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A new subclass of analytic functions associated with generalized telephone number

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The aim of this paper is to find the initial coefficient bounds $|a_2|$ and $|a_3|$ of Taylor-Maclaurin's series and Fekete-Szegö estimate for the function belongs to the newly defined subclasses defined by the subordination to generalized telephone number. A same results have been obtained to the inverse function, f^{-1}. Additionally, application to our results is poission distribution is defined and analysed using Hadamard product.

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Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty} a_n \mathfrak{z}^n, \qquad \mathfrak{z} \in \mathcal{U},$$

which are analytic in the open unit disk $\mathcal{U} = \{\mathfrak{z} \in \mathbb{C} | |\mathfrak{z}| < 1\}$. Furthermore, let $\mathcal{S}$ represent the subclass of all functions in $\mathcal{A}$ that are univalent in $\mathcal{U}$. It is well known that for any two functions f_1 and f_2 in $\mathcal{A}$, we say that f_1 is subordinate to f_2 in $\mathcal{U}$ if there exists an analytic function ψ with $\psi(0) = 0$ and $|\psi(\mathfrak{z})| < 1$ such that $f_1(\mathfrak{z}) = f_2(\psi(\mathfrak{z}))$. This subordination is denoted by $f_1 \prec f_2$. If f_2 is univalent in $\mathcal{U}$, then

$$f_1 \prec f_2 \quad (\mathfrak{z} \in \mathcal{U}) \quad \Leftrightarrow \quad f_1(0) = f_2(0) \quad \text{and} \quad f_1(\mathcal{U}) \subset f_2(\mathcal{U}).$$

The class of starlike functions, $\mathcal{S}^*$ and the class of convex functions, $\mathcal{C}$ are among the most well-studied subclasses of $\mathcal{S}$. These subclasses are defined as follows:

$$\mathcal{S}^* = \left\{ f \in \mathcal{S} : \mathrm{Re}\left(\frac{\mathfrak{z} f'(\mathfrak{z})}{f(\mathfrak{z})}\right) > 0, \quad \mathfrak{z} \in \mathcal{U} \right\}$$

and

$$\mathcal{C}=\left\{f\in\mathcal{S}:\mathrm{Re}\left(1+\frac{\mathfrak{z}f''(\mathfrak{z})}{f'(\mathfrak{z})}\right)>0,\quad \mathfrak{z}\in\mathcal{U}\right\}.$$

Ma and Minda unified subclasses of starlike and convex functions by introducing the following two general classes by means of subordination:

$$\mathcal{S}^*(\phi)=\left\{f\in\mathcal{A}:\ \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})}\prec\phi(\mathfrak{z}),\ \mathfrak{z}\in\mathcal{U}\right\}$$

and

$$\mathcal{C}(\phi)=\left\{f\in\mathcal{A}:1+\frac{\mathfrak{z}f''(\mathfrak{z})}{f'(\mathfrak{z})}\prec\phi(\mathfrak{z}),\ \mathfrak{z}\in\mathcal{U}\right\},$$

where $\phi(\mathfrak{z})=1+B_1\mathfrak{z}+B_2\mathfrak{z}^2+B_3\mathfrak{z}^3+\cdots$ with $B_1>0$ is an analytic function which maps the unit disk $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis such that $\phi(0)=1$ and $\phi'(0)>0$. We observe that several familiar classes were introduced for the special choices of ϕ considering the conditions of the Ma and Minda , for example see .

The conventional telephone numbers (or involution numbers) are computed using the recurrence relation

$$\mathcal{T}(n)=\mathcal{T}(n-1)+(n-1)\mathcal{T}(n-2),\quad \text{for}\quad n\geq 2,$$

with the initial conditions $\mathcal{T}(0)=\mathcal{T}(1)=1$.

The connection between these numbers and symmetric groups was first observed in 1800 by Heinrich August Rothe, who pointed out that $\mathcal{T}(n)$ represents the number of involutions (self-inverse permutations) in symmetric groups (for details, see). Since involutions correspond to standard Young tableaux, it follows that the n^{th} involution number also represents the number of Young tableaux on the set $\{1,2,3,\ldots,n\}$ (for details, see).

It is worth mentioning that the telephone number interpretation of this recurrence is attributed to John Riordan , who observed that $\mathcal{T}(n)$ represents the number of connection patterns in a telephone system with n subscribers. Włoch and Wołowiec-Musiał introduced a new version of telephone numbers, defined recursively for integers $n\geq 0$ and $\lambda\geq 1$ as

$$\mathcal{T}(\lambda,n)=\lambda\mathcal{T}(\lambda,n-1)+(n-1)\mathcal{T}(\lambda,n-2),$$

with initial conditions $\mathcal{T}(\lambda,0)=1$ and $\mathcal{T}(\lambda,1)=\lambda$.

Furthermore, in 2019, Bednarz and Wołowiec-Musiał proposed a new generalization of telephone numbers (GTN), given by

$$\mathcal{T}_\lambda(n)=\mathcal{T}_\lambda(n-1)+\lambda(n-1)\mathcal{T}_\lambda(n-2),\quad n\geq 2,\quad \lambda\geq 1,$$

with initial conditions $\mathcal{T}_\lambda(0)=1$ and $\mathcal{T}_\lambda(1)=1$. For these numbers, they derived the generating function, an explicit formula, and matrix representations. Additionally, they provided combinatorial interpretations and examined various congruence properties of these numbers. Notably, the exponential generating function for $\mathcal{T}_\lambda(n)$ and the assumed formula for GTN are given by

$$e^{\left(x+\lambda\frac{x^2}{2}\right)}=\sum_{n=0}^{\infty}\mathcal{T}_\lambda(n)\frac{x^n}{n!},\quad (\lambda\geq 1).$$

We observe that for $\lambda = 1$, this reduces to the classical telephone numbers $\mathcal{T}(n)$ as defined in [TN-GF].

Particular values of $\mathcal{T}_\lambda(n)$*, for some values of* n*.*

n	$\mathcal{T}_\lambda(n)$
0	1
1	1
2	$1+\lambda$
3	$1+3\lambda$
4	$1+6\lambda+3\lambda^2$
5	$1+10\lambda+15\lambda^2$
6	$1+15\lambda+45\lambda^2+15\lambda^3$

Now, we consider the function

$$\mathcal{X}_\lambda(\mathfrak{z}) := e^{\left(\mathfrak{z}+\lambda\frac{\mathfrak{z}^2}{2}\right)} = 1+\mathfrak{z}+\left(\frac{1+\lambda}{2}\right)\mathfrak{z}^2+\left(\frac{1+3\lambda}{6}\right)\mathfrak{z}^3+\left(\frac{3\lambda^2+6\lambda+1}{24}\right)\mathfrak{z}^4+\cdots$$

which is analytic in $\mathcal{U}$ and satisfies $\mathcal{X}_\lambda(0) = 1$ and $(\mathcal{X}_\lambda)'(0) > 0$. Moreover, it maps the unit disk $\mathcal{U}$ onto a starlike region with respect to 1, which is symmetric about the real axis. Significant attention has been devoted to this function in geometric function theory (GFT), as discussed in and the references therein.

Inspired by the work of the aforementioned authors and utilizing GTN, we introduce a novel subclass of $\mathcal{S}$. For functions belonging to this class, we derive initial coefficient bounds for $|a_2|$ and $|a_3|$, including the Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$, as well as coefficient inequalities for the inverse function f^{-1} within the function classes considered. As an application of our results, we also explore connections with the Poisson distribution.

Definition 1. Let $0 \le k \le 1$ and $f \in \mathcal{A}$ then f be of the form [Taylor-1] is said to be in the class $\mathcal{M}(\lambda,k)$ if it satisfies the following condition,

$$\left(f'(\mathfrak{z})\right)^k\left(\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})}\right)^{1-k} \prec \mathcal{X}_\lambda(\mathfrak{z}), \quad \mathfrak{z} \in \mathcal{U}.$$

Definition 2. Let $0 \le k \le 1$ and $f \in \mathcal{A}$ then f be of the form [Taylor-1] belongs to the class $\mathcal{N}(\lambda,k)$ if it satisfies the following condition,

$$\left(f'(\mathfrak{z})\right)^k\left(1+\frac{\mathfrak{z}f''(\mathfrak{z})}{f'(\mathfrak{z})}\right)^{1-k} \prec \mathcal{X}_\lambda(\mathfrak{z}), \quad \mathfrak{z} \in \mathcal{U}.$$

Remark 1. By assigning a particular value in the above definitions, we get:

1. $\mathcal{M}(\lambda,0) \equiv \mathcal{S}^*(\lambda)$ consists of function $f \in \mathcal{A}$ of the form [Taylor-1] defined by,

$$\mathcal{S}^*(\lambda) = \left\{f \in A: \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} \prec \mathcal{X}_\lambda(\mathfrak{z}), \quad \mathfrak{z} \in \mathcal{U}\right\}.$$

2. $\mathcal{N}(\lambda,0) \equiv \mathcal{K}(\lambda)$ consists of function $f \in \mathcal{A}$ of the form [Taylor-1] defined by,

$$\mathcal{K}(\lambda) = \left\{f \in A: 1+\frac{\mathfrak{z}f''(\mathfrak{z})}{f'(\mathfrak{z})} \prec \mathcal{X}_\lambda(\mathfrak{z}), \quad \mathfrak{z} \in \mathcal{U}\right\}.$$

3. Setting $k=1$ in Definition 1 or Definition 2, we get the class of functions $f \in \mathcal{A}$ of the form [Taylor-1] defined by

$\mathcal{R}(\lambda)=\{f \in A: f'(\mathfrak{z}) \prec \mathcal{X}_\lambda(\mathfrak{z}), \quad \mathfrak{z} \in \mathcal{U}\}.$

Let us denote $\mathcal{P}$ the class of Carathéodory functions p that are analytic of the form:

$$p(\mathfrak{z})=1+\sum_{n=1}^{\infty} c_n \mathfrak{z}^n, \quad \mathfrak{z} \in \mathcal{U},$$

with positive real part ($\operatorname{Re}(p(\mathfrak{z}))>0$) in $\mathcal{U}$ and $p(0)=1$. Next, we consider the following lemmas to discuss our results.

Lemma 1. *Let $p \in \mathcal{P}$ be of the form [p-eq1]. Then, for all $n \in \mathbb{N}:=\{1,2,3,\ldots\}$, we have* $|c_n| \leq 2$.

Lemma 2. *Suppose $p \in \mathcal{P}$ is given by [p-eq1]. If μ is any complex number, then the following inequality holds:* $\left|c_2-\mu c_1^2\right| \leq 2\max\{1,|2\mu-1|\}$.

Lemma 3. *Consider a function $p \in \mathcal{P}$ of the form [p-eq1]. If μ is a real number, then the following bound holds:* $\left|c_2-\mu c_1^2\right| \leq \begin{cases} -4\mu+2 & : \quad \mu \leq 0 \\ 2 & : \quad 0 \leq \mu \leq 1 \\ 4\mu-2 & : \quad \mu \geq 1 \end{cases}$

coefficient estimates

In the following theorem, we obtain initial coefficient estimates for the function f in the class of $\mathcal{M}(\lambda,k)$ and $\mathcal{N}(\lambda,k)$.

Theorem 1. *If $0 \leq k \leq 1$ and $f \in \mathcal{A}$ is in the form [Taylor-1] belong to the class $\mathcal{M}(\lambda,k)$, then* $|a_2| \leq \frac{1}{1+k}$, $|a_3| \leq \frac{1}{(2+k)}\max\left\{1,\left|\frac{(\lambda+1)(1+k)^2+(2+k)(k-1)}{2(1+k)^2}\right|\right\}$.

Proof. Consider $p \in \mathcal{P}$ by $p(\mathfrak{z})=\frac{1+w(\mathfrak{z})}{1-w(\mathfrak{z})}=1+c_1\mathfrak{z}+c_2\mathfrak{z}^2+c_3\mathfrak{z}^3+\cdots$. Writing $w(\mathfrak{z})$ in terms of $p(\mathfrak{z})$, we get $w(\mathfrak{z})=\frac{p(\mathfrak{z})-1}{p(\mathfrak{z})+1}=\frac{c_1}{2}\mathfrak{z}+\left(\frac{c_2}{2}-\frac{c_1^2}{4}\right)\mathfrak{z}^2+\left(\frac{c_1^2}{8}-\frac{1}{2}c_1c_2+\frac{c_3}{2}\right)\mathfrak{z}^3+\cdots$. Using the above relation in the representation of $\mathcal{X}_\lambda(w(\mathfrak{z}))$ stated in [GTN-fucntion], we obtain $\mathcal{X}_\lambda(w(\mathfrak{z}))=1+\frac{c_1}{2}\mathfrak{z}+\left(\frac{c_2}{2}+\frac{(\lambda-1)c_1^2}{8}\right)\mathfrak{z}^2+\left(\frac{c_3}{2}+(\lambda-1)\frac{c_1c_2}{4}+\frac{(1-3\lambda)}{48}c_1^3\right)\mathfrak{z}^3+\cdots$. Consider $f \in \mathcal{M}(\lambda,k)$. By Definition 1, there exist an analytic function $w(\mathfrak{z})$, such that $\left(f'(\mathfrak{z})\right)^k\left(\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})}\right)^{1-k}=\mathcal{X}_\lambda(w(\mathfrak{z})) \quad (\mathfrak{z} \in \mathcal{U})$. In the view of [Taylor-1], we have $$\begin{aligned} \label{eq8} \notag\left(f^{\prime}(\mathfrak{z})\right)^{{k}} {\left(\frac{\mathfrak{z}f^{\prime}(\mathfrak{z})}{f(\mathfrak{z})}\right)^{1-{k}}} &=1+(1+{k})a_{2}\mathfrak{z}+(2+{k})\left(a_{3}-\frac{1-{k}}{2}a_{2}^{2}\right)\mathfrak{z}^{2}\\&\qquad +\frac{(3+{k})}{6}\left[6a_{4}-6(1-k)a_{2}a_{3}+(1-{k})(2-{k})a_{2}^{3}\right] \mathfrak{z}^{3}+\cdots. \end{aligned}$$ In view of [eq6], we compare the corresponding coefficients of [eq7] and [eq8], which leads to $(1+k)a_2=\frac{c_1}{2}$, $(2+k)\left(a_3-\frac{1-k}{2}a_2^2\right)=\frac{c_2}{2}+\frac{(\lambda-1)c_1^2}{8}$. Using Lemma 1 in [eq9], we get $|a_2| \leq \frac{1}{(1+k)}$. Rewriting [eq10], we get

$a_3 = \frac{1}{2(2+k)}\left[c_2-\alpha c_1^2\right]$ where $\alpha = \frac{(1-\lambda)(1+k)^2+(2+k)(k-1)}{4(1+k)^2}$. *Applying Lemma 2 in [eq10] we obtain,* $|a_3| \leq \frac{1}{(2+k)}\max\left\{1,\left|\frac{(\lambda+1)(1+k)^2+(2+k)(k-1)}{2(1+k)^2}\right|\right\}$. *This completes the proof of theorem.* □

Theorem 2. *If* $0 \leq k \leq 1$ *and the function* $f \in \mathcal{A}$ *of the form [Taylor-1] is in the class* $\mathcal{N}(\lambda,k)$*, then* `$$\left|a_{2}\right| \leq \frac{1}{2}, \\ $$` $|a_3| \leq \frac{1}{3(2-k)}\max\left\{1,\left|\frac{(1-\lambda)+(2k-4)}{2}\right|\right\}$.

Proof. Consider the function f *given by [Taylor-1] which is in the class* $\mathcal{N}(\lambda,k)$ *and by the defination [defination-2], there exists an analytic function* $w(\mathfrak{z})$*, such that* $\left(f'(\mathfrak{z})\right)^k\left(1+\frac{\mathfrak{z}f''(\mathfrak{z})}{f'(\mathfrak{z})}\right)^{1-k} = \mathcal{X}_\lambda(w(\mathfrak{z})) \quad (\mathfrak{z} \in \mathcal{U})$. *From [Taylor-1],* `$$\begin{aligned} \label{eq12} \notag{\left[f^{\prime}(\mathfrak{z})\right]^{k}\left[1+\frac{\mathfrak{z}f^{\prime\prime}(\mathfrak{z})}{f^{\prime}(\mathfrak{z})}\right]^{1-k}} &=1+2a_{2}\mathfrak{z}+\left[3(2-k) a_{3}-4(1-k) a_{2}^{2}\right] \mathfrak{z}^{2} \\&\qquad +\left[8(1-k) a_{2}^{3}-18(1-k) a_{2} a_{3}-4(3-2 k) a_{4}\right] \mathfrak{z}^{3}+\cdots. \end{aligned}$$` *substituting [eq7], [eq12] in [eq11] and comparing corresponding coefficients* $\mathfrak{z}^2$, $\mathfrak{z}^3$ *we obtain,* `$$\label{eq13} a_{2}=\frac{c_{1}}{4} \\ $$` $a_3 = \frac{1}{6(2-k)}\left[c_2-\frac{[(1-\lambda)-2(1-k)]}{4}c_1^2\right]$. *Using Lemma [lemma-1] and Lemma [lemma-2] respectively in [eq13] and [eq14] gives the bounds of* a_2 *and* a_3*. Hence the proof of theorem is complete.* □

In the view of Remark 1, we present the following corollaries

Corollary 1. *If* $f \in \mathcal{S}^*(\lambda)$ *and is of the form [Taylor-1], then* $|a_2| \leq 1$ *and* $|a_3| \leq \frac{1}{2}\max\left\{1,\left|\frac{\lambda-1}{2}\right|\right\}$.

Corollary 2. *If* $f \in \mathcal{K}(\lambda)$ *and is of the form [Taylor-1], then* $|a_2| \leq \frac{1}{2}$ *and* $|a_3| \leq \frac{1}{6}\max\left\{1,\left|\frac{\lambda+3}{2}\right|\right\}$.

Corollary 3. *If* $f \in \mathcal{R}(\lambda)$ *and is of the form ([Taylor-1]), then* $|a_2| \leq \frac{1}{2}$ *and* $|a_3| \leq \frac{1}{3}\max\left\{1,\left|\frac{\lambda+1}{2}\right|\right\}$.

Remark 2. The result $|a_2|$ obtained in corollary 1 and corollary 2 coincide with the findings of and $|a_3|$ is an improvement with the result of .

Fekete-Szegö Estimates

Now in this section, for the class $\mathcal{M}(\lambda,k)$ and $\mathcal{N}(\lambda,k)$ we are estimating the Fekete-Szegö inequality $\left|a_3-\mu a_2^2\right|$, when μ is both real and complex.

Theorem 3. *Let* $0 \leq k \leq 1$, $\mu \in \mathbb{C}$. *If* $f \in \mathcal{M}(\lambda,k)$, *then* $\left|a_3-\mu a_2^2\right| \leq \frac{1}{(2+k)}\max\left\{1,\left|\frac{(\lambda+1)(1+k)^2+(2+k)(k+2\mu-1)}{2(1+k)^2}\right|\right\}$.

Proof. *From [eq9] and [eq10], we get* $\left|a_3-\mu a_2^2\right| = \frac{1}{2(2+k)}\left[c_2-\eta c_1^2\right]$ *where* $\eta = \frac{(1-\lambda)(1+k)^2+(2+k)(k-1)+2(2+k)\mu}{4(1+k)^2}$ *Using Lemma 2 in [eq15],* $\left|a_3-\mu a_2^2\right| \leq \frac{1}{(2+k)}\max\left\{1,\left|\frac{(\lambda+1)(1+k)^2+(2+k)(k+2\mu-1)}{2(1+k)^2}\right|\right\}$ *Therefore, Theorem 3 is complete.* □

Theorem 4. *Let* $0 \leq k \leq 1$, $\mu \in \mathbb{C}$. *If* $f \in \mathcal{N}(\lambda,k)$, *then* $\left|a_3-\mu a_2^2\right| \leq \frac{1}{3(2-k)}\max\left\{1,\left|\frac{2(k-1)+3\mu(2-k)-2(\lambda+1)}{4}\right|\right\}$.

Proof. From equation [eq13] and [eq14], we attain, $\left|a_3-\mu a_2^2\right| = \frac{1}{6(2-k)}\left|c_2-\eta_1 c_1^2\right|$ *where* $\eta_1 = \frac{2(1-\lambda)+2(k-1)+3\mu(2-k)}{8}$ *Using Lemma [lemma-2] in [eq16] gives the required result.* □

Corollary 4. *Let* $0 \leq k \leq 1$, $\mu \in \mathbb{C}$.*If* $f \in \mathcal{S}^*(\lambda)$*and is of the form [Taylor-1],then* $\left|a_3-\mu a_2^2\right| \leq \frac{1}{2}\max\left\{1,\left|\frac{4\mu+(\lambda-1)}{2}\right|\right\}$.

Corollary 5. *Let* $0 \leq k \leq 1$, $\mu \in \mathbb{C}$. *If* $f \in \mathcal{K}(\lambda)$ *and is of the form [Taylor-1], then* $\left|a_3-\mu a_2^2\right| \leq \frac{1}{6}\max\left\{1,\left|\frac{3\mu-(\lambda+2)}{2}\right|\right\}$.

Corollary 6. *Let* $0 \leq k \leq 1$, $\mu \in \mathbb{C}$.*If* $f \in \mathcal{R}(\lambda)$ *and is of the form ([Taylor-1]), then* $\left|a_3-\mu a_2^2\right| \leq \frac{1}{3}\max\left\{1,\left|\frac{3\mu+2(\lambda+1)}{4}\right|\right\}$,

Theorem 5. *Let* $0 \leq k \leq 1$, $\mu \in \mathbb{R}$. *If* $f \in \mathcal{M}(\lambda,k)$, *then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} -\frac{(1+\lambda)(1+k)^2-(2+k)(k+2\mu-1)}{2(2+k)(1+k)^2} & :\mu \leq -S_1 \\ \frac{1}{2+k} & :-S_1 \leq \mu \leq S_2 \\ -\frac{(1+\lambda)(1+k)^2+(2+k)(k+2\mu-1)}{2(2+k)(1+k)^2} & :\mu \geq S_2 \end{cases}$$

where

$S_1 = \frac{(1-\lambda)(1+k)^2+(2+k)(k-1)}{2(2+k)}$ *and* $S_2 = \frac{(\lambda+3)(1+k)^2-(2+k)(k-1)}{2(2+k)}$

Proof. By using Lemma [lemma-3] to the equation [eq15], gives the required result. □

Theorem 6. *Let* $0 \leq k \leq 1$, $\mu \in \mathbb{R}$. *If* $f \in \mathcal{N}(\lambda,k)$, *then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{2(\lambda+1)+2(1-k)+3\mu(k-2)}{2} & :\mu \leq \frac{2(\lambda-k)}{3(2-k)} \\ \frac{1}{3(2-k)} & :\frac{2(\lambda-k)}{3(2-k)} \leq \mu \leq \frac{8-2(k-\lambda)}{3(2-k)} \\ \frac{2(k-\lambda)+3\mu(2-k)-4}{2} & :\mu \geq \frac{8-2(k-\lambda)}{3(2-k)} \end{cases}$$

Proof. Using Lemma [lemma-3] in the equation [eq16] we obtain above result. □

Coefficient Inequalities for the inverse Function

From Koebe one-quarter theorem , for every function $f \in \mathcal{A}$, there exist inverse function

$$f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} b_n\, \omega^n,$$

on a disk with $|\omega| < \frac{1}{4}$. Since f is in $\mathcal{M}(\lambda,k)$ and $\mathcal{N}(\lambda,k)$ and are univalent functions in $\mathcal{U}$,there exist $f^{-1}(\omega)$ as in the above equation with $|\omega| < r_0$, where $r_0 > \frac{1}{4}$ where r_0 is greater than the radius of the Koebe domain of these classes. In our next results, we are finding the bounds of $|b_2|$, $|b_3|$ and Fekete-Szegö inequality $|b_3-\mu_1 b_2|$ of f^{-1}.

Theorem 7. *If* $f \in \mathcal{M}(\lambda,k)$ *and* $f^{-1}(w)=w+\sum_{n=2}^{\infty} b_n w^n$ *is the inverse function of* f *with* $|w|<r_0$ *where* $r_0>\frac{1}{4}$ *, then for any complex number* μ_1, $|b_2|\leq\frac{1}{1+k}$, $|b_3|\leq\frac{1}{(2+k)}\max\left\{1,\left|\frac{(\lambda+1)(k+1)-(k+2)}{2(1+k)}\right|\right\}$ *and* $\left|b_3-\mu_1 b_2^2\right|\leq\frac{1}{(2+k)}\max\left\{1,\left|\frac{(1+\lambda)(1+k)^2-(2+k)(1+k-2\mu_1)}{2(1+k)^2}\right|\right\}$.

Proof. Since $f^{-1}(w)=w+\sum_{n=2}^{\infty} b_n w^n$ *it is clear that,* $f^{-1}(f(\mathfrak{z}))=f\left(f^{-1}(\mathfrak{z})\right)=\mathfrak{z}$ *From [Taylor-1] and [eq18],* $f^{-1}\left(\mathfrak{z}+\sum_{n=2}^{\infty} a_n \mathfrak{z}^n\right)=\mathfrak{z}$ *after simplification,* $\mathfrak{z}+(a_2+b_2)\mathfrak{z}^2+(a_3+2a_2b_2+b_3)\mathfrak{z}^3+\cdots=\mathfrak{z}$ *Comparing the coefficients of* $\mathfrak{z}^2$ *and* $\mathfrak{z}^3$ *in [eq20], we get* $b_2=-a_2$ *and* $b_3=-a_3-2a_2b_2=2a_2^2-a_3$ *Considering the values of* a_2 *and* a_3 *from [eq9] and [eq10] and substituting in [eq21] and [eq22], we obtain* $b_2=-\frac{c_1}{2(1+k)}$ *and* $b_3=-\frac{1}{2(2+k)}\left[c_2-\frac{[(1-\lambda)(1+k)+(2+k)]}{4(1+k)}c_1^2\right]$ *Using Lemma [lemma-1] and [lemma-2] in equation [eq23] and [eq24],* $|b_2|\leq\frac{1}{1+k}$, $|b_3|\leq\frac{1}{(2+k)}\max\left\{1,\left|\frac{(\lambda+1)(k+1)-(k+2)}{2(1+k)}\right|\right\}$ *And, for any complex number* μ_1, *we have* $\left|b_3-\mu_1 b_2^2\right|=\frac{-1}{2(2+k)}\left|c_2-\frac{(1-\lambda)(1+k)^2+(2+k)(1+k)+2(2+k)\mu_1}{4(1+k)^2}c_1^2\right|$ *Using Lemma [lemma-2] in [eq25],* $\left|b_3-\mu_1 b_2^2\right|\leq\frac{1}{(2+k)}\max\left\{1,\frac{|(1+x)(1+k)^2-(2+k)(1+k-2\mu_1)|}{2(1+k)^2}\right\}$. □

Hence this completes the proof.

Theorem 8. *If* $f \in \mathcal{N}(\lambda,k)$ *and* $f^{-1}(w)=w+\sum_{n=2}^{\infty} b_n w^n$ *is the inverse function of* f *with* $|w|<r_0$ *where* $r_0>\frac{1}{4}$ *then for any complex number* μ_1, $|b_2|\leq\frac{1}{2}$, $|b_3|\leq\frac{1}{3(2-k)}\max\left\{1,\left|\frac{(1-\lambda)+(2-k)}{2}\right|\right\}$ *and* $|b_3-\mu_1 b_2^2|\leq\frac{1}{3(2-k)}\max\{1,|\frac{2(1-\lambda)+(2-k)(2-3\mu_1)}{4}|\}$.

Proof. Proceeding as in theorem [T7] and substituting the values of a_2 *and* a_3 *from [eq13] and [eq14] into [eq21] and [eq22], we obtain* $b_2=-\frac{c_1}{4}$ *and* $b_3=-\frac{1}{6(2-k)}\left[c_2-\frac{(1-\lambda)-(k-4)}{4}c_1^2\right]$ $\left|b_3-\mu_1 b_2^2\right|=\frac{-1}{6(2-k)}\left|c_2-\frac{2(1-\lambda)+4(k-1)+3(2-k)(2-\mu_1)}{8}c_1^2\right|$. *Applying Lemma [lemma-1] and [lemma-2] in to [eq26] , [eq27] and [eq28], we can obtain* b_2, b_3 *and* $\left|b_3-\mu_1 b_2^2\right|$ *as required. Hence the proof.* □

Application of the Poisson Distribution

Now, we discuss the application to functions $\mathcal{M}(\lambda,k)$ and $\mathcal{N}(\lambda,k)$ based on the Poisson distributions.

Definition 3. Let X be a discrete random variable and λ be the mean. Then, X is said to be a Poisson distribution and defined by,

$$P(X=y)=\frac{e^{-\lambda}\lambda^y}{y!},\quad where\quad y=0,1,2,3,\cdots\quad and\quad \lambda>0.$$

Porwal expressed a Poisson distribution in terms of power series given by,

$$P(\lambda,\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} e^{-\lambda} \mathfrak{z}^{n}, \quad \mathfrak{z} \in \mathcal{U}.$$

By ratio test the radius of the convergence of the series [eq29] is infinity. Also Porwal defined a linear operator $\mathfrak{G}^{\lambda}:\mathcal{A}\rightarrow\mathcal{A}$ by

$$\mathfrak{G}^{\lambda} f(\mathfrak{z}) = P(\lambda,\mathfrak{z}) * f(\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} e^{-\lambda} a_n \mathfrak{z}^{n} = \mathfrak{z} + \sum_{n=2}^{\infty} \eta_n \, a_n \mathfrak{z}^{n}$$

where $\eta_n = \eta_n(\lambda) = \frac{\lambda^{n-1} e^{-\lambda}}{(n-1)!}$ and $*$ denotes the Hadamard product (or convolution) between two analytic functions.

Now let us define the class $\mathcal{M}(\lambda,k,\eta)$ and $\mathcal{N}(\lambda,k,\eta)$ as given below,

$\mathcal{M}(\lambda,k,\eta) = \{f \in \mathcal{A}:\mathfrak{G}^{\lambda} f \in \mathcal{M}(\lambda,k)\}$

and

$\mathcal{N}(\lambda,k,\eta) = \{f \in \mathcal{A}:\mathfrak{G}^{\lambda} f \in \mathcal{N}(\lambda,k)\}$.

Different subclasses of analytic and univalent functions are discussed based on Poisson distribution by many authors . In the next results we are finding Fekete-Szegö inequality, for the class $\mathcal{M}(\lambda,k,\eta)$ and $\mathcal{N}(\lambda,k,\eta)$.

Theorem 9. *Let* $0 \leq k \leq 1$ *and* $\mu \in \mathbb{C}$, *if* $f \in \mathcal{M}(\lambda,k,\eta)$ *and* $\mathfrak{G}^{\lambda} f$ *then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{(2+k)\eta_3} \max\left\{1, \left|\frac{(1+\lambda)(1+k)^2\eta_2^2 - (2+k)(k-1) - 2(2+k)\eta_3\mu}{2(1+k)^2\eta_2^2}\right|\right\}.$$

Proof. Since $f \in \mathcal{M}(\lambda,k,\eta)$, *from [eq31]* $\left[\left(\mathfrak{G}^{\lambda} f(\mathfrak{z})\right)'\right]^{k}\left[\frac{\mathfrak{z}\left(\mathfrak{G}^{\lambda} f(\mathfrak{z})\right)'}{\left(\mathfrak{G}^{\lambda} f(\mathfrak{z})\right)}\right]^{1-k} = \mathcal{X}_{\lambda}(\omega(\mathfrak{z})), \quad \mathfrak{z} \in \mathcal{U}$ *using [eq30], we have $$\begin{aligned} \label{eq34} \notag{\left[\left(\mathfrak{G}^{\lambda} f(\mathfrak{z})\right)^{\prime}\right]^{k}\left[\frac{\mathfrak{z}\left(\mathfrak{G}^{\lambda} f(\mathfrak{z})\right)^{\prime}}{\left(\mathfrak{G}^{\lambda} f(\mathfrak{z})\right)}\right]^{1-k}} &=1+(1+k) \eta_{2} a_{2} \mathfrak{z}+(2+k)\left(\eta_{3} a_{3}-\frac{1-k}{2} \eta_{2}^{2} a_{2}^{2}\right) \mathfrak{z}^{2} \\&\qquad +\frac{(3+k)}{6}\left[6 \eta_{4} a_{4}-6(1-k) \eta_{2} a_{2} \eta_{3} a_{3}+(1-k)(2-k) \eta_{2}^{3} a_{2}^{3}\right] \mathfrak{z}^{3}+\cdots \end{aligned}$$ Substituting [eq7] and [eq34] in [eq33] and equating the coefficients of* $\mathfrak{z}$ *and* $\mathfrak{z}^{2}$, *we have* $a_2 = \frac{c_1}{2\eta_2(1+k)}$ *and* $a_3 = \frac{1}{2(2+k)\eta_3}\left[c_2 - \frac{(1-\lambda)(1+k)^2 + (2+k)(k-1)}{4(1+k)^2} c_1^2\right]$. *Also from [eq35] and [eq36] we get,* $\left|a_3 - \mu a_2^2\right| = \frac{1}{(2+k)\eta_3}\left[c_2 - \lambda c_1^2\right]$, *where* $\lambda = \frac{[(1-\lambda)(1+k)^2 + (2+k)(k-1)]\eta_2^2 + 2(2+k)\eta_3\mu}{4(1+k)^2\eta_2^2}$ *Using Lemma [lemma-2] in [eq37] gives the required result.This complites the proof of Theorem 9.* □

Theorem 10. *Let* $0 \leq k \leq 1$ *and* $\mu \in \mathbb{C}$,*if* $f \in \mathcal{N}(\lambda,k,\eta)$ *then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{3(2-k)\eta_3} \max\left\{1, \left|\frac{2(x+1)\eta_2^2 + 4(1-k)\eta_2^2 + 3(k-2)\eta_3\mu}{4\eta_2^2}\right|\right\}.$$

Proof. From the equation [eq32] and [eq30], $\left[\left(\mathfrak{G}^{\lambda} f(\mathfrak{z})\right)'\right]^{k}\left[1+\frac{\mathfrak{z}\left(\mathfrak{G}^{\lambda} f(\mathfrak{z})\right)''}{\left(\mathfrak{G}^{\lambda} f(\mathfrak{z})\right)'}\right]^{1-k}=\mathcal{X}_{\lambda}(\omega(\mathfrak{z})), \quad \mathfrak{z} \in \mathcal{U}$ $$\begin{aligned} \label{eq39} \notag{\left[\left(\mathfrak{G}^{\lambda} f(\mathfrak{z})\right)^{\prime}\right]^{k}\left[1+\frac{\mathfrak{z}\left(\mathfrak{G}^{ \lambda} f(\mathfrak{z})\right)^{\prime \prime}}{\left(\mathfrak{G}^{\lambda}f(\mathfrak{z})\right)^{\prime}}\right]^{1-k}} &= 1+2 \eta_{2} a_{2}\mathfrak{z}+\left[3(2-k) \eta_{3} a_{3}-4(1-k) \eta_{2}^{2} a_{2}^{2}\right] \mathfrak{z}^{2}\\&\qquad +\left[8(1-k) \eta_{2}^{3} a_{2}^{3}-18(1-k) \eta_{2} \eta_{3} a_{2} a_{3}-4(3-2 k) \eta_{4} a_{4}\right] \mathfrak{z}^{3}+\cdots \end{aligned}$$ *Using [eq7] and [eq39] in [eq38] and comparing the coefficients of* $\mathfrak{z}$ *and* $\mathfrak{z}^2$, $a_2=\frac{c_1}{4\eta_2}$ *and* $a_3=\frac{1}{6(2-k)\eta_3}\left[c_2-\frac{(1-\lambda)+2(k-1)}{4}c_1^2\right]$. *Appling Lemma [lemma-1] and [lemma-2] in equations [eq40] and [eq41] respectively we have bounds* a_2 *and* a_3. *Also* $\left|a_3-\mu a_2^2\right|=\frac{1}{6(2-k)\eta_3}\left|c_2-\frac{2(1-x)\eta_2^2+4(k-1)\eta_2^2+3(2-k)\eta_3\mu}{8\eta_2^2}c_1^2\right|$ *By Lemma [lemma-2] the above equation gives the required inequality.* ⍰

Theorem 11. *Let* $0\le k\le 1$ *and* $\mu\in\mathbb{R}$, *if* $f\in\mathcal{M}(\lambda,k,\eta)$ *then we have,*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}-\frac{(\lambda+1)(1+k)^2\eta_2^2-(2+k)(k-1)\eta_2^2-2(2+k)\eta_3\mu}{2(2+k)(1+k)^2\eta_2^2\eta_3} & :\mu\le -Q_1\\ \frac{1}{(2+k)\eta_3} & :-Q_1\le\mu\le Q_2\\ -\frac{\left[(1+\lambda)(1+k)^2+(2+k)(k-1)\right]\eta_2^2+2(2+k)\eta_3\mu}{2(2+k)(1+k)^2\eta_2^2\eta_3} & :\mu\ge Q_2\end{cases}$$

where

$$Q_1=\frac{\left[(1-\lambda)(1+k)^2+(2+k)(k-1)\right]\eta_2^2}{2(2+k)\eta_3} \quad \text{and} \quad Q_2=\frac{\left[(\lambda+3)(1+k)^2-(2+k)(k-1)\right]\eta_2^2}{2(2+k)\eta_3}$$

Proof. Using Lemma [lemma-3] in [eq37], we have requried result. ⍰

Theorem 12. *Let* $0\le k\le 1$ *and* $\mu\in\mathbb{R}$, *if* $f\in\mathcal{N}(\lambda,k,\eta)$ *then,*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}-\frac{2(1+x)\eta_2^2-4(k-1)\eta_2^2-3\mu(2-k)\eta_3}{12(2-k)\eta_2^2\eta_3} & :\mu\le-\frac{2(1-\lambda)\eta_2^2-4(k-1)\eta_2^2}{3(2-k)\eta_3}\\ \frac{1}{3(2-k)\eta_3} & :-\frac{2(1-\lambda)\eta_2^2+4(1-k)\eta_2^2}{3(2-k)\eta_3}\le\mu\le\frac{2(3+\lambda)\eta_2^2-4(k+1)\eta_2^2}{3(2-k)\eta_3}\\ \frac{-2(\lambda+1)\eta_2^2+4(k-1)\eta_2^2+3\mu(2-k)\eta_3}{12(2-k)\eta_2^2\eta_3} & :\mu\ge\frac{(2(3+\lambda)\eta_2^2-4(k-1)\eta_2^2}{3(2-k)\eta_3}\end{cases}$$

Proof. Make use of Lemma [lemma-3] in equation [eq42] and simplifying the equation we can obtain the desired result. □

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