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Generalization on Value distribution of L-functions and differential polynomials

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In this article we study the value distribution of L-functions and differential polynomials sharing some value, which generalize and extend recent results of V. Priyanka, S. Rajeshwari and V. Husna .

[1]

Introduction

Value distribution of L-functions concerns distribution of the zeros of L-functions L and more generally, the c-points of L, i.e., the roots of the equation $L(s)=c$, or the values in the pre-image $L^{-1}=\{s\in\mathbb{C}:L(s)=c\}$, where and throughout the paper, s denotes the complex variable in the complex plane $\mathbb{C}$ and c denotes a complex value. L-functions can be analytically continued as meromorphic functions in $\mathbb{C}$.

Two meromorphic functions f and g in the complex plane are said to share a value $c\in\mathbb{C}\cup\{\infty\}$ IM (ignoring multiplicities) if $f^{-1}(c)=g^{-1}(c)$ as two sets in $\mathbb{C}$. Moreover, f and g are said to share a value CM (counting multiplicities) if they share the value c and if the roots of the equations $f(s)=c$ and $g(s)=c$ have the same multiplicities. In terms of sharing values, two non-constant meromorphic functions in $\mathbb{C}$ must be identically equal if they share five values IM, and one must be a Mobius transform of the other if they share four values CM.

The Riemann hypothesis as one of the millenium problems has been given a lot of attention by mathematical workers for a long time. Selberg guessed that the Riemann hypothesis is also true for L-functions in the selberg class. Such an L-function based on Riemann zeta function as the prototype is defined to be a Dirichlet series. $L(s)=\sum_{n=1}^{\infty}a(n)n^{-s}$, satisfying the following axioms (): (i) Ramanujan hypothesis. $a(n)\ll n^{\varepsilon}$ for every $\varepsilon>0$ (ii) Analytic continuation. There is a non-negative integer k such that $(s-1)^{k}L(s)$ is an entire function of finite order. (iii) Functional equation. L satisfies a functional equation of type

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$\Lambda_L(s) = \omega\overline{\Lambda_L(1-s)}$,

where

$$\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{k} \Gamma\left(\lambda_j s + v_j\right)$$

with positive real numbers Q,λ_j and complex numbers v_j,ω with $\text{Re}v_j \geq 0$ and $|\omega| =$ 1. (iv) Euler product hypothesis. $L(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$ with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$, where the product is taken over all prime numbers p. In this paper, a meromorphic function always mean a function which is meromorphic in the whole complex plane $\mathbb{C}$. We denote by $N_{k)}\left(r, \frac{1}{(f-a)}\right)$ the counting function for zeros of $f-a$ with multiplicity $\leq k$, and by $\overline{N}_{k)}\left(r, \frac{1}{(f-a)}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r, \frac{1}{(f-a)}\right)$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}\left(r, \frac{1}{(f-a)}\right)$ the corresponding one for which multiplicity is not counted.

Let z_0 be a zero of $f-a$ of multiplicity p and a zero of $g-a$ of multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p > q \geq 1$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r,a;f)$ the counting function of those a-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,a;g), N_E^{1)}(r,a;g), \overline{N}_E^{(2}(r,a;g)$.

Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f,g share the value a with weight k.

The definition implies that if f,g share the value a with weight k then z_0 is an a point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a point of g with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f,g share (a,k) to mean that f,g share the value a with weight k. Clearly if f,g share (a,k) then f,g share (a,p) for any integer $p, 0 \leq p < k$. Also we note that f,g share a value a IM or CM if and only if f,g share $(a,0)$ or (a,∞) respectively. We denote $\rho(f)$ for order of $f(z)$.

$$\rho(f) = \limsup_{r\to\infty} \frac{logT(r,f)}{logr}.$$

We first recall the following result due to Steuding , which actually holds without the Euler product hypothesis:

Theorem 1. *(, page 152) If two L-functions L_1 and L_2 with $a(1) = 1$ share a complex value $c \neq \infty$ CM, then $L_1 = L_2$.*

Later on, Li proved the following result to deal with a question posed by Chung-Chun Yang :

Theorem 2. *Let a and b be two distinct finite values, and let f be a meromorphic function in the complex plane such that f has finitely many poles in the complex plane. If f and a non-constant L-function L share a CM and b IM, then $L = f$.*

In 1997, Lahiri posed the following question:

What can be said about the relationship between two meromorphic functions f and g, when two differential polynomials, generated by f and g respectively, share some nonzero finite value? In this direction, Fang and Yang-Hua respectively proved the following results:

Theorem 3. *Let f and g be two non-constant entire functions, and let n and k be two positive integers such that $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants, satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f =$ tg for a constant t such that $t^n = 1$.*

Theorem 4. *Let f and g be two non-constant meromorphic functions, and let $n \geq 11$ be a positive integer. If f^nf' and g^ng' share 1 CM, then either $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$, where c_1, c_2, and c are three constants, satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

In 2017, Fang Liu, Xiao-Min Li and Hong-Xun Yi proved the following results.

Theorem 5. *Let f be a non-constant meromorphic function, let L be an L - function, and let n and k be two positive integers with $n > 3k + 6$. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $1CM$, then $f = tL$ for a constant t satisfying $t^n = 1$.*

Theorem 6. *Let f be a non-constant meromorphic function, let L be an L - function, and let n and k be two positive integers with $n > 3k + 6$. If $(f^n)^{(k)}(z)-z$ and $(L^n)^{(k)}(z)-z$ share $0\ CM$, then $f = tL$ for a constant t satisfying $t^n = 1$.*

Now it is natural to ask the following question which is the motivation of the paper. QUESTION. Can a CM shared value be replaced by $(p(z),l)$ in Theorems E and F ? In 2018, Harina Pandit Waghamore and Naveenkumar S H proved the following results.

Theorem 7. *Let f be a non-constant meromorphic function in $\mathbb{C}$, let L be an non-constant L-function and let n and k be two positive integers. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $(p(z),l)$, where $p(z)$ be a nonzero polynomial with deg $(p) = m$ and f and L share $(\infty,0)$. Suppose one of the following conditions hold:*

a. $l \geq 3$ and $n > 3k + 4$;

b. $l = 2$ and $n > 3k + 6$;

c. $l = 1$ and $n > 3k + 7$;

d. $l = 0$ and $n > 7k + 11$. Then $f = tL$ for a constant t satisfying $t^n = 1$.

Theorem 8. *Let f be a non-constant entire function in $\mathbb{C}$, let L be an non-constant L-function, and let n and k be two positive integers. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $(p(z),l)$, where $p(z)$ be a nonzero polynomial with deg $(p) = m$ and f and L share $(\infty,0)$. Suppose one of the following conditions hold:*

a. $l \geq 2$ and $n > 2k + 4$;

b. $l = 1$ and $n > \frac{5k+9}{2}$;

c. $l = 0$ and $n > 5k + 7$.

Then $f = tL$ for a constant t satisfying $t^n = 1$.

Regarding Theorem 7 and 9 one may ask the following question. What happens if $(f^n)^{(k)}$ and $(L^n)^{(k)}$ is replaced by $(f^nP(f))^{(k)}$ and $(L^nP(L))^{(k)}$ in Theorems 7 and 9 ?

In 2023, V. Priyanka, S. Rajeshwari and V. Husna obtained analogous results to answer the above question affirmatively, and proved the following results.

Theorem 9. *Let f be a non-constant meromorphic function in $\mathbb{C}$, let L be an non-constant L-function and let n and k be two positive integers. If $(f^n P(f))^{(k)}$ and $(L^n P(L))^{(k)}$ share $(p(z),l)$, where $p(z)$ be a nonzero polynomial with deg $(p) = m$ and f and L share $(\infty,0)$. Suppose one of the following conditions hold:*

a. $l \geq 3$ *and* $n > 3k + m + 4$*;*

b. $l = 2$ *and* $n > 3k + m + 6$*;*

c. $l = 1$ *and* $n > 3k + m + 7$*;*

d. $l = 0$ *and* $n > 7k + 3m + 11$*.*

Then $f = tL$ *for a constant* t *satisfying* $t^n = 1$*.*

Theorem 10. *Let f be a non-constant entire function in $\mathbb{C}$, let L be an non-constant L-function, and let n and k be two positive integers. If $(f^n P(f))^{(k)}$ and $(L^n P(L))^{(k)}$ share $(p(z),l)$, where $p(z)$ be a nonzero polynomial with deg $(p) = m$ and f and L share $(\infty,0)$. Suppose one of the following conditions hold:*

a. $l \geq 2$ *and* $n > 2k + m + 4$*;*

b. $l = 1$ *and* $n > \frac{5k+m+9}{2}$*;*

c. $l = 0$ *and* $n > 5k + m + 7$*.*

Then $f = tL$ *for a constant* t *satisfying* $t^n = 1$*.*

Main Result

Theorem 11. *Let f be a non-constant meromorphic function in $\mathbb{C}$, let L be an non-constant L-function and let n and k be two positive integers. If $((f^n)^s P(f))^{(k)}$ and $((L^n)^s P(L))^{(k)}$ share $(p(z),l)$, where $p(z)$ be a nonzero polynomial with deg $(p) = m$ and f and L share $(\infty,0)$. Suppose one of the following conditions hold:*

a. $l \geq 3$ *and* $ns > k + s(2k + 4) + m$*;*

b. $l = 2$ *and* $ns > k + s(2k + 4) + m + 2$*;*

c. $l = 1$ *and* $ns > k + s(5k + 9) + 2m + 5$*;*

d. $l = 0$ *and* $ns > k + 2s(3k + 1) + 6m + 9$.

Then $f = tL$ *for a constant* t *satisfying* $t^n = 1$.

Theorem 12. *Let* f *be a non-constant entire function in* $\mathbb{C}$, *let* L *be an non-constant* L-*function, and let* n *and* k *be two positive integers. If* $((f^n)^s P(f))^{(k)}$ *and* $((L^n)^s P(L))^{(k)}$ *share* $(p(z),l)$, *where* $p(z)$ *be a nonzero polynomial with deg* $(p) = m$ *and* f *and* L *share* $(\infty,0)$. *Suppose one of the following conditions hold:*

a. $l \geq 2$ *and* $ns > s(2k + 4) + m$;

b. $l = 1$ *and* $ns > \frac{s(5k+9)+2m}{2}$;

c. $l = 0$ *and* $ns > k + 2s(2k + 1) + 5m + 5$.

Then $f = tL$ *for a constant* t *satisfying* $t^n = 1$.

Some Lemmas

In this section, we present some lemmas which will be needed later on. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We will denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right)\left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right).$$

Lemma 1. *Let* f *be a non-constant meromorphic function and let* $a_n(z)(\not\equiv 0), a_{n-1}(z), \ldots, a_0(z)$ *be meromorphic functions such that* $T(r, a_i(z)) = S(r,f)$ *for* $i = 0,1,2,\ldots,n$. *Then* $T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r,f) + S(r,f)$.

Lemma 2. *Suppose that* f *is a non-constant meromorphic function in the complex plane and* k *is a positive integer. Then* $N(r,0;f^{(k)}) \leq N(r,0;f) + k\overline{N}(r,\infty,f) + O(log(T(r,f)) + logr)$, *as* $r \to \infty$, *outside of a possible exceptional set of finite linear measure.*

Lemma 3. *Let f be a non-constant meromorphic function, and p,k be positive integers. Then*
$N_p\left(r,0;f^{(k)}\right) \le T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f)$,
$N_p\left(r,0;f^{(k)}\right) \le k\overline{N}(r,\infty,f) + N_{p+k}(r,0;f) + S(r,f)$.

Lemma 4. *If $N(r,0;f^{(k)}|f \ne 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then* $N(r,0;f^{(k)}|f \ne 0) \le k\overline{N}(r,\infty,f) + N(r,0;f|<k) + k\overline{N}(r,0;f| \ge k) + S(r,f)$.

Lemma 5. *Suppose that f and g be two non-constant meromorphic functions. Let $F = [f^n]^{(k)}, G = [g^n]^{(k)}$, where n,k are positive integers. If f,g share ∞ IM and $V \equiv 0$, then* $F \equiv G$

Lemma 6. *Let f_1 and f_2 be two non-constant meromorphic functions satisfying $\overline{N}(r,0;f_i) + \overline{N}(r,\infty;f_i) = S(r;f_1,f_2)$ for $i = 1,2$. If $f_1^s f_2^t - 1$ is not identically zero for arbitrary integers s and $t(|s|+|t|>0)$, then for any positive ε, we have $N_0(r,1;f_1,f_2) \le \varepsilon T(r) + S(r;f_1,f_2)$, where $N_0(r,1;f_1,f_2)$ denotes the reduced counting function related to the common 1 points of f_1 and f_2 and $T(r) = T(r,f_1) + T(r,f_2), S(r;f_1,f_2) = o(T(r))$ as $r \to \infty$ possibly outside a set of finite linear measure.*

Lemma 7. *Let f and g be two non-constant meromorphic functions sharing $(1,k_1)$, where $2 \le k_1 \le \infty$. Then*

$$\overline{N}(r,1;f|=2) + 2\overline{N}(r,1;f|=3) + \ldots + (k_1-1)\overline{N}(r,1;f|=k_1) + k_1\overline{N}_L(r,1;f) + (k_1+1)\overline{N}_L(r,1;g) + k_1\overline{N}_E^{(k_1+1}(r,1;g) \le N(r,1;g) - \overline{N}(r,1;g).$$

Lemma 8. *Let F and G be two non-constant meromorphic functions sharing $(1,1)$ and $H \not\equiv 0$. Then*

$$T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) + S(r,F) + S(r,G).$$

Lemma 9. *Let F and G be two non-constant meromorphic functions sharing $(1,0)$ and $H \not\equiv 0$. Then*

$$T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,F) + S(r,G).$$

Lemma 10. *Let f and g be two non-constant meromorphic functions. If $(f^n)^{(k)} = (g^n)^{(k)}$ and $n > k+1$, then $f = tg$ for a constant t such that $t^n = 1$.*

Lemma 11. *Let f be a non-constant meromorphic function in $\mathbb{C}$, let L be an non-constant L-function, and let n and k be two positive integers with $H \not\equiv 0$. If $F = [(f^n)^s]^{(k)}$ and $G = [(L^n)^s]^{(k)}$ share $(1,k_1)$, and f and L share $(\infty,0)$, then* $(n-1)\overline{N}(r,\infty;f) \leq (k+1)\{T(r,f)+T(r,L)\} + \overline{N}_*(r,1;F,G) + O(logr)$.

Proof. Suppose ∞ is an evp of f and L, then the lemma follows immediately. Next suppose ∞ is not an evp of f and L. Since $H \neq 0$ from Lemma 3.5 we have $V \not\equiv 0$. We suppose that z_0 is a pole of f with multiplicity q and a pole of L with multiplicity r. Clear;y z_0 is a pole of F with multiplicity $nsq+k$ and a pole of L with multiplicity $nsr+k$. Noting that f and L share $(\infty,0)$ from the definition of v it is clear that z_0 is a zero of v with multiplicity $ns+k-1$. Now using the Milloux theorem [[8], p.55] and Lemma 3.1, we obtain from the definition of v that $m(r,v) = o(logr)$. Then using Valiron-Mokhonko lemma(cf. [22]) and Lemma 3.3, we get

$$
\begin{aligned}
(ns+k-1)\overline{N}(r,\infty;f) &\leq N(r,0;v) \leq N(r,v) + O(1) \\
&\leq N(r,\infty;v) + m(r,0;v) + O(1) \\
&\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N_*}(r,1;F,G) + O(logr) \\
&\leq N_{k+1}(r,0;(f^n)^s) + k\overline{N}(r,\infty;f) + N_{k+1}(r,0;(L^n)^s) + k\overline{N}(r,\infty,L) + \overline{N_*}(r,1;F,G) + O(logr) \\
&\leq s(k+1)\overline{N}(r,0;f) + s(k+1)\overline{N}(r,0;L) + k\overline{N}(r,\infty;f) + \overline{N_*}(r,1;F,G) + O(logr)
\end{aligned}
$$

This gives $(ns-1)\overline{N}(r,\infty;f) \leq s(k+1)T(r,f) + T(r,L) + \overline{N_*}(r,1;F,G) + O(logr)$ *This completes the proof of the Lemma.*

Lemma 12. *Let f be a non-constant meromorphic function in $\mathbb{C}$, let L be an non-constant L-function and $F = \frac{(f^n)^{(k)}}{p(z)}, G = \frac{(L^n)^{(k)}}{p(z)}$, where $p(z)$ be a nonzero polynomial with deg $(p) = m, n$ and k be two positive integers such that $n > 3k+2$. If f and L share $(\infty,0)$ and $H \equiv 0$ then either $[f^n]^{(k)}[L^n]^{(k)} \equiv p^2$ or $f^n \equiv L^n$.*

Lemma 13. *Let $f(z)$ and $g(z)$ be two non-constant finite-order meromorphic functions, let P(w) be as defined in (3.1) , let $k(>0), m(\geq 0)$ be integers satisfying $n > 2k+m+5$. If* $[f^nP(f)]^{(k)} \equiv [g^nP(g)]^{(k)}$ $f^nP(f) \equiv g^nP(g)$.

Lemma 14. *Let $f(z)$ and $g(z)$ be two non-constant finite order meromorphic functions,and let $k(>0)$ be an integer satisfying $n > k+5$. Also let $P(w)$ be as defined in (3.1). Suppose that $a(z)(\not\equiv 0,\infty)$ is a small function with respect to $f(z)$ with finitely many zeros and poles.*

If $(f^nP(f))^{(k)}(g^nP(g))^{(k)} = \alpha^2(z)$ *and, in addition,* $f(z)$ *and* $g(z)$ *share* ∞ *IM, then* $P(w)$ *reduces to a nonzero monomial namely,* $P(w) = a_iw^i \not\equiv 0$ *for some* $i \in 0,1,...,m$.

Proof of Main Results

Proof of Theorem 2.1.. Suppose that d is the degree of L. Then $d = 2\sum_{i=1}^{k}\lambda_j$, where k and λ_j are respectively the positive integer and the positive real number in the axiom (iii) of the definition of L - function.
Then we have that

$$T(r,L) = \frac{d}{\Pi} rlogr + O(r)$$

(, p. 150). Clearly, f and L are transcendental meromorphic functions (, p. 43). Note that an L - function at most has one pole $z = 1$ in the complex plane. Let $F_1 = \frac{(F)^{(k)}}{\eta(z)}$ and $G_1 = \frac{(G)^{(k)}}{\eta(z)}$ where $F = f^nP(f)$ and $G = L^nP(L)$. It follows that F_1 and G_1 share $(1,l)$ except the zeros of $p(z)$ and f,g share $(\infty,0)$.

Case 1. Let $H \not\equiv 0$.

Subcase 1.1. $l \geq 1$

From (3.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) poles of F and G with different multiplicities, (iv) zeros of $F'(G')$ which are not the zeros of $F(F-1)(G(G-1))$.

Since H has only simple poles we get

$$N(r,\infty;H) \leq \overline{N}_*(r,\infty;f,g) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'),$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Let z_0 be a simple zero of $F-1$ but $a(z_0) \neq 0,\infty$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F| = 1) \leq N(r,0;H) \leq N(r,\infty;H) + O(logr).$$

While $l \geq 3$, using (4.2) and (4.3) we get

$$\begin{aligned}\overline{N}(r,1;F) \leq\ & N(r,1;F| = 1) + \overline{N}(r,1;F| \geq 2) \\ \leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + \overline{N}_*(r,1;F,G) \\ & + \overline{N}(r,1;F| \geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + O(logr).\end{aligned}$$

Now in view of Lemmas 3.4 and 3.7 we get

$$\begin{aligned}&\overline{N}_0(r,0;G') + \overline{N}(r,1;F| \geq 2) + \overline{N}_*(r,1;F,G) \\ &\leq \overline{N}_0(r,0;G') + \overline{N}(r,1;F| = 2) + \overline{N}(r,1;F| = 3) + \ldots + \overline{N}(r,1;F| = l) \\ &\quad + \overline{N}_E^{(l+1)}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_*(r,1;F,G) \\ &\leq \overline{N}_0(r,0;G') - \overline{N}(r,1;F| = 3) - \ldots - (l-2)\overline{N}(r,1;F| = l) - (l-1)\overline{N}_L(r,1;F) \\ &\quad - l\overline{N}_L(r,1;G) - (l-1)\overline{N}_E^{(l+1)}(r,1;F) + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_*(r,1;F,G) \\ &\leq \overline{N}_0(r,0;G') + N(r,1;G) - \overline{N}'(r,1;G) - (l-2)\overline{N}_L(r,1;F) - (l-1)\overline{N}_L(r,1;G) \\ &\leq N(r,0;G'|G \neq 0) - (l-2)\overline{N}_L(r,1;F) - (l-1)\overline{N}_L(r,1;G) \\ &\leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) - (l-2)\overline{N}_*(r,1;F,G) - \overline{N}_L(r,1;G) \\ &\leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) - \overline{N}_*(r,1;F,G) - \overline{N}_L(r,1;G)\end{aligned}$$

Hence using (4.4), (4.5), Lemmas 3.3 and 3.11 we get from the second fundamental theorem that

$$\begin{aligned}&(ns + m)T(r,f) \\ &\leq T(r,F) + N_{k+2}(r,0;(f^n)^s P(f)) - N_2(r,0;F) + O(logr) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + N_{k+2}(r,0;(f^n)^s P(f)) - N_2(r,0;F) - \overline{N}_0(r,0;F') \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;F) + N_{k+2}(r,0;(f^n)^s P(f)) + \overline{N}(r,0;F| \geq 2) + \\ &\overline{N}(r,0;G| \geq 2) + \overline{N}(r,1;F| \geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;G') - N_2(r;0;F) + O(logr) \\ &\leq 2\overline{N}(r,\infty;f) + \overline{N}(r,\infty;L) + N_{k+2}(r,0;(f^n)^s P(f)) + N_2(r;0;G) - \overline{N}_*(r,1;F,G) \\ &- \overline{N}_L(r,1;G) + O(logr) \\ &\leq 2\overline{N}(r,\infty;f) + N_{k+2}(r;0;(f^n)^s P(f)) + N_2\left(r;0;((L^n)^s P(L))^{(k)}\right) - \overline{N}_*(r,1;F,G) + O(logr) \\ &\leq \left[s(k + 2) + m + \frac{s(2 + k)\,(k + 1)}{ns-1}\right]\{T(r,f) + T(r,L)\} + O(logr).\end{aligned}$$

Correspondingly we get

$$(ns + m)T(r,L) \leq \left[s(k + 2) + m + \frac{s(k + 1)^2}{ns-1}\right]\{T(r,f) + T(r,L)\} + O(logr).$$

Combining equation (4.6) and (4.7),we get

$$\begin{aligned}&(ns + m)T(r,f) + T(r,L) \\ &\qquad \leq \left[ns + m - 2s(k + 2) - 2m - \frac{s(k + 2)\,(k + 1)}{ns-1} - \frac{s(k + 1)^2}{ns-1}\right]\{T(r,f) + T(r,L)\} \\ &\qquad \leq O(logr).\end{aligned}$$

the bracket term quantity can be written as

$$\left[\frac{(ns-1)^2 - (2s(k + 2) + m)\,(ns-1) - s(k + 1)\,(2k + 3)}{ns-1}\right],$$

by a simple calculation one can easily find out that when

$$ns-1>2s(k+2)+m-1>\frac{2s(k+2)+m+\sqrt{(2s(k+2)+m)^2+4s(2k+3)\,(k+1)}}{2}$$

i.e., $ns>s(2k+4)+m+k$, we get a contradiction from (4.8). While $l\geq 2$, like (4.4), (4.5) and not using Lemma 3.11 in (4.6) we can deduce a contradiction when $ns>s(2k+4)+m+k+2$. So we omit the detail. While $l=1$ from lemma 3.3 we obtain

$$\begin{aligned}N_2(r,0;F) &\leq N_2(r,0;(f^n)^sP(f))^{(k)})+S(r,f)\\ &\leq T\left(r,((f^n)^s)^{(k)}\right)+T\left(r,P(f)^{(k)}\right)-nT(r,f)\\ &+N_{k+2}(r;0;f^n)^s)+N_{k+2}(r;0;P(f))+S(r,f)\end{aligned}$$

which implies

$$\begin{aligned}&(ns+m)T(r,f)\\ &\quad\leq N_{k+2}(r;0;f^n)^s)+N_{k+2}(r;0;P(f))+N_2(r,0;F)+T(r,F)+O(logr)+S(r,f)\end{aligned}$$

Using (4.9) and Lemma 3.8, we get

$$\begin{aligned}(ns+m)T(r,f) &\leq \frac{5}{2}\overline{N}(r,\infty;f)+\frac{1}{2}\overline{N}(r,0;F)+N_{k+2}\left(r,0;((f^n)^s)^{(k)}\right)+N_2\left(r,0;(((L^n)^s)P(L))^{(k)}\right)+O(logr)\\ &\leq \frac{5}{2}\overline{N}(r,\infty;f)+\frac{1}{2}[s(k+1)\overline{N}(r,0;f)+mN(r,0;f)+k\overline{N}(r,\infty;f)]\\ &\quad N_{k+2}(r,0;(L^n)^sP(L))+s(k+2)\overline{N}(r,0;f)+O(logr)\\ &\leq \left(\frac{5+k+3m+5ks+9s}{2}\right)T(r)+O(logr)\end{aligned}$$

where $T(r)=\max\{T(r,f),T(r,g)\}$.

In a corresponding way we can obtain

$$(ns+m)T(r,L)\leq\left(\frac{5+k+3m+5ks+9s}{2}\right)T(r)+O(logr)$$

Combining (4.10) and (4.11) we see that

$$(ns+m)-\left(\frac{5+k+2m+5ks+9s}{2}\right)T(r)\leq O(logr).$$

Since $ns>\frac{5+k+2m+5ks+9s}{2}$, (4.12) leads to a contradiction.

Subcase 1.2 $l=0$. Using (4.9) and Lemma 3.9, we get

$$\begin{aligned}(ns+m)T(r,f) &\leq 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;L)+2\overline{N}(r,0;F)+N_{k+2}(r,0;(f^n)^sP(f))+N_2\left(r,0;((L^n)^sP(L))^{(k)}\right)\\ &\quad+\overline{N}\left(r,0;((L^n)^sP(L))^{(k)}\right)+O(logr)\\ &\leq (2ks+2m+4)\overline{N}(r,\infty;f)+(2(k+1)s+3m+k+2)\overline{N}(r,0;f)+(2ks+2m+3)\overline{N}(r,0;L)+O(logr)\\ &\leq (6ks+7m+2s+k+9)T(r)+O(logr),\end{aligned}$$

where $T(r) = \max\{T(r,f),T(r,g)\}$.

In a corresponding way we can obtain

$(ns + m)T(r,L) \leq (6ks + 7m + 2s + k + 9)T(r) + O(logr)$

Combining (4.13) and (4.14) we see that

$(ns - 6ks - 6m - 2s - k - 9)T(r) \leq O(logr)$.

Since $ns > 6ks + 6m + 2s + k + 9$, (4.15) leads to a contradiction.

Case 2. Let $H \equiv 0$. Then by Lemma 3.14, we obtain either

$((f^n)^s P(f))^{(k)} (L^n)^s P(L))^{(k)} \equiv p^2$

or

$(f^n)^s P(f) \equiv (L^n)^s P(L)$.

For $s = 1$

$f^n P(f) \equiv L^n P(L)$.

We consider the following two cases:

Case 2.1. Suppose that $((f^n)^s P(f))^{(k)} ((L^n)^s P(L))^{(k)} \equiv p^2$. Then,

$F_1 G_1 \equiv 1$

where

$$F_1 = \frac{((f^n)^s P(f))^{(k)}}{\eta(z)}, \quad G_1 = \frac{((L^n)^s P(L))^{(k)}}{\eta(z)}.$$

First of all, we prove that 0 is a Picard exceptional value of f and L. Indeed, suppose that $z_0 \notin (z{:}p(z) = 0)$ is a zero of f with multiplicity m. Then, by the view of (4.16) we can find that $z_0 = 1$ is a pole of L with multiplicity, say p_1, such that $mn - k = (m + n)p_1 + k$, and so $(m - p_1)n - mp_1 = 2k$ and so we have $n \leq 2k/s$, which contradicts the assumption $ns > k + s(2k + 4) + m$. Similarly, we can prove that 0 is a Picard exceptional value of L. On the other hand, by (4.1) and (4.16), Valiron-Mokhonko lemma (), a result from Whittaker [(), p. 82] and the definition of the order of a meromorphic function we have

$\rho(f) = \rho(f^n P(f)) = \rho((f^n P(f))^{(k)}) = \rho((L^n P(L))^{(k)}) = \rho(L^n P(L)) = \rho(L) = 1$.

Noting that L has at most one pole $z = 1$ in the complex plane, we have by (4.16),(4.18) and Lemma 3.2 that

$$(ns+k+m)\overline{N}(r,\infty;f) \leq N\big(r,0;((L^n(^sP(L))^{(k)}\big)$$
$$\leq N(r,0;(L^n)^s)+(k+m)\overline{N}(r,\infty;(L^n)^s)+O(logr)=O(logr).$$

Therefore,

$$\overline{N}(r,\infty;f)+\overline{N}(r,\infty;L)\leq O(logr).$$

Now we set

$$f_1=\frac{((f^n)^sP(f))^{(k)}}{((L^n)^sP(L))^{(k)}},\quad f_2=\frac{((f^n)^sP(f))^{(k)}-1}{((L^n)^sP(L))^{(k)}-1}.$$

By (4.21) and the assumption that f and L are transcendental meromorphic functions. We have $f_1\not\equiv 0$ and $f_2\not\equiv 0$. Suppose that one of f_1 and f_2 is a nonzero constant. Then, by (4.21) we see that $((f^n)^sP(f))^{(k)}$ and $((L^n)^sP(L))^{(k)}$ share ∞ CM. Combining this with (4.16) we deduce that ∞ is a Picard exceptional value of f and L. Next we suppose that f_1 and f_2 are non-constant meromorphic functions. Then, by (4.16) and (4.21) we have

$$F_2=\frac{f_1(1-f_2)}{f_1-f_2},\quad G_2=\frac{1-f_2}{f_1-f_2}.$$

By (4.22) we can find that there exists a subset $I\subset(0,+\infty)$ with infinite linear measure such that $S(r)=o(T(r))$ and

$$T(r,F_2) \leq 2(T(r,f_1)+T(r,f_2))+S(r)$$
$$\leq 8T(r,F_1)+S(r).$$

These give $S(r,F_2)=S(r;f_1,f_2)$. Also we note that

$$\overline{N}(r,0;f_i)+\overline{N}(r,\infty;f_i)=S(r;f_1,f_2),$$

for $i=1,2$.

We note that $\overline{N}(r,-1;F_2)\neq S(r,F_2)$, since otherwise by the second fundamental theorem, F_1 will be a constant.

Also we see that

$$\overline{N}(r,-1;F_2)\leq N_0(r,1;f_1,f_2).$$

Thus we have

$$T(r,f_1)+T(r,f_2)\leq 4N_0(r,1;f_1,f_2)+S(r,F_1).$$

Then, by Lemma 3.6 there exists two mutually prime integers s and $t(|s|+|t|>0)$ such that

$f_1^s f_2^t \equiv 1,$

i.e.,

$$\left[\frac{F_2}{G_2}\right]^s \left[\frac{F_2-1}{G_2-1}\right]^t \equiv 1.$$

If either s or t is zero then we arrive at contradiction and so $st \neq 0$.

By (4.24) we consider the following two subcases:

Subcase 2.1.1. Suppose that $st < 0$, say $s > 0$ and $t < 0$, say $t = -t_1$, where t_1 is some Positive integer. Then, (4.24) can be written as

$$\left[\frac{F_1}{G_1}\right]^s \equiv \left[\frac{F_1-1}{G_1-1}\right]^{t_1}.$$

Let z_1 be a pole of F_2 of multiplicity p_1. Then from (4.25) we see that z_1 must be a zero of G_1 of multiplicity p_1. Now from (4.25) we get $2s = t_1$, which is impossible. Hence F_2 has no pole. Similarly we can prove that G_2 also has no poles.

Subcase 2.1.2. Suppose that $st < 0$ or $st > 0$. Then by (4.25) we can see that F_2 and G_2 share ∞ CM. This together with (4.16) and (4.18) implies that ∞ is a Picard Exceptional value of f and L. Combining this with the obtained result that 0 is a Picard Exceptional value of f and L, we have

$$L(z) = e^{A_2 z + B_2},$$

where $A_2 \neq 0$ and B_2 are constants. By (4.26) and Hayman [], p. 7] we have

$$T(r,L) = T(r,e^{A_2 z + B_2}) = \frac{|A_2|r}{\pi}(1 + o(1)).$$

Which contradicts (4.1).

Case 2.2. Suppose that $f^n = L^n$. Then, we have $f = tL$, where t is a constant satisfying $t^n = 1$.

This completes the proof of Theorem 2.1.

Proof of Theorem 2.2

Proof. Noting that $\overline{N}(r,\infty;f) = \overline{N}(r,\infty;L) = 0$, and proceeding in the like manner as the proof of Theorem 2.1 we obtain the proof of the Theorem 2.2. □

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