ASYMPTOTIC STABILIZATION FOR DISCRETE TYPE STOCHASTIC DYNAMIC SYSTEM INCORPORATING DELAY

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ABSTRACT

This paper investigates the stability of differential equations of stochastic type by achieving the stability condition for the corresponding stochastic difference equation. By considering the stochastic differential equation that characterises the dynamics of a single isolated neurone involving delay, the system formulation is created. In order to discretise the stochastic differential equation, the Euler-Maruyama Method is utilised. And with the aid of theorems and appropriate assumptions, the desired stability is attained. To demonstrate the effectiveness of the proposed asymptotic stability result for the obtained theoretical results, we provide a numerical example.

Keyword: Difference equation, stochastic difference equation, martingale sequence, Lyapunov-krasovkii functional, Neural networks.

Subject Code: 39A10, 39A50, 60G42, 37B25, 92B20.

1. INTRODCTION

The fact that difference equations are widely used in many different real-world situations has drawn a lot of attention to them. Differential equations in neural networks with delays are known to have a wide range of applications in specialised fields like pattern recognition, image processing, signal processing, associative memory, and optimisation problems. Applications like these rely heavily on the system's equilibrium point being stable. The stability of the system's specified equilibrium point is crucial to these applications. As a result, stability analysis is required for digital signal processing architecture and applications. Time delays occur in hardware enactments of recurrent neural networks because of the amplifiers' limited switching speed and communication duration. Research on the global asymptotic or exponential stability of neural networks with temporal delays has been conducted extensively in recent years. For these applications, the stability of the system's designated equilibrium point is indispensable. As a result, stability analysis is required for digital signal processing architecture and applications. Time delays occur in hardware enactments of recurrent neural networks because of the amplifiers limited switching speed and communication duration. Recently, a lot of work has been done to study the global asymptotic or exponential stability of neural networks with temporal delays. In particular, there is growing interest in studying differential equations with discrete and distributed delays [6,7]. Furthermore, an intended system's stability might be disrupted by numerous random disturbances that occur during computation execution. Convincing stochastic inputs have the potential to stabilise or destabilise an intended system. But real systems also have impulsive effects in addition to stochastic ones. Therefore, it is important to take into account both the impulsive and stochastic effects on the system of difference equations [11,12]. When working with linear stochastic differential equations, a number of authors have investigated the mean-square asymptotic stability of numerical methods, such as [7,12,14,15]. Nevertheless, the nearly certain asymptotic stability of numerical methods has received little attention.

Motivated by the previous conversations, the primary goal of The purpose of this study is to examine the global asymptotic stability of a system of differential equations that characterises a neuron's dynamics. This paper investigates the practically assured asymptotic stability of the strong Euler-Maruyama approach for the nonlinear scalar stochastic differential equation. Since the solutions to the continuous issue exhibit the same asymptotic behaviour under the requirements from Theorem 3.1, it is evident that the A worthy discrete model is the realised difference equation. The obtained difference equation is a worthy discrete model that is based on a number of well-known inequalities and the Lyapunov-Krasovskii functional approach thus construct novel stability requirements for the stochastic difference equation. The usefulness of the suggested stability conclusion is demonstrated with an example using numerical simulation data.

2. MATHEMATICAL FORMULATION

The system of stochastic differential equations that uses delay differential equations to explain the dynamics of an isolated neurone is represented by the following.

$$dy(t) = [-y(t) + \vartheta\varphi(y(t) - \mu y(t-\tau)]dt + \psi\big(t, y(t-\tau)\big)dW_t, \qquad t \geq 0 \qquad (1)$$

where $y(t) -$ is the amount of activation of a neurone at time t.

$\vartheta -$ is constant that characterises the $y(t)$ variable's range

$\mu -$ is a measure used to characterise the impact of historical events.

$\tau -$indicates the delay

$\varphi -$ The neuron's activation function

$\psi -$ intensity of the noise.

W_t −Wiener process

as well as the constants $\vartheta \in \mathbb{R}^+, \mu \geq 0$ and $\tau \in [0, \infty)$. Additionally, the aforementioned system builds a neural network model.

Examining the associated discretised form of the equation yields the stability of the previously mentioned system. Therefore, using Euler-Maruyama, the discretisation of the abovementioned model is provided by

$$y_{k+1} = (1-d)y_k + \vartheta d\phi(y_k - \mu y_{k-\tau}) + \sqrt{d}\psi_{k,y_{k-\tau}}\xi_{k+1} \qquad , k \in \mathbb{N}_0 \qquad (2)$$

using the arbitrary nonrandom beginning value $y_0 \in \mathbb{R}$. The size of the mesh is $d \in (0,1]$. Since ϕ −is a continuous real-valued function, So that

$$|\varphi(x)| \leq |x| \qquad (3)$$

and

$$|\vartheta|(1+|\mu|) < 1 \qquad (4)$$

ξ_k −are the independent random variables appropriately selected in relation to the mean $E\xi_k = 0$ and variance expressed in terms of unit, $E{\xi_k}^2 = 1$.

3. MAIN RESULTS:

Lemma 3.1: Cosider $\{x_k\}_{k\in\mathbb{N}}$ to be a sequence of independent $\mathcal{F}_k$-measurable random variables, where $E[x_k] = 0$ and $\boldsymbol{E}|x_k| < \infty$. Also consider $\{y_k\}_{k\in\mathbb{N}}$ to be a sequence of $\mathcal{F}_k$-measurable random variables chosen so that $\boldsymbol{E}|y_{k-1}|x_k| < \infty$ for all $k \geq 1$. Then$\{Z_k\}_{k\in\mathbb{N}}$,

$$Z_k = \sum_{r=1}^{k} y_{r-1}x_r$$

Is true for every $k \in \mathbb{N}$, to be a $\mathcal{F}_k$- *martingale* and $\{y_{k-1}x_k\}$ is to be a $\mathcal{F}_k$- *martingale difference*.

Lemma 3.2: Consider $\{W_k\}_{k\in\mathbb{N}}$ to be a non-negative $\mathcal{F}_k - martingale$ process, $\mathbb{E}|W_k| < \infty \; \forall k \in \mathbb{N}$ and

$$W_{k+1} \leq W_k + u_k - v_k + v_{k+1}\,, \qquad k \in \mathbb{N}_{0,}$$

Such that $\{v_k\}_{k\in\mathbb{N}}$ is an $\mathcal{F}_k - martingale\, difference$, $\{u_k\}_{k\in\mathbb{N}}, \{v_k\}_{k\in\mathbb{N}}$ are considered as nonnegative $\mathcal{F}_k - martingale$ process, $\mathbb{E}|u_k|, \mathbb{E}|v_k| < \infty \; \forall k \in \mathbb{N}$. Then

$$\left\{\omega: \sum_{k=1}^{\infty} u_k < \infty l\right\} \subseteq \left\{\omega: \sum_{k=1}^{\infty} v_k < \infty\right\} \cap \{W_k \rightarrow\}.$$

where $\{W_k \rightarrow\}$ refers to the set of all $\omega \in \Omega$, provided $\lim_{k\to\infty} W_k(\omega) = 0$ will exist and it is finite.

Theorem 3.1: Assume that y_k is the equation (2) solution under the given conditions.

$$\left|\psi_{k,y_k}\right| \le \gamma_k |y_k|^2 + {\eta_k}^2, \qquad \psi_{k,0} = 0 \tag{5}$$

Where $\sum_{i=1}^{\infty} {\eta_i}^2 < \infty$ and

$$\vartheta^2(1+|\mu|)^2 + \gamma_k < 1 \tag{6}$$

Th Equation (4) is met. Then, practically $\lim_{k\to\infty} y_k = 0$ is almost everywhere.

Proof:

Consider equation (2),

$$y_{k+1} = (1-d)y_k + \vartheta d\phi(y_k - \mu y_{k-\tau}) + \sqrt{d}\psi_{k,y_{k-\tau}}\xi_{k+1} \qquad , k \in \mathbb{N}_0$$

When we square both sides, we obtain

$$\begin{aligned}{y_{k+1}}^2 &= [(1-d)y_k + \vartheta d\phi(y_k - \mu y_{k-\tau})]^2 + 2[(1-d)y_k + \vartheta d\phi(y_k - \mu y_{k-\tau})]\sqrt{d}\psi_{k,y_{k-\tau}}\xi_{k+1} \\ &\quad + d\psi^2{}_{k,y_{k-\tau}}{\xi_{k+1}}^2 \\ &= [(1-d)y_k + \vartheta d\phi(y_k - \mu y_{k-\tau})]^2 + d\psi^2{}_{k,y_{k-\tau}} + \delta_{k+1}. \end{aligned} \tag{7}$$

where $\{\delta_k\}_{k\in\mathbb{N}}$ is characterised as a $\mathcal{F}_n$-martingale difference, and its value is

$$\delta_{k+1} = 2[(1-d)y_k + \vartheta d\phi(y_k - \mu y_{k-\tau})]\sqrt{d}\psi_{k,y_{k-\tau}}\xi_{k+1} + d\psi^2{}_{k,y_{k-\tau}}\{{\xi_{k+1}}^2 - 1\} \tag{8}$$

Based on our assumption, we have ${\xi_k}^2 = 1$. Therefore, $\{E{\xi_k}^2 - 1\}$ $\mathcal{F}_{k+1}$-represents a martingale difference. From the lemma, it can be concluded that δ_{k+1} constitutes a $\mathcal{F}_{k+1}$-martingale-difference.
Now

$$\sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{n} |x_i y_i| \le \sum_{i=1}^{n} \sqrt{|x_i|}(\sqrt{|x_i|}\,|y_i|)$$

According to Holder's inequality, we can derive the following:

$$\sum_{i=1}^{n}|x_i y_i| \leq \sum_{i=1}^{n}\sqrt{|x_i|}(\sqrt{|x_i|}\,|y_i|) \leq \sqrt{\sum_{i=1}^{n}|x_i|}\sqrt{\sum_{i=1}^{n}|x_i|\,|y_i|^2}$$

Now consider,

$$|(1-d)y_k + \vartheta d\phi(y_k - \mu y_{k-\tau})| \leq |(1-d)y_k| + |\vartheta|d|\phi(y_k - \mu y_{k-\tau})|$$

$$\leq |(1-d)y_k| + |\vartheta|d|y_k - \mu y_{k-\tau}|$$

$$\leq (1-d)|y_k| + |\vartheta|d[|y_k| + |\mu||y_{k-\tau}|]$$

$$\leq (1-d+|\vartheta|d)|y_k| + |\vartheta|d|\mu||y_{k-\tau}|$$

Here let $n = 2 \quad x_1 = 1-d+|\vartheta|d, \quad y_1 = |y_k|, \quad x_2 = |\vartheta|d|\mu|, \quad y_2 = |y_{k-\tau}|,$

$$\leq [1-d+|\vartheta|d+|\vartheta|d|\mu|]\,[(1-d+|\vartheta|d)|y_k|^2 + |\vartheta|d|\mu||y_{k-\tau}|^2] \qquad (9)$$

substituting (9) in (7) we have

$$y_{k+1}{}^2 \leq [1-d+|\vartheta|d+|\vartheta|d|\mu|]\,[(1-d+|\vartheta|d)|y_k|^2 + |\vartheta|d|\mu||y_{k-\tau}|^2] + d\psi^2{}_{k,y_{k-\tau}} + \delta_{k+1}$$

$$\leq [1-d+|\vartheta|d+|\vartheta|d|\mu|]\,[(1-d+|\vartheta|d)|y_k|^2 + |\vartheta|d|\mu||y_{k-\tau}|^2] + d\gamma_k|y_k|^2 + d\eta_k{}^2 + \delta_{k+1}$$

$$\leq [(1-d+|\vartheta|d+|\vartheta|d|\mu|)]\,[(1-d+|\vartheta|d)|y_k|^2 + |\vartheta|d|\mu||y_{k-\tau}|^2] + d\gamma_k|y_k|^2 + d\eta_k{}^2 + \delta_{k+1}$$

$$\leq [(1-d+|\vartheta|d+|\vartheta|d|\mu|)(1-d+|\vartheta|d) + d\gamma_k]|y_k|^2 + [d|\vartheta||\mu|(1-d+|\vartheta|d+|\vartheta|d|\mu|)]|y_{k-\tau}|^2 + d\eta_k{}^2 + \delta_{k+1}$$

Let $a = d|\vartheta||\mu|(1-d+|\vartheta|d+|\vartheta|d|\mu|)$

Let us take

$$v(k)^{(i)} = a\sum_{s=k-\tau}^{k-1} y_s{}^2$$

$$v(k) = y_k{}^2 + v(k)^{(i)}$$

Consider,

$$\Delta v(k)^{(i)} = v(k+1)^{(i)} - v(k)^{(i)}$$

$$= a\sum_{s=k+1-\tau}^{k} {y_s}^2 - a\sum_{s=k-\tau}^{k-1} {y_s}^2$$

$$= a{y_k}^2 - a{y_{k-\tau}}^2$$

Now consider

$$\Delta v(k) = {y_{k+1}}^2 - {y_k}^2 + \Delta v(k)^{(i)}$$

$$= {y_{k+1}}^2 - {y_k}^2 + a{y_k}^2 - a{y_{k-\tau}}^2$$

$$\leq [(1-d+|\vartheta|d+|\vartheta|d|\mu|)(1-d+|\vartheta|d)+d\gamma_k]|y_k|^2 + [d|\vartheta||\mu|(1-d+|\vartheta|d+|\vartheta|d|\mu|)]|y_{k-\tau}|^2 + d{\eta_k}^2 + \delta_{k+1} - {y_k}^2 + a{y_k}^2 - a{y_{k-\tau}}^2$$

$$\leq [a-1+(1-d+|\vartheta|d+|\vartheta|d|\mu|)(1-d+|\vartheta|d)+d\gamma_k]|y_k|^2 + d{\eta_k}^2 + \delta_{k+1}$$

$$\leq [d|\vartheta||\mu|(1-d+|\vartheta|d+|\vartheta|d|\mu|) - 1 + (1-d+|\vartheta|d+|\vartheta|d|\mu|)(1-d+|\vartheta|d)+d\gamma_k]|y_k|^2 + d{\eta_k}^2 + \delta_{k+1}$$

$$\leq \left(\left[1-d\left(1-|\vartheta|(1+|\mu|)\right)\right]^2 - 1 + d\gamma_k\right)|y_k|^2 + d{\eta_k}^2 + \delta_{k+1}) \qquad (10)$$

From (6) and for every $d \in (0,1]$ we obtain,

$0 < 1-|\vartheta|(1+|\mu|) < 1$,$0 < d[1-|\vartheta|(1+|\mu|)] < 1$

$\Rightarrow 0 < 1-d[1-|\vartheta|(1+|\mu|)] < |\vartheta|(1+|\mu|)$

Therefore,

$$(1-d[1-|\vartheta|(1+|\mu|)])^2 + d\gamma_k \leq \vartheta^2(1+|\mu|)^2 + d\gamma_k < 1.$$

Let us denote

$$\xi = \left(1 - \left[1-d\left(1-|\vartheta|(1+|\mu|)\right)\right]^2 - d\gamma_k\right)$$

From (10), we have

$$\Delta v(k) \leq \left(\left[1-d\left(1-|\vartheta|(1+|\mu|)\right)\right]^2 - 1 + d\gamma_k\right)|y_k|^2 + d{\eta_k}^2 + \delta_{k+1}$$

$$v(k+1) \leq v(k) - \xi {y_k}^2 + d{\eta_k}^2 + \delta_{k+1} \qquad (11)$$

And let, $W_k = v(k)$, $u_k = d{\eta_k}^2$,$v_k = \xi {y_k}^2$ and $v_{k+1} = \delta_{k+1}$

By applying lemma 4.2, we obtain the following results:

$$\lim_{k\to\infty} v(k) \text{ and } \xi \lim_{k\to\infty} \sum_{s=1}^{k} {y_s}^2$$

Such that they exist and almost surely finite

The objective is to demonstrate that $\lim_{k\to\infty} y_k = 0$. Assume that the limit as k approaches infinity of $y_k$is not equal to zero with a non-zero probability. We can identify a set Ω such ` ${y_{k_l}}^2 > \varsigma(x), l \in \mathbb{N}, x \in \Omega$

Let us now define,

$$\psi(k,x) = number\ of\ \ sequence\ \{k_n(x)\} \leq k$$

For all k in the set of natural numbers and x in the set Ω. As k approaches infinity, $\psi(k,x)$ approaches infinity. Thus, a contradiction is reached, as x is an element of Ω.

$$\infty > \xi \sum_{s=1}^{k} {y_s}^2(x) \geq \xi \sum_{s_l \leq k} {y_{s_l}}^2(x) \geq \varsigma(x)\xi\psi(k,x) \to \infty, \qquad k \to \infty$$

which contradicts our assumption. Hence we conclude that

$$\lim_{k\to\infty} y_k = 0$$

Hence the theorem is proved.

4. NUMERICAL EXAMPLE

This section presents a numerical example to illustrate the efficiency of the suggested asymptotic stability result outcome.

Example 5.1 The following stochastic recurrent neural networks are analysed utilising two neurones and impulses.

$$dy(t) = [-y(t) + \vartheta\varphi(y(t) - \mu y(t-\tau)]dt + \psi\big(t, y(t-\tau)\big)dW_t, \qquad t \geq 0\ , t \neq t_k \quad (12)$$

$\Delta y(t_k) = dy(t_k), t = t_k, k = 1,2, ...$

The neuron's activation function is provided by $\varphi(x) = \tanh(0.7) - 0.1\, sin\, x$, delay $\tau(t) = 05 + 0.5 \sin t$, $\vartheta = \begin{bmatrix} -0.4 & 0.3 \\ 0.5 & 0.1 \end{bmatrix}$, $\mu = \begin{bmatrix} 0.2 & -0.4 \\ -0.1 & 0.5 \end{bmatrix}$

Here, it is evident that our assumptions are satisfied by the activation function and constants.

Numerical Simulation

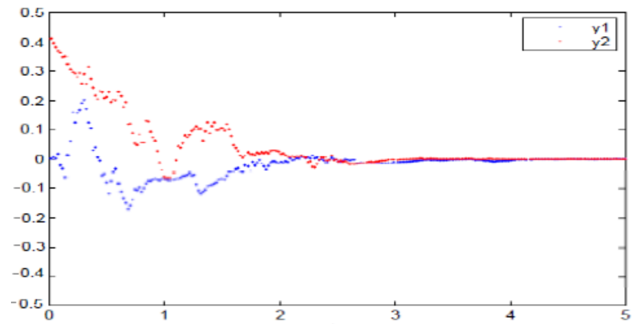

Figure: 2 Impulsive effects in the state responses of $y_1(t), y_2(t)$ of the system (12).

The system (12) fulfils all requirements specified in Theorem 3.1. Therefore, the stochastic neural network containing impulsive effects is globally asymptotically stable.

5. CONCLUSION

This study employs the Lyapunov method to analyse the stability behaviour of a system of stochastic differential equations that represents the dynamics of a single isolated neurone with delay considerations. A new set of sufficient conditions that validate the global asymptotic stability of the neural networks under study has been discovered through the use of stochastic analysis and the construction of a suitable Lyapunov function. This study could be expanded to investigate the equilibrium point's global exponential stability. Lastly, a numerical example is provided to show how our stability result can be employed.

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