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An Innovative Approach to Solve Linear Mixed Partial Fractional Differential Equations Containing More Than Two Independent Variables

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Abstract: This work introduces the Laplace Substitution Method (LSM) for solving linear mixed partial fractional differential equations involving more than two independent variables. The method transforms such complex equations into ordinary fractional differential equations by employing an analytical approach that combines repeated substitution with the Laplace transform of fractional derivatives. LSM stands out as a highly effective, practical, and robust tool for addressing these types of problems. Furthermore, its implementation is relatively straightforward. This study is expected to facilitate the analysis of linear mixed partial fractional differential equations, which frequently arise in various fields of research and technological innovation. To validate the proposed method, illustrative examples are provided. The method's effectiveness and numerical stability are further assessed through graphical analysis.

Keywords: Linear Fractional Differential Equation, Mixed Partial Fractional Differential equations, Laplace Substitution Method, Fractional Differential Equation of More than two Variables

1. Introduction

A new area of Mathematics called fractional calculus extends differentiation and integration to non-integer orders. Fractional calculus has a history nearly as long as that of classical calculus, starting with some conjectures by Leibniz (1695, 1697) and Euler (1730). On the other hand, fractional differential equations (FDEs) and fractional calculus have gained popularity recently. The gradually evolving chronicle of this ancient and yet innovative subject matter can be located in [1–5]. Fractional calculus offers more potent mathematical modelling than classical calculus for several significant phenomena. In numerous scientific and engineering fields, numerous applications have been recorded in the past few decades, such as physical, chemical, and natural systems [6], such as anomalous convection [7], ecological effects [8], infectious disease spreading [9], blood flow issues [10], control phenomenon [11], etc.

Formulating a precise solution for a particular differential equation of a physical model is a significant task in many real-world situations. Strong analytical ideas along with techniques and algorithms that yield consistent results are required. As the presence of fractional differential equations in various fields is increasing, it is essential to investigate the stability of solutions of different fractional order differential equations. Their solutions have a great deal of interest.

Our study primarily focuses on introducing the Laplace Substitution Method, a cutting-edge and reliable approach for solving linear mixed partial fractional differential equations containing more than two independent variables. S. S. Handibag and B. D. Karande initially presented the Laplace Substitution Method for partial differential equations [12]. Laplace substitution is a popular approach among mathematicians for solving several kinds of nonlinear partial differential equations with mixed derivatives. It has been demonstrated to be an effective technique for handling linear partial differential equations concerning general and specific cases. They have demonstrated the suitability and effectiveness of this technique by showing that it can solve partial integro-differential equations with mixed derivatives, as well as higher-order linear and nonlinear equations. The Laplace substitution method is applied to construct the bright and dark optical solitons solutions of the Schäfer–Wayne short-pulse equation. [13]. The approach eliminates a significant amount of calculation labour, is quickly convergent, and requires neither linearization nor discretization. The outcomes show substantial concurrence with the precise solution.

The structure of this paper is as follows. Section 2 provides the fundamental concepts, theorem, and initial insights needed for the sequel. Section 3 dealt with the Laplace substitution method for linear fractional differential equations with mixed partial derivatives, and Section 4 provided the application of method. The conclusion, which is summed up in Section 5, comes next.

2. Basic definitions

There is an extensive literature on various definitions of fractional derivatives. In this section, we give some definitions, theorems and properties of the fractional calculus theory [14, 15].

Definition 2.1: For the case of Riemann-Liouville we have the following definition:

$$D_x^{\xi}(f(x)) = \frac{1}{\Gamma(n-\xi)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\xi-1}\ f(t)dt \qquad (2.1)$$

where $'\Gamma'$ denotes gamma Function which is Mellin transform of exponential Function and is defined as

$$\Gamma y = \int_o^{\xi}\ t^{y-1}\, e^{-t}\ \mathrm{dt} \qquad \mathrm{Re}\,[y] > 0. \qquad (2.2)$$

Definition 2.2: The fractional order derivative of Function x^{σ}, $\sigma >$ -1 is given as,

$$D_x^{\xi}(x^{\sigma}) = \frac{x^{-\xi+\sigma}\,\Gamma(1+\sigma)}{\Gamma(1-\xi+\sigma)} \qquad (2.3)$$

Definition 2.3: The Laplace transform of Fractional R-L derivative is

$$L\left\{D_x^{\xi}\ (F\ (x))\right\} = s^{\xi}\mathrm{F}\ (\mathrm{S}) - \sum_{k=0}^{n-1} s^k D_x^{\xi-k-1}(0) \quad , n\text{-}1 < \xi \le n \qquad (2.4)$$

$$\text{Where } \mathrm{F(S)} = L\{(F\ (x)\} = \int_o^{\xi}\ e^{-st}F\ (t)\ dt$$

Theorem: Let f, g be $\xi-$differentiable at a point $t > 0$.

$$D^{\xi}(af + bg) = aD^{\xi}(f) + bD^{\xi}(g), \text{ for all } a, b \in \mathrm{R} \qquad (2.5)$$

3. Laplace Substitution Method:

The general from of linear fractional differential equations with mixed partial derivatives with initial conditions is given below.

$$\text{Lu}\,(x,y,t) + \text{Ru}\,(x,y,t) = \text{h}\,(x,y,t) \qquad (3.1)$$

$$D_x^{\xi-1}(0,y,t) = c_1,\ D_x^{\xi-2}\,(0,y,t) = c_2, \ldots, D_x^{\xi-n}\,(0,y,t)\,) = c_n \qquad (3.2)$$

$$D_y^{\sigma-1}\,(x,0,t) = b_1,\ D_y^{\sigma-2}\,(x,0,t) = b_2, \ldots, D_y^{\sigma-n}\,(x,0,t) = b_n \qquad (3.3)$$

$$\&\ D_t^{\tau-1}\,(x,y,0) = a_1,\ D_t^{\tau-2}\,(x,y,0) = a_2, \ldots, D_t^{\tau-n}\,(x,y,0) = a_n \qquad (3.4)$$

where, $L = \frac{\partial^{\xi+\sigma+\tau}}{\partial x^{\xi}\,\partial y^{\sigma}\partial t^{\tau}}$, Ru (x,t) is a group of remaining linear terms and h(x,y,t) is source term. $[n\text{-}1< \xi \le n,\ n\text{-}1< \sigma \le n\ \&\ n\text{-}1< \tau \le n]$

We can write (1) in the following form,

$$\frac{\partial^{\xi+\sigma+\tau}\,u(x,y,t)}{\partial x^{\xi}\,\partial y^{\sigma}\,\partial t^{\tau}} + \text{Ru}\,(x,y,t) = \text{h}\,(x,y,t)$$

$$\frac{\partial^{\xi}}{\partial x^{\xi}}\left(\frac{\partial^{\sigma}}{\partial y^{\sigma}}\left(\frac{\partial^{\tau}\,u(x,y,t)}{\partial t^{\tau}}\right)\right) + \text{Ru}\,(x,y,t) = \text{h}\,(x,y,t) \qquad (3.5)$$

Substituting $\frac{\partial^{\sigma}}{\partial x^{\sigma}}\left(\frac{\partial^{\tau}\,u(x,y,t)}{\partial t^{\tau}}\right) = U(x,y,t)$ in (3.5), we get,

$$\frac{\partial^{\xi}U(x,y,t)}{\partial x^{\xi}} + \text{Ru}\,(x,y,t) = \text{h}\,(x,y,t) \qquad (3.6)$$

Applying the Laplace transform to (3.6) with respect to $x$and initial conditions from (3.2), we get

$$U(s,y,t) = \frac{C_1}{s^{\xi}} + \frac{C_2}{s^{\xi-1}} + \ldots\ldots + \frac{C_n}{s^{\xi-n+1}} + \frac{1}{s^{\xi}}\,L_x\,\{h\,(x,y,t) - Ru\,(x,y,t)\}$$

Applying Inverse Laplace transform w. r. t. x on both sides, we get,

$$U(x,y,t) = \frac{C_1\,x^{\xi-1}}{\Gamma\xi} + \frac{C_2\,x^{\xi-2}}{\Gamma(\xi-1)} + \ldots\ldots + \frac{C_n\,x^{\xi-n}}{\Gamma(\xi-n+1)} + L_x^{-1}\left\{\frac{1}{s^{\xi}}\,L_x\,\{h\,(x,y,t) - Ru\,(x,y,t)\}\right\} \qquad (3.8)$$

But $U(x,y,t) = \frac{\partial^{\sigma}}{\partial y^{\sigma}}\left(\frac{\partial^{\tau}\,u(x,y,t)}{\partial t^{\tau}}\right)$,

$$\frac{\partial^{\sigma}}{\partial y^{\sigma}}\left(\frac{\partial^{\tau}\,u(x,y,t)}{\partial t^{\tau}}\right) = \frac{C_1\,x^{\xi-1}}{\Gamma\xi} + \frac{C_2\,x^{\xi-2}}{\Gamma(\xi-1)} + \ldots + \frac{C_n\,x^{\xi-n}}{\Gamma(\xi-n+1)} + L_x^{-1}\left\{\frac{1}{s^{\xi}}\,L_x\,\{h\,(x,y,t) - Ru\,(x,y,t)\}\right\} \qquad (3.9)$$

Put $\frac{\partial^{\tau}\,u(x,y,t)}{\partial t^{\tau}} = V(x,y,t)$ in (3.9), we get,

$$\frac{\partial^{\sigma}V(x,y,t)}{\partial y^{\sigma}} = \frac{C_1\,x^{\xi-1}}{\Gamma\xi} + \frac{C_2\,x^{\xi-2}}{\Gamma(\xi-1)} + \ldots + \frac{C_n\,x^{\xi-n}}{\Gamma(\xi-n+1)} + L_x^{-1}\left\{\frac{1}{s^{\xi}}\,L_x\,\{h\,(x,y,t) - Ru\,(x,y,t)\}\right\} \qquad (3.10)$$

Applying Laplace transform w. r. t. y to (3.10) on both sides and initial conditions from (3.3), we get,

$$s^{\sigma}V(x,s,t) = b_1 + sb_2 \ldots + s^{n-1}b_n + \ldots + \frac{1}{s}\left[\frac{C_1\,x^{\xi-1}}{\Gamma\xi} + \frac{C_2\,x^{\xi-2}}{\Gamma(\xi-1)} + \cdots + \frac{C_n\,x^{\xi-n}}{\Gamma(\xi-n+1)}\right] + L_y\left\{L_x^{-1}\left\{\frac{1}{s^{\xi}}\,L_x\,\{h\,(x,y,t) - Ru\,(x,y,t)\}\right\}\right\} \qquad (3.11)$$

Applying Inverse Laplace transform w. r. t. y to (3.11) on both sides, we get,

$$V(x,y,t)=\frac{b_1 y^{\sigma-1}}{\Gamma\sigma}+\frac{b_2 y^{\sigma-2}}{\Gamma(\sigma-1)}+\ldots+\frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)}+\frac{y^{\sigma}}{\Gamma(\sigma+1)}\left[\frac{C_1 x^{\xi-1}}{\Gamma\xi}+\frac{C_2 x^{\xi-2}}{\Gamma(\xi-1)}+\cdots+\frac{C_n x^{\xi-n}}{\Gamma(\xi-n+1)}\right]+L_y^{-1}\left\{\frac{1}{s^{\sigma}}L_y\left\{L_x^{-1}\left\{\frac{1}{s^{\xi}}L_x\{h(x,t)-Ru(x,t)\}\right\}\right\}\right\} \quad (3.12)$$

But $V(x,y,t)=\frac{\partial^{\tau}u(x,y,t)}{\partial t^{\tau}}$,

$$\frac{\partial^{\tau}u(x,y,t)}{\partial t^{\tau}}=\frac{b_1 y^{\sigma-1}}{\Gamma\sigma}+\frac{b_2 y^{\sigma-2}}{\Gamma(\sigma-1)}+\ldots\ldots+\frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)}+\frac{y^{\sigma}}{\Gamma(\sigma-1)}\left[\frac{C_1 x^{\xi-1}}{\Gamma\xi}+\frac{C_2 x^{\xi-2}}{\Gamma(\xi-1)}+\cdots+\frac{C_n x^{\xi-n}}{\Gamma(\xi-n+1)}\right]+L_y^{-1}\left\{\frac{1}{s^{\sigma}}L_y\left\{L_x^{-1}\left\{\frac{1}{s^{\xi}}L_x\{h(x,t)-Ru(x,t)\}\right\}\right\}\right\} \quad (3.13)$$

Applying Laplace transform to (3.13) w. r. t. t on both sides and initial conditions from (3.4), we get,

$$u(x,y,s)=\frac{a_1}{s^{\tau}}+\frac{a_2}{s^{\tau-1+}}+\cdots+\frac{a_n}{s^{\tau}-n+1}+\frac{1}{s^{\tau+1}}\left\{\frac{b_1 y^{\sigma-1}}{\Gamma\sigma}+\frac{b_2 y^{\sigma-2}}{\Gamma(\sigma-1)}+\cdots+\frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)}+\frac{y^{\sigma}}{\Gamma(\sigma-1)}\left\{\frac{C_1 x^{\xi-1}}{\Gamma\xi}+\frac{C_2 x^{\xi-2}}{\Gamma(\xi-1)}+\cdots+\frac{C_n x^{\xi-n}}{\Gamma(\xi-n+1)}\right\}\right\}+\frac{1}{s^{\tau}}L_t\left\{L_y^{-1}\left\{\frac{1}{s^{\sigma}}L_y\left\{L_x^{-1}\left\{\frac{1}{s^{\xi}}L_x\{h(x,t)-Ru(x,t)\}\right\}\right\}\right\}\right\} \quad (3.14)$$

Applying Inverse Laplace transform w. r. t. t to (3.14) on both sides, we get,

$$u(x,y,t)=\frac{a_1 t^{\tau-1}}{\Gamma\tau}+\frac{a_2 t^{\tau-2}}{\Gamma(\tau-1)}+\cdots+\frac{a_n t^{\tau-n}}{\Gamma(\tau-n+1)}+\frac{t^{\tau}}{\Gamma(\tau+1)}\left\{\frac{b_1 y^{\sigma-1}}{\Gamma\sigma}+\frac{b_2 y^{\sigma-2}}{\Gamma(\sigma-1)}+\cdots+\frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)}+\frac{y^{\sigma}}{\Gamma(\sigma-1)}\left\{\frac{C_1 x^{\xi-1}}{\Gamma\xi}+\frac{C_2 x^{\xi-2}}{\Gamma(\xi-1)}+\cdots+\frac{C_n x^{\xi-n}}{\Gamma(\xi-n+1)}\right\}\right\}+L_t^{-1}\left\{\frac{1}{s^{\tau}}L_t\left\{L_y^{-1}\left\{\frac{1}{s^{\sigma}}L_y\left\{L_x^{-1}\left\{\frac{1}{s^{\xi}}L_x\{h(x,t)-Ru(x,t)\}\right\}\right\}\right\}\right\}\right\} \quad (3.15)$$

is the required solution of equation (3.1).

4. Applications:

Example 1: $\frac{\partial^{\xi+\sigma+\tau}u(x,y,t)}{\partial x^{\xi}\partial y^{\sigma}\partial t^{\tau}}=0$ (4.1.1)

with initial conditions

$$D_x^{\xi-1}(0,y,t)=c_1,\ D_x^{\xi-2}(0,y,t)=c_2,\ldots\ldots,\ D_x^{\xi-n}(0,y,t)=c_n,$$
$$D_t^{\sigma-1}(x,0,t)=b_1, D_t^{\sigma-2}(x,0,t)=b_2,\ \ldots\ldots, D_t^{\sigma-n}(x,0,t)=b_n$$
$$\&\ D_t^{\tau-1}(x,y,0)=a_1,\ D_t^{\tau-2}(x,y,0)=a_2,\_\_\_\_\_\ D_t^{\tau-n}(x,y,0)=a_n$$

where c_i is either constant or a function of *y, t*, b_i is either constant or a function of *x, t* and a_i is either constant or a function of *x, y.*

Let us assume $\frac{\partial^{\sigma+\tau}u}{\partial y^{\sigma}\partial t^{\tau}}=\mathrm{U}\Rightarrow\frac{\partial^{\xi}U}{\partial x^{\xi}}=0$ (4.1.2)

which is a homogeneous fractional differential equation.
Applying Laplace transform on both sides to equation (4.1.2) w. r. t. x and initial conditions, we get,

$$\mathrm{U}(s,y,t)=\frac{c_1}{S^{\xi}}+\frac{c_2}{S^{\xi-1}}+\ldots..+\frac{c_n}{S^{\xi-n+1}} \tag{4.1.3}$$

Applying inverse Laplace transform on both sides to equation (4.1.3) w. r. t. x,

$$U(x,y,t)=\frac{C_1\,x^{\xi-1}}{\Gamma\xi}+\frac{C_2\,x^{\xi-2}}{\Gamma(\xi-1)}+\ldots..+\frac{C_n x^{\xi-n}}{\Gamma(\xi-n+1)} \tag{4.1.4}$$

But $U(x,y,t)=\frac{\partial^{\sigma}}{\partial y^{\sigma}}\left(\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}}\right)$,

$$\frac{\partial^{\sigma}}{\partial y^{\sigma}}\left(\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}}\right)=\frac{C_1\,x^{\xi-1}}{\Gamma\xi}+\frac{C_2\,x^{\xi-2}}{\Gamma(\xi-1)}+\ldots+\frac{C_n x^{\xi-n}}{\Gamma(\xi-n+1)} \tag{4.1.5}$$

Substituting $\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}}=V(x,y,t)$ in (4.1.5)
Applying Laplace transform on both sides of equation (4.1.5) w. r. t. y, and initial conditions, we get

$$V(x,s,t)=\frac{b_1}{S^{\sigma}}+\frac{b_2}{S^{\sigma-1}}+\ldots+\frac{b_n}{S^{\sigma-n+1}}+\frac{1}{S^{\sigma+1}}\left[\frac{c_1\,x^{\xi-1}}{\Gamma\xi}+\frac{c_2\,x^{\xi-2}}{\Gamma(\xi-1)}+\ldots..+\frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)}\right]$$

Applying Inverse Laplace transform w. r. t. y on both sides, we get,

$$V(x,y,t)=\frac{b_1\,y^{\sigma-1}}{\Gamma\sigma}+\frac{b_2\,y^{\sigma-2}}{\Gamma(\sigma-1)}+\ldots..+\frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)}+\frac{y^{\sigma}}{\Gamma(\sigma+1)}\left[\frac{c_1\,x^{\xi-1}}{\Gamma\xi}+\frac{c_2\,x^{\xi-2}}{\Gamma(\xi-1)}+\ldots..+\frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)}\right]$$

But $V(x,y,t)=\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}}$,

$$\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}}=\frac{b_1\,y^{\sigma-1}}{\Gamma\sigma}+\frac{b_2\,y^{\sigma-2}}{\Gamma(\sigma-1)}+\ldots..+\frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)}+\frac{y^{\sigma}}{\Gamma(\sigma+1)}\left[\frac{c_1\,x^{\xi-1}}{\Gamma\xi}+\frac{c_2\,x^{\xi-2}}{\Gamma(\xi-1)}+\ldots..+\frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)}\right]$$

Applying Laplace transform on both sides of equation w. r. t. t and initial conditions, we get,

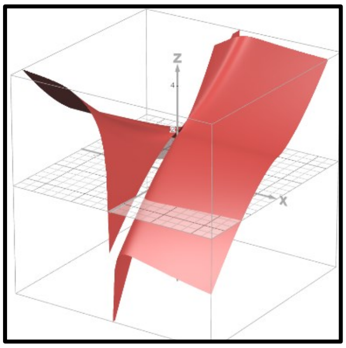
$$u(x,y,s)=\frac{a_1}{s^{\tau}}+\frac{a_2}{s^{\tau-1+}}+\cdots+\frac{a_n}{s^{\tau}-n+1}+\frac{1}{s^{\tau+1}}\left\{\frac{b_1\,y^{\sigma-1}}{\Gamma\sigma}+\frac{b_2\,y^{\sigma-2}}{\Gamma(\sigma-1)}+\cdots+\frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)}+\frac{y^{\sigma}}{\Gamma(\sigma-1)}\left\{\frac{c_1\,x^{\xi-1}}{\Gamma\xi}+\frac{c_2\,x^{\xi-2}}{\Gamma(\xi-1)}+\ldots+\frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)}\right\}\right\}$$

Applying Inverse Laplace transform w. r. t. t on both sides, we get,

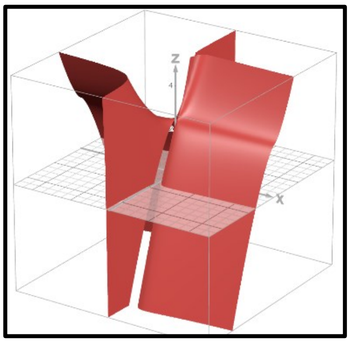
$$u(x,y,t)=\frac{a_1 t^{\tau-1}}{\Gamma\tau}+\frac{a_2 t^{\tau-2}}{\Gamma(\tau-1)}+\cdots+\frac{a_n t^{\tau-n}}{\Gamma(\tau-n+1)}+\frac{t^{\tau}}{\Gamma(\tau+1)}\left\{\frac{b_1\,y^{\sigma-1}}{\Gamma\sigma}+\frac{b_2\,y^{\sigma-2}}{\Gamma(\sigma-1)}+\cdots+\frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)}+\frac{y^{\sigma}}{\Gamma(\sigma-1)}\left\{\frac{c_1\,x^{\xi-1}}{\Gamma\xi}+\frac{c_2\,x^{\xi-2}}{\Gamma(\xi-1)}+\ldots..+\frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)}\right\}\right\}$$

is the solution of given equation (4.1.1).
For different values of t, the answers are obtained in graphical pattern as follows.
Consider $a_i=1, b_i=0, c_i=1, \xi=1.2, \sigma=1.4, \tau=1.7$

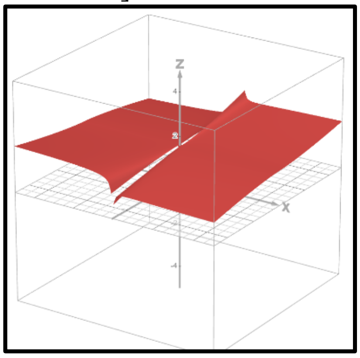
Case I] $t = 1$



Case II] $t = -2$



Case III] $t = 0.3$



Example 2: $\frac{\partial^{\xi+\sigma+\tau} u\,(x,y,t)}{\partial x^{\xi}\, \partial y,^{\sigma} \partial t^{\tau}} = x$ (4.2.1)

with initial conditions,

$$D_x^{\xi-1}(0,y,t) = c_1,\ D_x^{\xi-2}(0,y,t) = c_2, \ldots.., D_x^{\xi-n}(0,y,t) = c_n,$$
$$D_t^{\sigma-1}(x,0,t) = b_1, D_t^{\sigma-2}(x,0,t) = b_2,\ \ldots..,D_t^{\sigma-n}(x,0,t) = b_n$$
$$\&\ D_t^{\tau-1}\,(x,y,0) = a_1,\ D_t^{\tau-2}\,(x,y,0) = a_2,\ \ldots..\ D_t^{\tau-n}\,(x,y,0) = a_n$$

where c_i is either constant or a function of y, t, b_i is either constant or a function of x, t and a_i is either constant or a function of x, y.

$$\text{Let us assume } \frac{\partial^{\sigma+\tau} u}{\partial y^{\sigma}\, \partial t^{\tau}} = \mathrm{U}\ \Rightarrow \frac{\partial^{\xi}\, U}{\partial x^{\xi}} = x \tag{4.2.2}$$

which is non-homogeneous fractional differential equation.

Applying Laplace transform on both sides of equation (4.2.2) w. r. t. x, and initial conditions, we get.

$$\mathrm{U}(s,y,t) = \frac{c_1}{s^{\xi}} + \frac{c_2}{s^{\xi-1}} + \ldots..+\frac{c_n}{s^{\xi-n+1}} + \frac{1}{s^{\xi+2}} \tag{4.2.3}$$

Applying inverse Laplace transform on both sides of equation (4.2.3) w. r. t. x,

$$U(x,y,t) = \frac{c_1\, x^{\xi-1}}{\Gamma\xi} + \frac{c_2\, x^{\xi-2}}{\Gamma(\xi-1)} +\ldots+ \frac{c_n\, x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{x^{\xi+1}}{\Gamma(\xi+2)} \tag{4.2.4}$$

But $U(x,y,t) = \frac{\partial^{\sigma}}{\partial y^{\sigma}}\left(\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}}\right)$,

$$\frac{\partial^{\sigma}}{\partial y^{\sigma}}\left(\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}}\right) = \frac{C_1\, x^{\xi-1}}{\Gamma\xi} + \frac{C_2\, x^{\xi-2}}{\Gamma(\xi-1)} +\ldots..+ \frac{C_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{x^{\xi+1}}{\Gamma(\xi+2)} \tag{4.2.5}$$

Substituting $\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}} = V(x,y,t)$ in (4.2.5)

Applying Laplace transform on both sides of equation (4.2.5) w. r. t. y, and initial conditions, we get.

$$V\,(x,s,t) = \frac{b_1}{s^{\sigma}} + \frac{b_2}{s^{\sigma-1}} + \ldots. + \frac{b_n}{s^{\sigma-n+1}} + \frac{1}{s^{\sigma+1}}\left[\frac{c_1\, x^{\xi-1}}{\Gamma\xi} + \frac{c_2\, x^{\xi-2}}{\Gamma(\xi-1)} + \ldots.. + \frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{x^{\xi+1}}{\Gamma(\xi+2)}\right]$$

Applying Inverse Laplace transform w. r. t. y on both sides, we get,

$$V(x,y,t) = \frac{b_1\, y^{\sigma-1}}{\Gamma\sigma} + \frac{b_2\, y^{\sigma-2}}{\Gamma(\sigma-1)} +\ldots+ \frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)} + \frac{y^{\sigma}}{\Gamma(\sigma+1)}\left[\frac{c_1\, x^{\xi-1}}{\Gamma\xi} + \frac{c_2\, x^{\xi-2}}{\Gamma(\xi-1)} + \ldots + \frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{x^{\xi+1}}{\Gamma(\xi+2)}\right]$$

But $V(x,y,t) = \frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}}$,

$$\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}} = \frac{b_1\, y^{\sigma-1}}{\Gamma\sigma} + \frac{b_2\, y^{\sigma-2}}{\Gamma(\sigma-1)} +\ldots+ \frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)} + \frac{y^{\sigma}}{\Gamma(\sigma+1)}\left[\frac{c_1\, x^{\xi-1}}{\Gamma\xi} + \frac{c_2\, x^{\xi-2}}{\Gamma(\xi-1)} +\ldots + \frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{x^{\xi+1}}{\Gamma(\xi+2)}\right]$$

Applying Laplace transform on both sides of equation w. r. t. t and initial conditions, we get,

$$u(x,y,s) = \frac{a_1}{s^{\tau}} + \frac{a_2}{s^{\tau-1+}} + \cdots + \frac{a_n}{s^{\tau-n+1}} + \frac{1}{s^{\tau+1}}\left\{\frac{b_1 y^{\sigma-1}}{\Gamma\sigma} + \frac{b_2 y^{\sigma-2}}{\Gamma(\sigma-1)} + \cdots + \frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)} + \frac{y^{\sigma}}{\Gamma(\sigma-1)}\left\{\frac{c_1 x^{\xi-1}}{\Gamma\xi} + \frac{c_2 x^{\xi-2}}{\Gamma(\xi-1)} + \ldots + \frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{x^{\xi+1}}{\Gamma(\xi+2)}\right\}\right\}$$

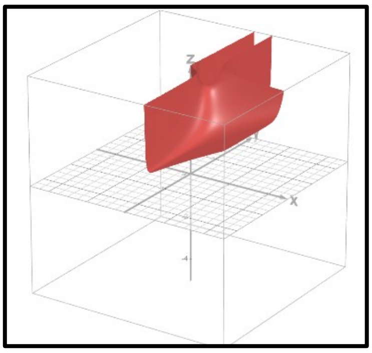
Applying Inverse Laplace transform w. r. t. t on both sides, we get,

$$u(x,y,t) = \frac{a_1 t^{\tau-1}}{\Gamma\tau} + \frac{a_2 t^{\tau-2}}{\Gamma(\tau-1)} + \cdots + \frac{a_n t^{\tau-n}}{\Gamma(\tau-n+1)} + \frac{t^{\tau}}{\Gamma(\tau+1)}\left\{\frac{b_1 y^{\sigma-1}}{\Gamma\sigma} + \frac{b_2 y^{\sigma-2}}{\Gamma(\sigma-1)} + \cdots + \frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)} + \frac{y^{\sigma}}{\Gamma(\sigma-1)}\left\{\frac{c_1 x^{\xi-1}}{\Gamma\xi} + \frac{c_2 x^{\xi-2}}{\Gamma(\xi-1)} + \ldots\ldots + \frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{x^{\xi+1}}{\Gamma(\xi+2)}\right\}\right\}$$

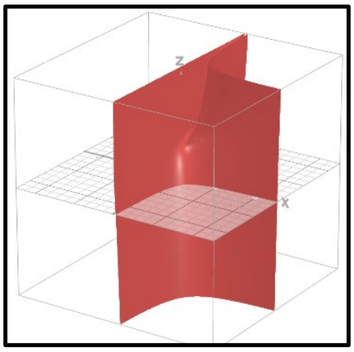
is the solution of given equation (4.2.1).

Consider $a_i = 1, b_i = 0, c_i = 1, \xi = 1.2, \sigma = 1.4, \tau = 1.7$

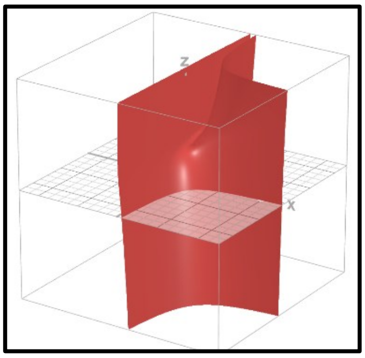
Case I] $x = 1$



Case II] $x = -2$



Case III] $x = 0.3$



Example 3: $\frac{\partial^{\xi+\sigma+\tau} u(x,y,t)}{\partial x^{\xi}\,\partial t^{\sigma}\partial t^{\tau}} = x^2 t$ (4.3.1)

with initial conditions,

$$D_x^{\xi-1}(0,y,t) = c_1,\ D_x^{\xi-2}(0,y,t) = c_2, \ldots\ldots, D_x^{\xi-n}(0,y,t) = c_n,$$
$$D_t^{\sigma-1}(x,0,t) = b_1, D_t^{\sigma-2}(x,0,t) = b_2,\ \ldots\ldots, D_t^{\sigma-n}(x,0,t) = b_n$$
$$\&\ D_t^{\tau-1}(x,y,0) = a_1,\ D_t^{\tau-2}(x,y,0) = a_2,\ \ldots\ldots D_t^{\tau-n}(x,y,0) = a_n$$

where c_i is either constant or a function of *y, t*, b_i is either constant or a function of *x, t* and a_i is either constant or a function of *x, y.*

Let us assume $\frac{\partial^{\sigma+\tau} u}{\partial y^{\sigma}\,\partial t^{\tau}} = \text{U} \Rightarrow \frac{\partial^{\xi} U}{\partial x^{\xi}} = x^2 t$ (4.3.2)

which is non-homogeneous fractional differential equation.

Applying Laplace transform on both sides of equation (4.3.2) w. r. t. x, and initial conditions, we get,

$$\text{U}(s,y,t) = \frac{c_1}{s^{\xi}} + \frac{c_2}{s^{\xi-1}} + \ldots\ldots + \frac{c_n}{s^{\xi-n+1}} + \frac{2t}{s^{\xi+3}} \tag{4.3.3}$$

Applying inverse Laplace transform on both sides of equation (4.3.3) w. r. t. *x,*

$$U(x,y,t) = \frac{c_1 x^{\xi-1}}{\Gamma\xi} + \frac{c_2 x^{\xi-2}}{\Gamma(\xi-1)} + \ldots\ldots + \frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{2tx^{\xi+2}}{\Gamma(\xi+3)} \tag{4.3.4}$$

But $U(x,y,t) = \frac{\partial^{\sigma}}{\partial y^{\sigma}}\left(\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}}\right)$,

$$\frac{\partial^{\sigma}}{\partial y^{\sigma}}\left(\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}}\right) = \frac{C_1 x^{\xi-1}}{\Gamma\xi} + \frac{C_2 x^{\xi-2}}{\Gamma(\xi-1)} + \ldots\ldots + \frac{C_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{2tx^{\xi+2}}{\Gamma(\xi+3)} \tag{4.3.5}$$

Substituting $\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}} = V(x,y,t)$ in (4.3.5)

Applying Laplace transform on both sides of equation (4.3.5) w. r. t. y and initial conditions, we get,

$$V\,(x,s,t) = \frac{b_1}{s^{\sigma}} + \frac{b_2}{s^{\sigma-1}} + \ldots + \frac{b_n}{s^{\sigma-n+1}} + \frac{1}{s^{\sigma+1}}\left[\frac{c_1 x^{\xi-1}}{\Gamma\xi} + \frac{c_2 x^{\xi-2}}{\Gamma(\xi-1)} + \ldots + \frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{2tx^{\xi+2}}{\Gamma(\xi+3)}\right]$$

Applying Inverse Laplace transform w. r. t. y on both sides, we get,

$$V(x,y,t) = \frac{b_1 y^{\sigma-1}}{\Gamma\sigma} + \frac{b_2 y^{\sigma-2}}{\Gamma(\sigma-1)} + \ldots + \frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)} + \frac{y^{\sigma}}{\Gamma(\sigma+1)}\left[\frac{c_1 x^{\xi-1}}{\Gamma\xi} + \frac{c_2 x^{\xi-2}}{\Gamma(\xi-1)} + \ldots + \frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{2tx^{\xi+2}}{\Gamma(\xi+3)}\right]$$

But $V(x,y,t) = \frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}}$,

$$\frac{\partial^{\tau} u(x,y,t)}{\partial t^{\tau}} = \frac{b_1 y^{\sigma-1}}{\Gamma\sigma} + \frac{b_2 y^{\sigma-2}}{\Gamma(\sigma-1)} + \ldots + \frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)} + \frac{y^{\sigma}}{\Gamma(\sigma+1)}\left[\frac{c_1 x^{\xi-1}}{\Gamma\xi} + \frac{c_2 x^{\xi-2}}{\Gamma(\xi-1)} + \ldots + \frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{2tx^{\xi+2}}{\Gamma(\xi+3)}\right]$$

Applying Laplace transform on both sides of equation w. r. t. t and initial conditions, we get,

$$u(x,y,s) = \frac{a_1}{s^{\tau}} + \frac{a_2}{s^{\tau-1+}} + \cdots + \frac{a_n}{s^{\tau}-n+1} + \frac{1}{s^{\tau+1}}\left\{\frac{b_1 y^{\sigma-1}}{\Gamma\sigma} + \frac{b_2 y^{\sigma-2}}{\Gamma(\sigma-1)} + \cdots + \frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)} + \frac{y^{\sigma}}{\Gamma(\sigma-1)}\left\{\frac{c_1 x^{\xi-1}}{\Gamma\xi} + \frac{c_2 x^{\xi-2}}{\Gamma(\xi-1)} + \ldots + \frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{2tx^{\xi+2}}{\Gamma(\xi+3)}\right\}\right\}$$

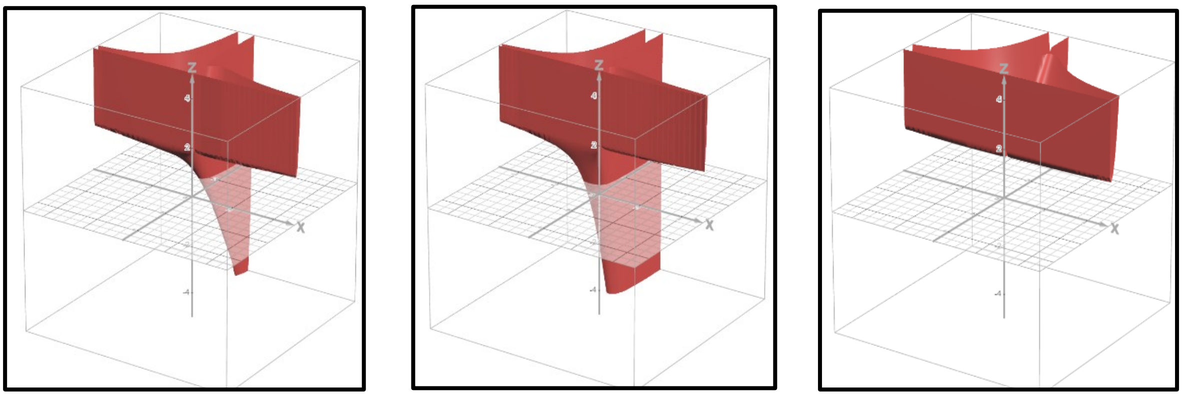
Applying Inverse Laplace transform w. r. t. t on both sides, we get,

$$u(x,y,t) = \frac{a_1 t^{\tau-1}}{\Gamma\tau} + \frac{a_2 t^{\tau-2}}{\Gamma(\tau-1)} + \cdots + \frac{a_n t^{\tau-n}}{\Gamma(\tau-n+1)} + \frac{t^{\tau}}{\Gamma(\tau+1)}\left\{\frac{b_1 y^{\sigma-1}}{\Gamma\sigma} + \frac{b_2 y^{\sigma-2}}{\Gamma(\sigma-1)} + \cdots + \frac{b_n y^{\sigma-n}}{\Gamma(\sigma-n+1)} + \frac{y^{\sigma}}{\Gamma(\sigma-1)}\left\{\frac{c_1 x^{\xi-1}}{\Gamma\xi} + \frac{c_2 x^{\xi-2}}{\Gamma(\xi-1)} + \ldots + \frac{c_n x^{\xi-n}}{\Gamma(\xi-n+1)} + \frac{2tx^{\xi+2}}{\Gamma(\xi+3)}\right\}\right\}$$

is the solution of the given equation (4.3.1).

Consider $a_i = 1, b_i = 0, c_i = 1, \xi = 1.2, \sigma = 1.4, \tau = 1.7$

Case I] $x = 1$ Case II] $x = -2$ Case III] $x = 0.3$



5. Conclusion:

Here is a polished and more academic version of your paragraph, keeping the meaning intact while improving clarity, flow, and formal tone: --- The resolution of linear mixed partial fractional differential equations involving more than two independent variables has opened numerous avenues for exploration. In this study, we investigated and demonstrated that the proposed method efficiently and accurately solves homogeneous and non-homogeneous linear mixed partial fractional differential equations with multiple independent variables. The method's effectiveness in addressing homogeneous linear fractional differential equations with mixed partial derivatives is illustrated in Example 1. Its applicability to non-homogeneous cases is further demonstrated through Examples 2 and 3. The proposed approach yields exact solutions and provides clear, reliable results for such equations. Graphical representations for specific cases are presented using Desmos software. The Laplace Substitution Method (LSM) has proven to be a powerful and effective technique for solving a broad class of problems. In conclusion, the LSM holds significant potential as a valuable tool for addressing more complex boundary value problems involving mixed partial derivatives than those considered in this work.

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