

# Spectral Analysis of the Adjacency Matrix and Hamiltonian Properties of Zero-divisor Graphs $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ and $\Gamma(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i])$

S. G. Jakkewad<sup>1</sup>, R. G. Metkar<sup>2</sup>, V Murgan<sup>3</sup>, P. Tekalkar<sup>4</sup>

<sup>1</sup> Karmaveer Bhaurao Patil College Vashi, Navi Mumbai

<sup>2</sup> Indira Gandhi (Sr) College Cidco, Nanded

<sup>3</sup> Karmaveer Bhaurao Patil College Vashi, Navi Mumbai

<sup>4</sup> Karmaveer Bhaurao Patil College Vashi, Navi Mumbai

<sup>1</sup>[shrikantjakkewad171991@gmail.com](mailto:shrikantjakkewad171991@gmail.com), <sup>2</sup>[rammetkarmath@gmail.com](mailto:rammetkarmath@gmail.com)

## Abstract

This paper investigates the spectral properties of complete bipartite zero-divisor graphs  $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$  and  $\Gamma(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i])$ . Our analysis's main focus is on the adjacency matrices of these graphs, proving several results on trace, singularity, and energy. Also, we demonstrate that these graphs are complete bipartite Hamiltonian graphs, containing Hamiltonian cycles of orders  $2(p - 1)$  and  $2(p^2 - 1)$  respectively.

**Keywords:** Complete bipartite Zero-divisor graph, Adjacency matrix, Spectral radius  $\rho(A)$ , Energy, Hamiltonian graph.

## 1 Introduction

Graph theory is a branch of mathematics that focuses on generating connections and relationships between various entities. A graph comprises vertices and edges, where each edge has two end vertices.

In the field of algebra, zero divisors play a significant role in understanding the structure of rings. A zero divisor is a nonzero element in a ring that, when the product of two non-zero elements, results in zero. The zero-divisor graph is a graphical representation in which each zero-divisor is considered a vertex, and an edge connects two vertices if their product is zero.

The concept of the zero-divisor graph  $\Gamma(R)$  I. Beck [3] first introduced the concept of a commutative ring, with the initial aim of exploring graph coloring. The study of zero-divisor graphs has received significant attention in algebra due to its ability to reveal the structural properties of algebraic systems.

Anderson, Frazier, Lauve, Levy, Livingston, and Shapiro [4,5,6] have extensively studied the zero-divisor graph of a commutative ring. The zero-divisor graph concept has recently been extended to non-commutative rings [7].

Graph theory encompasses a variety of concepts, with the analysis of matrices related to graphs, such as the adjacency and Laplacian matrices. Key mathematical properties derived from these matrices, including the trace and determinant, provide a deep understanding of the graph's structure and properties.

The concepts of determinant and trace have found significant applications in graph theory. William Rowan Hamilton introduced the idea of adjacency matrices, establishing a connection between determinants and graph properties through matrix representations. Later, Arthur Cayley extended

his foundational work on matrices and determinants to graph theory, particularly by studying adjacency and Laplacian matrices. James Joseph Sylvester is analysing the role of determinants. Alexander B. Kelmans further developed the application of determinants in analysing graph connectivity and the Laplacian matrix. Later, Pietro Franchi and Paolo Gronchi focused on the determinant and trace of the Laplacian matrix, emphasizing their role in finding spanning trees and studying graph connectivity. Collectively, these contributions highlight the profound interplay between algebraic tools and graph-theoretic properties.

The determinant of a matrix captures important structural information about the graph. In the case of adjacency matrices, the determinant is related to the graph's eigenvalues, which have applications in spectral graph theory.

The concept of graph energy has been a central focus in graph theory, drawing significant attention due to its roots in the spectral properties of graphs.

In recent years, the study of graph energy has achieved significant attention. Firstly, Ivan Gutman [8,9] introduced the concept of graph energy in 1978, defining it as the sum of the absolute values of all eigenvalues of a matrix derived from the graph. After that, various researchers have generated interest in the topic. Later, in 2009, Adiga and Smitha [10] put forward the concepts of the maximum degree matrix and the corresponding maximum degree energy of a graph. In 2012, Haemer [11] investigated Seidel switching and the Seidel energy of graphs. In 2014, Meenakshi and Lavanya [12] calculated twelve distinct types of energy for the complete graph  $K_4$ .

M. R. Ahmadi and R. Jahani-Nezhad [13] explored the Energy and Wiener Index of Zero-divisor graphs for the rings of integer modulo  $n$ , considering the cases for  $n = p^2$  and  $n = pq$ . Similarly, Reddy B S, Jain R, and Laxmikanth N. [14] analyzed the Eigenvalues and Wiener index of the Zero-divisor graph  $\Gamma(\mathbb{Z}_n)$ . Magi, P. M., Sr Magie Jose, and Anjaly Kishore [15] investigate  $\Gamma(\mathbb{Z}_n)$  for  $n = p^2q^2$  where  $p$  and  $q$  are primes. Additionally, Young, Matthew. [16] discussed Adjacency matrices of zero-divisor graphs of integers modulo  $n$ . In this context, Suthar, Sheela, and Om Prakash [17] focus on the neighborhood of vertex, neighborhood number and adjacency matrix of  $\Gamma(\mathbb{Z}_n)$ . Further research on zero-divisor graphs can be found in [1,2,18,19,20,21,22]. The Energy of graphs derived from finite commutative rings, specifically examining the energy of the zero-divisor graphs of the direct product of the ring of integers modulo  $p$  ( $\mathbb{Z}_p \times \mathbb{Z}_p$ ) and the ring of Gaussian integer modulo  $p$  ( $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$ ).

## 2 Preliminaries

This study introduces the essential concepts and definitions necessary for exploring the adjacency matrix of the zero-divisor graph, Hamiltonian graphs, and the complete bipartite zero-divisor graph. The following preliminaries establish the foundational framework for the main results presented in this paper. The zero-divisor graph of a commutative ring  $R$  is a graph whose vertices are nonzero zero-divisors of  $R$  such that the vertices  $x$  and  $y$  are adjacent if and only if  $xy = 0$ .

A special case let  $R$  be a commutative ring, and let  $Z(R)$  be the set of non-zero divisors of  $R$ . The complete bipartite zero-divisor graph  $\Gamma_b(R)$  is defined as

The vertex set is partitioned into two disjoint sets  $A$  and  $B$  such that  $V(\Gamma_b(R)) = A \cup B = Z(R) - \{0\}$  where every zero divisor is assigned to either set  $A$  and  $B$ .

There is an edge between every  $x \in A$  and every  $y \in B$  if and only if  $x y = 0$  in  $R$ .

The graph is complete bipartite, meaning every vertex in  $A$  is adjacent to every in  $B$ .

Another significant concept in graph theory is the Hamiltonian graph, A Hamiltonian graph is a graph that contains a Hamiltonian cycle, which is a cycle that visits every vertex of the graph exactly once and returns to the starting vertex.

The Eccentricity of the graph is the maximum distance between a vertex and all other vertices, and it is denoted by  $e(v)$ . The maximum value of eccentricity is called the diameter, whereas the minimum value of eccentricity is called the radius.

The least number of colours required for colouring of a graph  $G$  is called the chromatic number. We study the various properties of the adjacency matrix of the zero-divisor graphs. An adjacency matrix is a square matrix of  $n \times n$  size which represents a graph by showing edges between pairs of vertices. Example:  $A(a_{ij}) = 1$  if there is an edge between the vertex  $i$  and vertex  $j$ ,  $0$  if no edge exists between them. In the study of matrices, the trace of a square matrix  $A$ , denoted by  $tr(A)$ , is the sum of the diagonal elements of the matrix.

$tr(A) = a_{11} + a_{22} + \dots + a_{nn}$  where  $a_{11}, a_{22}, \dots, a_{nn}$  are the diagonal elements of the matrix  $A$ .

Another important matrix property is the spectral radius of a matrix  $A$ , denoted by  $\rho(A)$ , is the maximum absolute value of its eigenvalues. The energy of a graph  $E(G)$  is the sum of absolute values of the eigenvalues of the adjacency matrix of the zero-divisor graph.

### 3 Main results

#### 3.1 Theorem

**Statement:** let  $\mathbb{Z}_p$  be a ring with a characteristic prime, then:

- a) For all prime  $p$ , Trace of the adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p \times \mathbb{Z}_p$  is zero. i.e.  $\text{Trace}(A(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p))) = 0$
- b) For all primes  $p$  except 2, the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p \times \mathbb{Z}_p$  is singular.

**Proof:**

a) Let  $\mathbb{Z}_p = \{1,2,3, \dots, p - 1\}$  be a commutative ring with characteristic prime  $p$ .

Let  $\mathbb{Z}_p \times \mathbb{Z}_p$  be the direct product of the ring  $\mathbb{Z}_p$  with  $\text{Char}(\mathbb{Z}_p \times \mathbb{Z}_p) = p$ ,

where  $\mathbb{Z}_p \times \mathbb{Z}_p = \{(x, y) | x, y \in \mathbb{Z}_p\}$ .

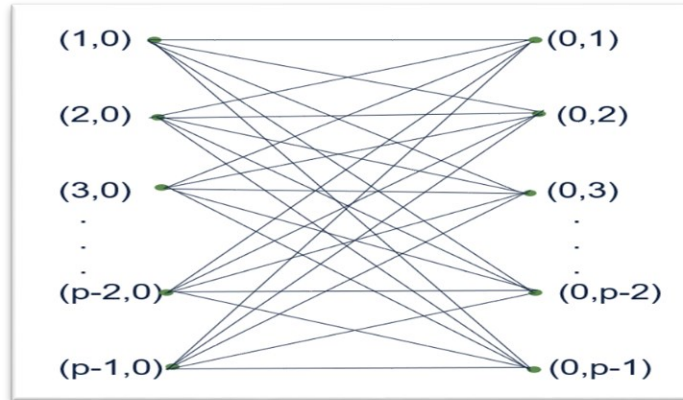
$$\text{Let } A = [a_{ij}]_{(2(p-1)) \times (2(p-1))} = \begin{bmatrix} 0 & 1 & 0 & \dots & 1 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 1 & \dots & 0 \end{bmatrix}_{(2(p-1)) \times (2(p-1))}$$

be the adjacency matrix corresponding to the zero-divisor graph of the ring  $\mathbb{Z}_p \times \mathbb{Z}_p$ . Trace of the adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p \times \mathbb{Z}_p$  i.e.  $\text{Trace}(A) = \sum_{i=1}^{2(p-1)} a_{ii}$ .

Fig.1.The zero-divisor graph  $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$

Here, we observe that there is no loop present in the zero-divisor graphs of  $\mathbb{Z}_p \times \mathbb{Z}_p$  (fig.1). Therefore, the Diagonal entries in the adjacency matrix are zero. Hence, the Trace of the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p \times \mathbb{Z}_p$  is 0.

b) Two vertices adjacent to the same set of vertices create identical rows in the zero-divisor graph  $\mathbb{Z}_p \times \mathbb{Z}_p$ . Therefore, The Adjacency matrix of the zero-divisor graph  $\mathbb{Z}_p \times \mathbb{Z}_p$  has identical rows



and columns. By the property of determinant, if the Matrix has identical rows or columns, the matrix is singular. Hence, for all primes except 2, the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p \times \mathbb{Z}_p$  is singular.

### 3.2 Theorem

**Statement:** Let  $\mathbb{Z}_p$  be a ring with a characteristic prime  $p$ , then

- a) For all prime  $p$  except 2, the trace of the adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$  is zero. i.e  $\text{Trace}(A(\Gamma(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]))) = 0$ .
- b) For all prime  $p$ , the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$  is singular.

**Proof:**

a) Let  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$  be the direct product of the ring  $\mathbb{Z}_p[i]$  with  $\text{Char}(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]) = p$ , where  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i] = \{(a + ib, c + id) \mid a, b, c, d \in \mathbb{Z}_p\}$ .

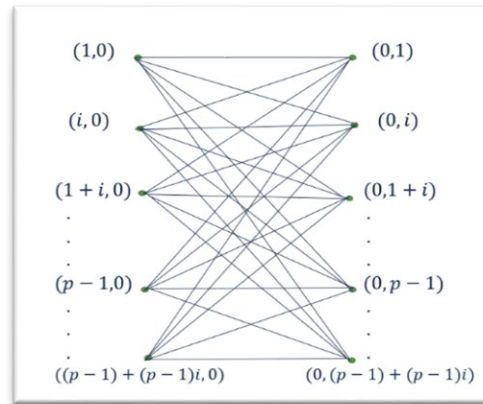
Let  $A = [a_{ij}]_{2(p^2-1) \times 2(p^2-1)}$  be adjacency matrix corresponding to the zero-divisor graph of the ring  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$ .

For all primes  $p$  except 2, the trace of the adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$ . i.e.  $\text{Trace}(A) = \sum_{i=1}^{2(p^2-1)} a_{ii}$

Here, in the following figure, we observe no loop in the zero-divisor graph of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$ .

Fig.2.The zero-divisor graph  $\Gamma(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i])$

Therefore, the diagonal entries in the adjacency matrix are zero. Hence, Trace of the Adjacency matrix of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$  is 0.



b) Let  $A = [a_{ij}]_{2(p^2-1) \times 2(p^2-1)}$  be adjacency matrix corresponding to the zero-divisor graph of the ring  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$ . In the zero-divisor graph of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$  for all primes  $p$ , two vertices adjacent to the same set of vertices create identical rows in the zero-divisor graph  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$ . Therefore, the adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$  has identical rows.

Hence, for all primes  $p$ , the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$  is singular.

### 3.3 Theorem

**Statement:** Let  $\mathbb{Z}_p$  be a ring with characteristic prime  $p$ , then for all primes  $p$ , the Energy of the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p \times \mathbb{Z}_p$  is given by:

$$E(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)) = 2(p - 1).$$

**Proof:** Let  $\mathbb{Z}_p \times \mathbb{Z}_p$  be the direct product of the ring  $\mathbb{Z}_p$  with  $\text{Char}(\mathbb{Z}_p \times \mathbb{Z}_p) = p$

Where  $\mathbb{Z}_p \times \mathbb{Z}_p = \{(x, y) \mid x, y \in \mathbb{Z}_p\}$

Let  $A = [a_{ij}]_{2(p-1) \times 2(p-1)}$  be adjacency matrix corresponding to the zero-divisor graph of the ring  $\mathbb{Z}_p \times \mathbb{Z}_p$  for all primes  $p$ .

For  $p$ , The Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p \times \mathbb{Z}_p$  is

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 1 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 1 & \dots & 0 \end{bmatrix}_{(2(p-1)) \times (2(p-1))}$$

The characteristic polynomial of the adjacency matrix  $A$  of the zero-divisor graph of  $\mathbb{Z}_p \times \mathbb{Z}_p$

$$|A - xI| = \begin{vmatrix} -x & 1 & 0 & \dots & 1 \\ 1 & -x & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 1 & 0 & 1 & \dots & -x \end{vmatrix} = x^{2(p-2)}(x - (p - 1))(x + (p - 1))$$

This implies that the eigenvalues are  $0, 0, \dots, 0, (p - 1), -(p - 1)$ .

Then by definition the energy of a of the zero-divisor graph  $E(\Gamma)$  is the sum of absolute values of the eigenvalues of the adjacency matrix of the zero-divisor graph. The energy of the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p \times \mathbb{Z}_p = E(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p))$

$$\Rightarrow E(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)) = 0 + 0 + \dots + 0 + |p - 1| + |-(p - 1)|$$

$$\Rightarrow E(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)) = (p - 1) + (p - 1) = 2(p - 1)$$

$$\Rightarrow E(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)) = 2(p - 1).$$

**Example**

Let  $p = 2$

The Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  is  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2 \times 2}$

The characteristic polynomial of the adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$|A - xI| = \begin{vmatrix} -x & 1 \\ 1 & -x \end{vmatrix} = x^2 - 1 = (x - 1)(x + 1)$$

This implies that the eigenvalues are  $1, -1$

Therefore, the Energy of the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$= E(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = |1| + |-1| = 1 + 1 = 2$$

Let  $p = 3$

The Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_3 \times \mathbb{Z}_3$  is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4 \times 4}$$

The characteristic polynomial of the adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_3 \times \mathbb{Z}_3$

$$|A - xI| = \begin{vmatrix} -x & 1 & 0 & 1 \\ 1 & -x & 1 & 0 \\ 0 & 1 & -x & 1 \\ 1 & 0 & 1 & -x \end{vmatrix} = x^2(x - 2)(x + 2)$$

This implies that the eigenvalues are  $0, 0, 2, -2$

Therefore, the Energy of the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_3 \times \mathbb{Z}_3 = E(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3))$

$$\Rightarrow E(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 0 + 0 + |2| + |-2| = 2 + 2 = 4$$

3.4 Theorem

**Statement:** For all prime  $p$  except 2, the Energy of the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$  is  $2(p^2 - 1)$

**Proof:** Let  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$  be the direct product of the ring  $\mathbb{Z}_p[i]$  with  $\text{Char}(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]) = p$

Where  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i] = \{(a + ib, c + id) \mid a, b, c, d \in \mathbb{Z}_p\}$

Let  $A = [a_{ij}]_{m \times m}$  The adjacency matrix corresponding to the zero-divisor graph of the ring  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$

In the graph of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$  for all prime  $p$  except 2

For prime  $p$ , the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$  is

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 1 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 1 & \dots & 0 \end{bmatrix}_{2(p^2-1) \times 2(p^2-1)}$$

The characteristic polynomial of the adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$

$$\begin{aligned} |A - xI| &= \begin{vmatrix} -x & 1 & 0 & \dots & 1 \\ 1 & -x & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 1 & \dots & -x \end{vmatrix} \\ &= x^{2(p^2-2)}(x - (p^2 - 1))(x + (p^2 - 1)) \end{aligned}$$

Eigenvalues are  $0, 0, \dots, 0, (p^2 - 1), -(p^2 - 1)$

The Energy of the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$

$$\begin{aligned} &= |0| + |(p^2 - 1)| + |-(p^2 - 1)| \\ &= (p^2 - 1) + (p^2 - 1) \\ &= 2(p^2 - 1) \end{aligned}$$

**Example**

Let  $p = 3$

The Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_3[i] \times \mathbb{Z}_3[i]$  is  $A = \begin{bmatrix} 0 & 1 & 0 & \dots & 1 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 1 & \dots & 0 \end{bmatrix}_{16 \times 16}$

The characteristic polynomial of the adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_3[i] \times \mathbb{Z}_3[i]$

$$|A - xI| = \begin{vmatrix} -x & 1 & 0 & \dots & 1 \\ 1 & -x & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 1 & 0 & 1 & \dots & -x \end{vmatrix} = x^{14}(x - 8)(x + 8)$$

Eigenvalues are 0(14th times), 8, -8

The Energy of the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_3[i] \times \mathbb{Z}_3[i]$

$$= |0| + |0| + \dots + |0| + |8| + |-8| = 8 + 8 = 2(8) = 16$$

For  $p = 5$

The Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_5[i] \times \mathbb{Z}_5[i]$  is  $A = \begin{bmatrix} 0 & 1 & 0 & \dots & 1 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 1 & \dots & 0 \end{bmatrix}_{48 \times 48}$

The characteristic polynomial of the adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_5[i] \times \mathbb{Z}_5[i]$

$$|A - xI| = \begin{vmatrix} -x & 1 & 0 & \dots & 1 \\ 1 & -x & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 1 & 0 & 1 & \dots & -x \end{vmatrix} = x^{46}(x - 24)(x + 24)$$

Eigenvalues are 0(46th times), 24, -24

The Energy of the Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_5[i] \times \mathbb{Z}_5[i] = |0| + |0| + \dots + |0| + |24| + |-24| = 24 + 24 = 2(24) = 48$

**Corollary 1:** For all primes  $p$ , the Spectral radius of the adjacency matrix of the zero-divisor graph  $\rho(A(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p))) = (p - 1)$ .

**Example:** Let  $\mathbb{Z}_3 \times \mathbb{Z}_3$  be a direct product of the ring  $\mathbb{Z}_3$ . The set of zero divisors of  $\mathbb{Z}_3 \times \mathbb{Z}_3$  is  $\{(1,0), (0,1), (0,2), (2,0)\}$ .

The Adjacency matrix of the zero-divisor graph of  $\mathbb{Z}_3 \times \mathbb{Z}_3$  is  $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4 \times 4}$

From the matrix we get,

Spectral radius adjacency matrix of the zero-divisor graph  $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$  is 2.

**Corollary 2:** For all primes except 2, the Spectral radius of the adjacency matrix of the zero-divisor graph  $\rho(A(\Gamma(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]))) = (p^2 - 1)$ .

### 3.5 Theorem

**Statement:** For all prime  $p$  except 2, let  $\mathbb{Z}_p$  and  $\mathbb{Z}_p[i]$  be two finite commutative rings with characteristic  $p$  then:

- a) The complete bipartite zero divisor graph  $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$  is a Hamiltonian graph and contains Hamiltonian cycles of order  $2(p - 1)$ .
- b) The complete bipartite zero divisor graph  $\Gamma(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i])$  is a Hamiltonian graph and contains Hamiltonian cycles of order  $2(p^2 - 1)$ .

**Proof:** a) Let  $\mathbb{Z}_p$  be the finite commutative rings, where  $\mathbb{Z}_p = \{1,2,3, \dots, p - 1\}$  be the commutative ring with characteristic prime  $p$ .

Let  $\mathbb{Z}_p \times \mathbb{Z}_p$  be the Direct product of the ring  $\mathbb{Z}_p$  with  $\text{Char}(\mathbb{Z}_p \times \mathbb{Z}_p) = p$

where  $\mathbb{Z}_p \times \mathbb{Z}_p = \{(x, y) \mid x, y \in \mathbb{Z}_p\}$  where  $\mathbb{Z}_p \times \mathbb{Z}_p$  has  $2(p - 1)$  non-zero-divisor elements

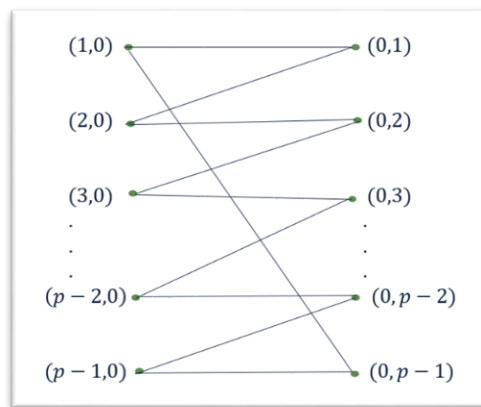
Fig.3.The Hamiltonian cycle of  $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$

For all prime  $p$  except 2, we observe that the  $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$  is a complete bipartite graph, i.e. the graph has two disjoint sets of vertices X and Y. By definition, a Hamiltonian graph contains a closed path, or cycle, that visits each vertex exactly once. This closed path is also known as a Hamiltonian cycle.

And precisely we observe that the  $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$  contain Hamiltonian cycle  $(1,0) - (0,1) - (2,0) - (0,2) - (3,0) - (0,3) - \dots - (p - 1,0) - (0,p - 1) - (1,0)$  of length  $2(p - 1)$ .

From this, we can say that the graph contains a Hamiltonian cycle, a closed path that visits each vertex exactly once and ends at the starting vertex. This implies that the zero-divisor graph  $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$  is the Hamiltonian graph for all primes except 2.

- b) Let  $\mathbb{Z}_p[i]$  be the finite commutative ring of Gaussian integers modulo  $p$ .



$\mathbb{Z}_p[i] = \{a + ib \mid a, b \in \mathbb{Z}_p\}$  be the commutative ring with characteristic prime  $p$ .

Let  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$  be the Direct product of the ring  $\mathbb{Z}_p[i]$  with  $\text{Char}(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]) = p$

Where  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i] = \{(a + ib, c + id) \mid a, b, c, d \in \mathbb{Z}_p\}$ .

We observe that the  $\Gamma(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i])$  is a complete bipartite graph i.e. the graph has two disjoint sets of vertices X and Y.

The number of vertices in each disjoint set is equal i.e. for  $K_{(p^2-1, p^2-1)}$ , where  $2(p^2 - 1)$  are the

number of vertices in disjoint sets.

Fig.4. The Hamiltonian cycle of  $\Gamma(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i])$

By definition of Hamiltonian graph, we know that a complete bipartite graph  $K_{m,n}$  is a Hamiltonian graph if and only if  $m = n$ . And precisely we observe that there is a cycle or closed walk present in the  $\Gamma(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i])$ ,

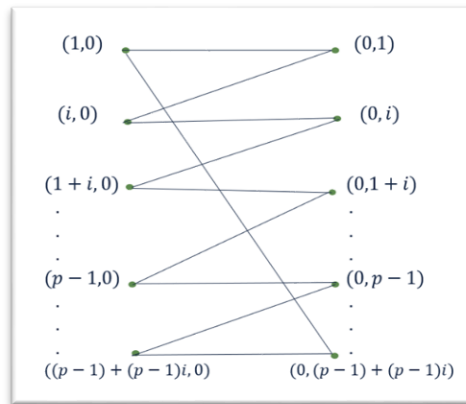
Where  $(1,0)$  is adjacent with  $(0,1)$ ,  $(0,1)$  is adjacent with  $(i, 0)$ ,  $(i, 0)$  is adjacent with  $(0, i)$ ,  $(0, i)$  is adjacent with  $(1 + i, 0)$ . This pattern continues such that,

$(p - 1, 0)$  is adjacent with  $(0, p - 1)$ , and so on,

$((p - 1) + (p - 1)i, 0)$  is adjacent with  $(0, (p - 1) + (p - 1)i)$ .

Finally,  $(0, (p - 1) + (p - 1)i)$  is adjacent with  $(1,0)$ . Completing the cycle of length  $2(p^2 - 1)$ .

Which is the largest Hamiltonian cycle, a closed path that visits each vertex exactly once and ends at the starting vertex. This implies that  $\Gamma(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i])$  is a Hamiltonian graph.



#### 4 Conclusion

This study contributes to the understanding of zero-divisor graphs of commutative rings  $\mathbb{Z}_p \times \mathbb{Z}_p$  and  $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$ , particularly spectral properties of their adjacency matrix and Hamiltonian properties.

These findings have potential applications in the zero-divisor graph theory of commutative rings. Future research could extend these results to higher products.

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