} + \frac{13.18\tau^2}{\Gamma(5.3)}\right] \le \sum_{j=3}^{+\infty} \frac{33.56\tau^j}{\Gamma(7.4)}$,

In order to $\tau^* \in (0.23, 0.24)$.

Let $\tau = 0.1$ and $b = 0.7$. Using an exact computation, we have $N(1) = 0.8714$, $N'(1) = 1.0313$, $q^* = 0.7261$, $\rho_1 = 0.8201$, $\rho_2 = 1.4781$, $\varrho_1 = 17.2372$, $\varrho_2 = 206.084$, $\psi_1(t) = 0.673t^{1.1}$, $\psi_2(t) = 0.0774t^{1.1}$, $\chi^* = 0.0592$, $\chi^*\varrho_2 = 13.3281$. Let $s_1 = 0.0534$, $s_2 = 1$, $s_3 = 5$, $s_4 = 89$, $s_5 = 1763$. The (H_1), (H_2) and (H_3) holds are not difficult to obtain. Direct calculation yields what we have

$$\varrho_2 \int_0^1 \phi\,(q, s_2, s_2) q(1-q)^{1.1} dq \approxeq 0.8338,$$

$$\psi^* \varrho_2 \int_{0.7}^1 q_2\,(q, s_3, s_4) q(1-q)^{1.1} dq > \chi^* \varrho_2 \int_{0.7}^1 \frac{q_3^4}{149} q(1-q)^{1.1} dq \approxeq 9.24,$$

$$\varrho_2 \int_0^1 \phi\,(q, s_3, s_5) q(1-q)^{1.1} dq < \varrho_2 \int_0^1 [s_3 \chi_2(q)]^{\frac{-1}{3}} \left[\frac{1}{149} + \frac{(1-q)^8}{59\sqrt{q}}\right] q(1-q)^{1.1} dq$$

$$+\varrho_2 \int_0^1 \left[\frac{q^8 \left(s_3 \chi_1(q)\right)^{\frac{-1}{3}}}{59\sqrt{(1-q)}} + \frac{(1-q)^8}{59\sqrt{q}} + \frac{s_5^{\frac{1}{2}} + 5^4 - \sqrt{5}}{149}\right] q(1-q)^{1.1} dq \approxeq 1562.6256$$

Because of this, (H_4), (H_5) and (H_6) hold. The resonant FIBVP(11) must therefore have at least three positive solutions, according to Theorem 3.1.

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