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On the growth of Random Fourier-Hermite series

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Abstract
The exact growth of the random Fourier–Hermite series $\sum_{k=0}^{\infty} a_k A_k(\omega)\phi_k(t)$, and its Fourier transform are established. Here $\phi_k(t)$ are considered to be both orthogonal Hermite functions in $L^2(\mathbb{R})$ and transformed Hermite function in $L^2[0,1]$. a_k are the Fourier–Hermite coefficients of functions in different L^2 spaces. $A_k(\omega)$ are random Fourier–Hermite coefficient associated with Wiener process and symmetric stable process of index 2.

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1. Introduction
Fourier Series in orthogonal functions e^{inx} and other orthogonal polynomials like Hermite polynomials, Jacobi polynomials etc. has a widespread application in physical sciences. The application of random Fourier series (RFS) in Hermite polynomials is found in image encryption and decryption in the work of Liu and Liu [2, 3] in 2007, who expected its more application in general signal and image processing. In fact the RFS they used is a random Fourier transform (RFT) with random coefficients choosen from the unit circle in $\mathbb{C}$ randomly. This motivated us to study more on random Fourier series in Hermite polynomials with different random coefficients. We have explored random Fourier Hermite series with random coefficient associated with continuous process like Wiener process and stable processes [4]. It is fascinating to know the rate at which the series converges. Attention has been paid to the rate of convergence of the series for smooth, non-pathological functions such as Hermite polynomials which are usually arise in engineering and physics problem. Earlier in 1997, Grawe[1] studied on asymptotic growth of Hermite series, where the scalars are Fourier - Hermite coefficients(FHC) of a function in $L^2(\mathbb{R})$ space.

Here the n^{th} Hermite function

$$\psi_n(t) := \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(t) e^{-\frac{t^2}{2}}, \tag{1.1}$$

where $n \in \mathbb{N}_0$, the set of non-negative integers and $t \in \mathbb{R}$. The normalized Hermite function satisfy the bound

$$|\psi_n(t)| \leq 0.816 \tag{1.2}$$

for all n and all real t.

Regarding growth of random Fourier series, Mishra etal. [5] have established the following weak law of large numbers:

$$\lim_{n\to\infty} P\left[\left|\frac{s_n}{J_n}\right| > \epsilon\right] = 0$$

for fixed t and $\epsilon > 0$, where s_n denotes the partial sum of the random series $\sum_{-\infty}^{\infty} A_k(\omega)e^{2\pi kit}$ with random coefficients $A_k(\omega)$, associated with stable process. J_n is a sequence of constants such that $\frac{J_n^2}{n} \to \infty$ as $n \to \infty$. Here J_n is higher order than $\sqrt{n}$. In [7], Nayak has established a stronger result for the growth of the series $\sum_{-\infty}^{\infty} A_k(\omega)e^{2\pi kit}$. They have shown that

$$P\left[\lim\ \sup \frac{s_n}{(2n+1)^{\frac{1}{\gamma}+\frac{1}{2}}(\log\log 2n+1)^{\frac{1}{\gamma}+\epsilon}} \leq 1\right] = 1$$

for $0 < \gamma < 2$ and for $0 < \gamma < 1$

$$P\left[\lim\ \sup \frac{s_n}{(2n+1)^{\frac{1}{\gamma}}(\log\log 2n+1)^{\frac{1}{\gamma}+\epsilon}} \leq 1\right] = 1.$$

The growth rate of the random Fourier-Stieltjes series,

$$\sum_{-\infty}^{\infty} a_k A_k(\omega)e^{2\pi kit}, \tag{1.3}$$

where the weight a_k are Fourier coefficients of an L^p function is studied by Mohanty [6]. [8] is referred to see the existence of its series. They established an upper class result for $1 \leq \gamma < 2$ and a Law of iterated logarithms, if the random coefficients $A_k(\omega)$ are Fourier coefficients of the Wiener process $X(t,\omega)$ and a_k are the Fourier coefficients of an $L^2[0,1]$ function. If the weight a_k are considered to be Fourier coefficient of bounded function $|f| \leq 1$ then the growth rate is sharper, he obtained the exact growth rate of $s_n(t,\omega)$.

In our recent work [4], we have established the convergence of the series

$$\sum_{k=0}^{\infty} a_k A_k(\omega)\phi_k(t), \tag{1.4}$$

in orthogonal Hermite functions $\phi_k(t)$. Here a_k are Fourier–Hermite coefficients of functions in different L^2 spaces, $A_k(\omega)$ are random Fourier-Hermite coefficients associated with Wiener process and symmetric stable process of index 2. Also we have shown that the Fourier transform of these series exist. The growth rate of these random Fourier series and their Fourier transforms are presented in this paper. The following definitions are needed to discuss the growth rate of these series.

Definition 1 : *A sequence of random variables X_n obeys the law of iterated logarithm if*

$$P\left[\lim\ \sup \frac{\sum_{k=1}^{n} X_k}{a_n} = 1\right] = 1$$

for a sequence of numbers $\{a_n\}$

When this holds good, the sum $\sum_{k=1}^{n} X_k$ is said to have a growth rate like a_n. The following lemma is required to prove the exact growth of a random series, related to a positive valued function in $L^2(\mathbb{R})$.

Lemma 1 *[[10], p. 250] If x_i, $i \geq 1$ is an independent and normally distributed random variables with mean zero and finite variance a^2, then*

$$P\left[\lim\ \sup \frac{\sum x_i}{\sqrt{2na^2 \log\log na^2}} = 1\right] = 1. \tag{1.5}$$

2. Growth of random Fourier series

Consider the series $\sum_{k=0}^{\infty} c_k C_k(\omega)\psi_k(t)$, where $\psi_k(t)$ are normalized Hermite–Gaussian functions, the scalars c_k are the FH coefficients of a function f in $L^2(\mathbb{R})$ defined as $\int_{-\infty}^{\infty} f(t)\psi_n(t)dt$ and $C_k(\omega)$ are random variables defined as

$$\int_{-\infty}^{\infty} \psi_k(s)dX(s,\ \omega) \tag{2.1}$$

which exist[4]. As we know the series converges, let its sum be denoted as

$$F(t,\ \cdot) = \int_{-\infty}^{\infty} f(s,\ t)dX(s,\ \cdot). \tag{2.2}$$

The following theorem establishes the exact growth rate of the series $\sum_{k=0}^{\infty} c_k C_k(\omega)\psi_k(t)$.

Theorem 2 *Let $X(t,\ \omega)$ be a symmetric stable process of index 2. Let $\sum_{k=0}^{\infty} c_k C_k(\omega)\psi_k(t)$ be the random Fourier–Hermite series related to a positive valued function $f \in L^2(\mathbb{R})$ space. Then*

$$P\left[\limsup \frac{S_n^{\psi}(t,\ \omega)}{\sqrt{E_n^2 c^2 \log(\log E_n^2 c^2)}} = 1\right] = 1,$$

where $S_n^{\psi}(t,\ \omega) := \sum_{k=0}^{n} c_k C_k(\omega)\psi_k(t)$, $E_n^2 := \sum_{k=0}^{n} |c_k \delta_k|^2$ and c^2 is a constant associated with the normal law of increment of symmetric stable process $X(t,\ \omega)$.

Proof. The process $X(t,\omega)$ is of index 2. So the random variables $C_k(\omega)$ are independent and normally distributed random variables with mean zero and finite variance. This implies $S_n^{\psi}(t,\omega)$ are independent and normally distributed with mean zero. Since f is a positive valued $L^2(\mathbb{R})$ function, there exists a sequence of functions $\{g_n\}$ in $C_c(\mathbb{R})$ such that $g_1 \subseteq g_2 \subseteq \ldots \subseteq g_n \ldots \subseteq f$. The fact that

$$E\left|\int_a^b f(t)dX(t,\ \omega)\right| = \int_a^b |f(t)|^2 dt, \qquad and -\infty < a < b < \infty, \tag{2.3}$$

and $g_n \neq 0$ in a compact set, the positivity of f implies

$$E\left|\int_{-\infty}^{\infty} f(t)dX(t,\ \omega)\right|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt.$$

Now, partial sum $S_n^{\psi}(t,\ \omega) := \sum_{k=0}^{n} c_k C_k(\omega)\psi_k(t)$ has variance

$$\begin{aligned}
&\sum_{k=1}^{n} E|c_k C_k(\omega)\psi_k(t)|^2 \\
&= \sum_{k=1}^{n} E\left|c_k\left(\int_{-\infty}^{\infty}\psi_k(s)dX(s,\ \omega)\right)\psi_k(t)\right|^2 \\
&= \sum_{k=1}^{n} E\left|\int_{-\infty}^{\infty} c_k\psi_k(t)\psi_k(s)dX(s,\ \omega)\right|^2 \\
&= c^2 \sum_{k=1}^{n} |c_k|^2|\psi_k(t)|^2 \int_{-\infty}^{\infty} |\psi_k(s)|^2 ds \\
&= \sum_{k=1}^{n} |c_k|^2|\delta_k|^2 c^2 < \infty, \text{ as } |\psi_k(t)| = |\delta_k| \leq 0.816.
\end{aligned}$$

Hence, by Lemma 1

$$P\left[\limsup \frac{S_n^{\psi}(t,\ \omega)}{\sqrt{E_n^2 c^2 \log(\log E_n^2 c^2)}} = 1\right] = 1.$$

Similarly, the exact growth rate of the random Fourier transform $\mathcal{F}(F(t,\ \omega)) = \sum_{k=0}^{n} c_k C_k(\omega)\lambda_k\psi_k(t)$ in Hermite functions described in section 1 can be estimated. Let the partial sum of the random Fourier–Hermite transform $\mathcal{F}(F(t,\omega))$ be $\mathcal{F}(S_n^{\psi}(t,\ \omega)) :=$

$\sum_{k=0}^{n} c_k C_k(\omega)\lambda_k\psi_k(t)$. The growth rate of $\mathcal{F}(F(t,\ \omega))$ is stated in the following theorem. Its proof is similar to that of the proof of the previous theorem, since λ_k are of absolute value 1.

Theorem 3 *The random Fourier–Hermite transform satisfies the following estimates*

$$P\left[\limsup \frac{|\mathcal{F}(S_n^{\psi}(t,\ \omega))|}{\sqrt{E_n^2\delta^2\log(\log E_n^2c^2)}} = 1\right] = 1$$

if $\mathcal{F}(S_n^{\psi}(t,\ \omega))$ is independent and normally distributed random variable having mean zero and finite variance. Here $E_n^2 := \sum_{k=0}^{n} c_k^2$ and c^2 and c_k are same as in the previous theorem.

Now we will consider the series $\sum_{k=0}^{\infty} b_k B_k(\omega)\tilde{\psi}_k^{\alpha}(t)$, where $\tilde{\psi}_n^{\alpha}(t)$ are the transformed Hermite functions equivalent to

$$\psi_n^{\alpha}\circ\phi(t) = \psi_n^{\alpha}(\phi(t)), \alpha > 0, \tag{2.4}$$

b_k are Fourier–Hermite coefficients of $\tilde{g}\in L^2(0,1)$ defined as

$$b_n = \int_0^1 \tilde{\psi}_n^{\alpha}(s)\tilde{g}(s)w(s)ds, \qquad s > 0$$

with $w(t) = \varphi'(t) = \frac{1}{t(1-t)}$[9]. Here

$$\psi_n^{\alpha}(t) = \frac{\sqrt{\alpha}}{\sqrt{2^n n!\sqrt{\pi}}} H_n(\alpha t)e^{-\frac{\alpha^2t^2}{2}}, \qquad n \geq 0, t \in \mathbb{R} \tag{2.5}$$

are generalized orthogonal Hermite functions. The random variables $B_k(\omega)$ are stochastic integrals

$$B_k(\omega) = \int_0^1 \tilde{\psi}_k^{\alpha}(s)\sqrt{w(s)}dW(s,\ \omega), \qquad k \in \mathbb{N}_0 \tag{2.6}$$

which are exist[4]. This series converges to the sum function $\tilde{F}(t,\ \omega) = \int_0^1 \tilde{g}(s,t)\sqrt{w(s)}dW(s,\ \omega)$ [4].

Theorem 4 *Let $W(t,\ \omega)$ be the Wiener process , $\tilde{S}_n(t,\ \omega)$ be the partial sum of the random Fourier–Hermite series $\sum_{k=0}^{\infty} b_k B_k(\omega)\tilde{\psi}_k^{\alpha}(t)$. Then the following estimate is satisfied*

$$P\left[\limsup \frac{|\tilde{S}_n(t,\ \omega)|}{\sqrt{E_n c^2\log(\log E_n c^2)}} = 1\right] = 1,$$

where $E_n := \sum_{k=0}^{n} b_k^2$ and c^2 is a constant associated with the normal law of increment of $W(t,\ \omega)$.

Proof. In ([4], Theorem 1), we have proved that, $B_k(\omega)$ are independent. These $B_k(\omega)$ are normally distributed with mean zero and with finite variance. This implies $\tilde{S}_n(t,\ \omega)$ are independent and normally distributed with mean zero. The variance of $\tilde{S}_n(t,\omega)$ is

$$\begin{aligned}
&\sum_{k=1}^{n} E|b_k B_k(\omega)\tilde{\psi}_k^{\alpha}(t)|^2 \\
&= \sum_{k=1}^{n} E|b_k(\int_0^1 \tilde{\psi}_k^{\alpha}(s)\sqrt{\mathrm{w}(s)}dX(s,\ \omega))\tilde{\psi}_k^{\alpha}(t)|^2 \\
&= \sum_{k=1}^{n} E|b_k(\int_0^1 \tilde{\psi}_k^{\alpha}(s)\sqrt{\mathrm{w}(s)}\tilde{\psi}_k^{\alpha}(t)dW(s,\ \omega))|^2 \\
&= c^2\sum_{k=1}^{n} \int_0^1 |\tilde{\psi}_k^{\alpha}(s)\tilde{\psi}_k^{\alpha}(t)\sqrt{\mathrm{w}(s)}|^2 ds \\
&= \sum_{k=1}^{n} b_k^2c^2 < \infty,
\end{aligned}$$

as b_k are the Fourier coefficients of transformed Hermite function in $L^2(0,\ 1)$ space. Thus,

$\tilde{S}_n(t,\ \omega)$ is a sum of independent random variables with finite variance and mean zero. Hence by using Lemma 1,

$$P\left[\limsup \frac{|\tilde{S}_n(t,\ \omega)|}{\sqrt{\sum_{k=1}^{n} b_k^2 c^2 \log(\log \sum_{k=1}^{n} b_k^2 c^2)}} = 1\right] = 1.$$

As $E_n = \sum_{k=1}^{n} b_k^2 c^2$, we obtain the result.

In ([4], Theorem 3), it is shown that, the Fourier transform of the random series $\tilde{F}(t, \omega) := \sum_{k=0}^{\infty} b_k B_k(\omega) \tilde{\psi}_k^{\alpha}(t)$ is

$$\mathcal{F}(\tilde{F}(t,\ \omega)) := \sum_{k=0}^{\infty} b_k B_k(\omega) \frac{1}{\sqrt{\alpha}} \lambda_k \psi_k(\frac{\phi(t)}{\alpha}).$$

Denote its partial sum as

$$\tilde{T}_n(t,\ \omega) := \sum_{k=0}^{n} b_k B_k(\omega) \frac{1}{\sqrt{\alpha}} \lambda_k \psi_k(\frac{\phi(t)}{\alpha}),$$

where λ_k are eigenvalues of the transformed Hermite function. The exact growth rate of $\mathcal{F}(\tilde{F}(t,\ \omega))$ is shown below.

Theorem 5 *The random Fourier transform $\mathcal{F}(\tilde{F}(t,\ \omega))$ of the transformed Hermite series $\tilde{F}(t,\ \omega)$ satisfies*

$$P\left[\limsup \frac{|\tilde{T}_n(t,\ \omega)|}{\sqrt{E_n^1 c^2 \log(\log E_n^1 c^2)}} = 1\right] = 1,$$

where $E_n^1 := \sum_{k=0}^{n} b_k^2$ and c^2 is a constant associated with the normal law of increment of $W(t,\omega)$.

Proof. Since $B_k(\omega) := \int_0^1 \tilde{\psi}_k^{\alpha}(s) dW(s,\ \omega)$, are independent and the variance of $\tilde{T}(t,\ \omega)$ is normally distributed with mean zero and finite variance, $\tilde{T}(t,\ \omega)$ are also independent and distributed with mean zero. The variance of $\tilde{T}(t,\omega)$ is

$$\begin{aligned}
&\sum_{k=1}^{n} E|b_k B_k(\omega) \frac{1}{\sqrt{\alpha}} \lambda_k \psi_k(\frac{\phi(t)}{\alpha})|^2 \\
&= \sum_{k=1}^{n} E| \int_0^1 (\frac{1}{\sqrt{\alpha}} b_k \tilde{\psi}_k^{\alpha}(s) \lambda_k \psi_k(\frac{\phi(t)}{\alpha}) \sqrt{\mathrm{w}(s)}) dW(s,\ \omega)|^2 \\
&= \sum_{k=1}^{n} \frac{1}{\sqrt{\alpha}} b_k^2 E| \int_0^1 \tilde{\psi}_k^{\alpha}(s) \lambda_k \psi_k(\frac{\phi(t)}{\alpha}) \sqrt{\mathrm{w}(s)} dW(s,\ \omega)|^2 \\
&= \frac{1}{\alpha} c^2 \sum_{k=1}^{n} |b_k|^2 \int_0^1 |\tilde{\psi}_k^{\alpha}(s) \lambda_k \psi_k(\frac{\phi(t)}{\alpha}) \sqrt{\mathrm{w}(s)}|^2 ds \\
&= \frac{1}{\alpha} \sum_{k=1}^{n} |b_k|^2 c^2 < \infty \\
&= \sum_{k=1}^{n} \frac{1}{\alpha} b_k^2 c^2,
\end{aligned}$$

This is finite. Hence, $\tilde{T}_n(t,\ \omega)$ is a sum of independent random variable with finite variance and mean zero. By using Lemma 1,

$$P\left[\limsup \frac{|\tilde{T}_n(t,\ \omega)|}{\sqrt{\sum_{k=1}^{n} \frac{1}{\alpha} b_k^2 c^2 \delta^2 \log(\log \sum_{k=1}^{n} \frac{1}{\alpha} b_k^2 c^2)}} = 1\right] = 1.$$

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References

[1] Grawe, M., 1997., Asymptotic growth of Hermite series and an application to the theory of the Riemann zeta function, Result. Math. **31**: 105–114;
[2] Liu, Z. and Liu, S., 2007. Random fractional Fourier transform, Opt. Lett. **32**: 2088–2090; DOI: https://doi.org/10.1364/OL.32.002088;
[3] Liu, Z. and Liu, S., 2007. Randomization of Fourier transform, Opt. Lett. **32**: 478–480; DOI: https://doi.org/10.1364/OL.32.000478;
[4] Mangaraj, B. and Sahoo, S., *On random Fourier–Hermite transform associated with stochastic process*, Aligarh J. of Statistics(Accepted);
[5] Mishra, M. N., Nayak, C. and Pattanayak, S., 1980. Some continuity properties of a random Fourier–Stieltjes series, J. of Math. and Phy. science, **14**(3), pp. 259–272;
[6] Mohanty, I. P, 1999. Studies on random Fourier–Stieltjes series related to stochastic integrals and a class of random algebraic polynomial, Phd Thesis, Sambalpur University;
[7] Nayak, C., 1990. Growth rate of random Fourier–Stieltjes series associated with stable process, J. of orissa Math. Soc. **1**, pp.31–39;
[8] Nayak, C., Pattanayak, S. and Mishra, M. N., 1987. Random Fourier-Stieltjes series associated with stable process, Tohoku Math. J. **39** (1): 1–15;
[9] Saadatmandi, A. and Akbari, Z., 2017. Transformed Hermite functions on a finite interval ad their applications to a class of singular boundary value problems, Comp. Appl. Math. **36**: 1085–1098; DOI: https://doi.org/10.1007/s40314-015-0284-6;
[10] Stout, W. P., 1974. Almost sure convergence, Academic press, New York;