

STRONGLY ISOLATE DOMINATION IN GRAPHS

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Abstract

This paper, we establish a newly discovered parameter called "Strongly Isolate Domination"(SID). A dominating set D^* of a graph G^+ is said to be an isolate dominating set (IDS) of G if $\langle D^* \rangle$ has at least one isolated vertex. The ID number of G^+ is represented by $\gamma_0(G^+)$. An ID-set D^* is considered as strongly isolate dominating set (SIDS) if there exists $y \in D^*$ such that $N_2(y) \cap D^* = \phi$, where $N_2(y) = \{y: d(y, b) \leq 2 \text{ and } y \neq b\}$. This paper involves some basic features of SIDS and compare SIDS with dominating set, ID-set and efficient dominating set(EDS). At the end, includes SID number of path, cycle, complete bipartite graph, complete b- partite graph and some group of graphs.

Keywords: Domination, isolate domination, strongly isolate domination.

Introduction

The general idea of ID in graphs has been exposed by I.Shahul Hamid, S.Balamurugan [1], in 2016. A dominating set D^* of a graph G^+ is considered as an isolate dominating set of G^+ if $\langle D^* \rangle$ has a minimum of one isolated vertex. The minimum cardinality of an ID- set of G^+ is known as ID number. The ID number of a graph G^+ is denoted by $\gamma_0(G^+)$. An ID- set of cardinality $\gamma_0(G^+)$ is called γ_0 - set. Through the application of the ID concept, we established a new domination criterion, specifically "Strongly Isolate Domination"(SID). An ID set D^* is called SID- set if there exists $v \in D^*$ such that $N_2(y) \cap D^* = \phi$, where $N_2(y) = \{y: d(y, b) \leq 2 \text{ and } y \neq b\}$. The minimum cardinality of a SID-set of G^* are known as the SID number. The SID number is represented by $\gamma_0^s(G^+)$. A SID set of cardinality $\gamma_0^s(G^+)$ is known as γ_0^s - set. An efficient dominating set (EDS) $D^* \subseteq V$ in a graph $G^+ = (V, E)$ is a dominating set with extra property that the closed neighbourhood $N[x]$ of every vertex $x \in V$ contains only one vertex in D^* . Let x is a point on a connected graph G^+ . The eccentricity $e(x)$ of x is defined by $e(x) = \max\{d(x, y)/x \in V (G^+)\}$. This paper explores certain properties of SID and the SID number in various groups of graphs.

2.Basics of strongly isolate dominating set.

This section focuses on deriving certain bounds for the SID number of connected graphs.

Theorem 2.1: For any graph G^+ that allows SIDS, $\gamma(G^+) \leq \gamma_0^s(G^+)$.

Proof 1: Since each and every SIDS of G^+ is also a dominating set of G^+ , we have $\gamma(G^+) \leq \gamma_0^s(G^+)$.

Theorem 2.2: For any graph G^+ that allows SIDS, $\gamma_0(G^+) \leq \gamma_0^s(G^+)$.

Proof 2 : Since each and every SIDS of G^+ is also an isolate dominating set of G^+ , we have $\gamma_0(G^+) \leq \gamma_0^s(G^+)$.

The next theorem gives the SID number of disconnected graphs which admit SID -sets.

Theorem 2.3: Let G^+ be a disconnected graph with p components F_1, F_2, \dots, F_p such that the first q components F_1, F_2, \dots, F_q admit SIDS. Then $\gamma_0^s(G^+) = \min_{1 \leq m \leq q} \{b_m\}$,

where $b_m = \gamma_0^s(F_m) + \sum_{1 \leq s \leq p, s \neq m} \{\gamma(F_s)\}$ for $1 \leq m \leq q$.

Proof 3: Let us assume $b_1 = \min_{1 \leq m \leq q} \{b_m\}$. Let T^+ be a γ_0^s -set of F_1 and let X_m^+ be a γ -set of F_m for every m with $2 \leq m \leq p$. Then the set $T^+ \cup (\cup_{m=2}^p X_m^+)$ is a SID-set of G^+ therefore $\gamma_0^s(G^+) \leq \gamma_0^s(F_1) + \sum_{m=2}^p \gamma(F_m) = b_1$. Now consider T' be a minimal SID set of G^+ . Hence the set T' should intersect the vertex set $V(F_m)$ for each m with $1 \leq m \leq p$. In addition, there exists r such that $T' \cap V(F_r)$ is a minimal SIDS of F_r and $1 \leq r \leq q$. Furthermore, every $1 \leq m \leq p, m \neq r$, the set $T' \cap V(F_m)$ is a dominating of F_m . Thus $|T'| \geq \gamma_0^s(F_r) + \sum_{1 \leq r \leq p, s \neq m} \gamma(F_m) \geq b_1$ and so $\gamma_0^s(G^+) = b_1$.

Remark 2.1 If the eccentricity of a vertex p of a graph G^+ is ≤ 2 , then the vertex is not SI and such vertex cannot be part of any SIDS of G^+ .

Theorem 2.4: Suppose G^+ be an ordered connected graph with $n \geq 4$. Then $\gamma_0^s(G^+) = 2$ if and only if there are two vertices $c, d \in D^* \subseteq V(G^+)$ therefore $N(c) \cap N(d) = \phi$ and $N[c] \cup N[d] = V(G^+)$.

Proof 4 : Suppose $\gamma_0^s(G^+) = 2$. Then there exists a SID-set $D^* = \{c, d\}$ and c is strongly isolated in $\langle D^* \rangle$. Suppose there exists $w \in N(c) \cap N(d)$. Then $d \in N_2(c)$, a contradiction to c is strongly isolated. Thus $N(c) \cap N(d) = \phi$. By the definition of dominating set $N[D^*] = V(G^+)$. That is, $N[c] \cup N[d] = V(G^+)$. Suppose that $N(c) \cap N(d) = \phi$ and $N[c] \cup N[d] = V(G^+)$. Clearly, $D^* = \{c, d\}$ is a SID-set of G^+ . Therefore, $\gamma_0^s(G^+) = 2$.

Theorem 2.5 A tree with $\dim(T^*) = 3$ admits strongly isolate dominating set iff T^* is a Mob graph.

Proof 5: Let G^+ be a connected graph. Assume that G^+ admits SID set, say D^* and let z be a strongly isolated vertex in $\langle D^* \rangle$. Figure 2.1 illustrate the longest path $P : u_1 - u_2 - u_3 - u_4$ in G^+ , since $\dim(G^+) = 3$.

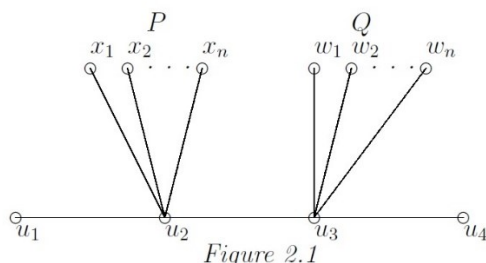


Figure 2.1

Let P represent the collection of vertices nearest to u_2 except u_1 and u_3 , Q represent the collection of vertices nearest to u_3 except u_2 and u_4 . In Remark 2.1, $u_2, u_3 \notin D^*$. Suppose $z = u_1$, then by the definition of SIDS, $u_2, N(u_2) \notin D^*$. Similarly, we can get contradiction when $z = u_4$.

Assume that P and Q are not empty.

Case 1: Suppose $z \in P$.

In any case imagine that $z = x_1$, then $u_2, u_1 \notin D^*$ and hence D^* cannot dominate u_1 , a contradiction. Similarly, $z \neq x_i$ for all $1 \leq i \leq |P|$.

Case 2: Suppose $z \in Q$.

In any case imagine that $z = w_1$, then $u_3, u_4 \notin D^*$ and hence D^* cannot dominate u_4 , a contradiction. Similarly, $z \neq w_i$ for all $1 \leq i \leq |Q|$. Thus either P or Q must be empty. Thus G^+ is a Mob graph.

Converse part is trivial.

Theorem 2.6 : Let G^+ be a graph such that $\dim(G^+) \leq 2$ also $\Delta(G^+) < n - 1$, then G^+ does not admit SID-set.

Proof 6 : On the contrary, suppose G^+ admits SIDS, let D be a SIDS. Let u be a strongly isolated vertex in $\langle D \rangle$. Since $degG(u) < n - 1$, there are some un dominated vertices in the graph $\langle G - N[u] \rangle$. Thus D must have one more element $v \neq u$. Since $diam(G^+) \leq 2$ and u is strongly isolated in $\langle D \rangle$, $N_2(u) \cap D = \phi$. But $N_2(u)$ contains a vertex v , a contradiction.

3 Comparative study of strongly isolate dominating set.

In this section, we compare SIDS with EDS set, dominating set and IDS.

Theorem 3.1 Let D^* be a SIDS in which each vertex of D^* is strongly isolated. Then D^* is an efficient dominating set.

Proof 7: Let D^* be a SIDS of a graph G in which each vertex is strongly isolated. Claim, D^* is an EDS .

Case 1: Let $v \in V - D^*$. Suppose there are two vertices $c, f \in D^* (c \neq f)$ such that v is adjacent with both c and f . Since c is strongly isolated, $N_2(c) \cap D^* = \phi$. But $N_2(c) \cap D^*$ contains a vertex f , a contradiction. Since D^* is a SIDS, v is adjacent with only one vertex of D^* . Thus $|N[v] \cap D^*| = 1$.

Case 2: If $v \in D^*$. Since v is strongly isolated, $|N[v] \cap D^*| = |\{v\}| = 1$. So D^* is an EDS.

Theorem 3.2 : Every efficient dominating set is a strongly isolate dominating set but however the opposite is not true.

Proof 8: Let G be a graph and D^* be EDS of G . To prove D^* is a SIDS, it is enough to prove that D^* has a strongly isolated vertex. Let $a \in D^*$. Suppose a is not strongly isolated, $N_2(a) \cap D^* \neq \phi$. Let $x \in N_2(a) \cap D^*$. Since D^* is independent, there exists $y \in V - D^*$ such that $(a, y), (y, x) \in E(G)$. Thus y is dominated twice by the vertex of D^* , namely, a and x , a contradiction. Thus a is strongly isolated and so D^* is SIDS.

Conversely, look at the following graph. Here in Figure 3.1, $D^* = \{a, d, f\}$ is a SIDS. Which is not an EDS.

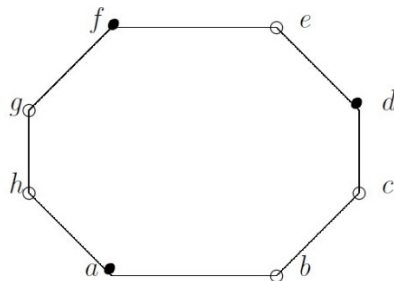


Figure 3.1

Theorem 3.3: For any integer $n \geq 1$, there exist a graph G^+ such that $\gamma(G^+) = \gamma_0(G^+) = \gamma_0^s(G^+) = n$.

Proof 9: Consider a path P_n with $V(P_n) = \{v_1, v_2, \dots, v_n\}$. For each $v_i (1 \leq i \leq n)$, attach a pendent vertex $u_i (1 \leq i \leq n)$ and let G^+ be the resultant graph. Note that $n = \gamma(G^+) = \gamma_0(G^+)$. Also $\{u_1, u_2, \dots, u_n\}$ is a SIDS with n elements and so $\gamma_0^s(G^+) \leq n$. By Theorem 2.1 and 2.2, $\gamma(G^+) = \gamma_0(G^+) = \gamma_0^s(G^+) = n$.

Theorem 3.4 : Let p and q are two integers such that $p < q$. Then there exist a graph G^+ such that $\gamma(G^+) = p$ and $\gamma_0^s(G^+) = q$.

Proof 10: Consider the complete graph K_p of order p with $V(K_p) = \{v_1, v_2, \dots, v_p\}$. For each $v_i (2 \leq i \leq p)$, attach a pendent vertex $u_i (2 \leq i \leq p)$. Let W be any graph which admits SIDS with $\gamma(W) = \gamma_0^s(W) = q - p + 1$ and join each and every vertex of W to v_1 and G^+ be the resultant graph. Let D^* be a dominating set of G^+ . To dominate the vertices of $V(W)$ and v_1 , D^* should take either v_1 or minimum of one vertex of $V(W)$. For each $i = 2$ to a ,

to dominate the vertices v_i and u_i , D^* should take in either u_i or vertex v_i and so $\gamma(G^+) \geq p$. Since $V(K_p)$ is a dominating set of G^+ , we have $\gamma(G^+) \leq p$ and so $\gamma(G^+) = p$. Let D^* be a SIDS G^+ and u be a strongly isolated vertex in $\langle D^* \rangle$. Since $e(v_i) = 2$ for all $1 \leq i \leq p$, $v_i \notin u$.

Case 1: Assume $u = u_i$ for any $2 \leq i \leq p$. Thus $V(K_p) \cap D^* = \emptyset$ and so $u_i \in D^*$ for all $2 \leq i \leq p$. Since $v_1 \notin D^*$, to dominate each and every of W , D^* should take $q - p + 1$ vertices (since $\gamma_0^s(W) = q - p + 1$). Thus $\gamma_0^s(G^+) \geq (p - 1) + (q - p + 1) = q$. Thus, $\gamma_0^s(G^+) \geq q$.

Case 2: Assume $u \in V(W)$. Thus $V(K_p) \cap D^* = \emptyset$ and so $u_i \in D^*$ for all $2 \leq i \leq p$. Since $v_1 \notin D^*$, to dominate each and every vertex of W , D^* should take in $q - p + 1$ vertices (since $\gamma_0^s(W) = q - p + 1$). Thus $\gamma_0^s(G^+) \geq (p - 1) + (q - p + 1) = q$. Thus, $\gamma_0^s(G^+) \geq q$. From Case 1 and Case 2, $\gamma_0^s(G^+) \geq q$. Let D' be a γ_0^s -set of W . Also $D' \cup \{u_2, u_3, \dots, u_p\}$ remains SIDS with $(p - 1) + (q - p + 1) = q$ - elements. Thus $\gamma_0^s(G^+) \leq q$ and so $\gamma_0^s(G^+) = q$.

Theorem 3.5: Let p and q are two integers such that $q > p$. Then there exists a graph G^+ such that $\gamma_0(G^+) = p$ and $\gamma_0^s(G^+) = q$.

Proof 11: Let C_{p+2} be a cycle of order $p + 2$ and $V(C_{p+2}) = \{1, 2, \dots, p + 2\}$. Let W be any graph which admits SIDS with $\gamma_0(W) = \gamma_0^s(W) = q - p + 1$. Consider the corona of two graphs C_{p+2} and W . For $1 \leq i^* \leq p + 2$, let W^{i^*} be the image of W whose vertices are connected to the vertex i^* . Let $G^+ = (C_p + 2 \circ W) - (W^p \cup W^{p+2})$ and D be an isolate dominating set of G^+ . Then for each $i^* = 1, 2, \dots, p - 1, p + 1$, to dominate the vertex i^* and the vertices of corresponding W^{i^*} , D should take in either i^* or at least one vertex of W^{i^*} and so $\gamma_0(G^+) \geq p$. Since $\{i^* : 1 \leq i \leq p - 1\} \cup \{p + 1\}$ is an isolate dominating set of G^+ , we have $\gamma_0(G^+) \leq p$ and so $\gamma_0(G^+) = p$. Let D be a SIDS of G^+ and u be a strongly isolated vertex in $\langle D \rangle$.

Case 1: Assume $u \in V(W^{i^*})$ for any i with $1 \leq i^* \leq p + 1$ and $i^* \neq p$. Then $i^* \notin D$. To dominate each and every vertex of $V(W^{i^*})$, D should take in as a minimum $q - p + 1$ vertices of $V(W^{i^*})$ (since $\gamma_0^s(W^{i^*}) = q - p + 1$). To dominate the vertices of $V(W^j)$ for any $1 \leq j \leq p + 1$ and $j \neq i^*$ and p , D must include as a minimum one vertex of $V(W^j)$ or j . Thus $|D| \geq (p - 1) + (q - p + 1) = q$.

Case 2: Assume $u = i^*$ for certain $1 \leq i^* \leq p + 2$ and $i^* \neq p + 1$. Then either $i^* + 1$ or $i^* - 1$ will adjacent with all the vertices of W^{i^*+1} or W^{i^*-1} respectively. In any case, let us assume $i^* + 1$ is dominates all the vertices of W^{i^*+1} . Note that $i^* + 1 \notin D$. In this case, to dominate the vertices of W^{i^*+1} , D should take $q - p + 1$ vertices of W^{i^*+1} (since $\gamma_0^s(W^{i^*+1}) = q - p + 1$). For any $j = 1, 2, \dots, p - 1, p + 1$, ($j \neq i^*$), to dominate j and W^j , D should take in j or at least one vertex of W^j . Thus $|D| \geq (q - p + 1) + (p - 1) = q$.

In all cases $|D| \geq q$ and hence $\gamma_0^s(G^+) \geq q$. Let X' be a γ_0^s -set of W^{p+1} . Also $D = X' \cup \{1, 2, \dots, p - 1\}$ is a SIDS of G^+ with $|D| = (p - 1) + (q - p + 1) = q$. Thus $\gamma_0^s(G^+) \leq q$ and so $\gamma_0^s(G^+) = q$.

4 Strongly isolate domination number of certain graphs.

This section, we obtain the SID number for paths, cycles and various group of graphs.

Theorem 4.1 : For a path P_m of order ($m \geq 4$), we have $\gamma_0^s(P_m) = \lceil m/3 \rceil$.

Proof 12: Let $V(P_m) = \{w_1, w_2, \dots, w_m\}$ and D be a SIDS of P_m . By Theorem 2.1, $\lceil m/3 \rceil = \gamma(P_m) \leq \gamma_0^s(P_m)$.

Case 1: When $m = 4$ or $m = 5$.

Since the set $D = \{w_1, w_4\}$ is a SIDS of P_m with 2 elements. Thus $\gamma_0^s(P_m) \leq \lceil m/3 \rceil$.

Case 2: When $m \geq 6$.

Sub case 1: Suppose $m = 3r$ for $r \geq 2$.

Since the set $D = \{w_{3i-1} : 1 \leq i \leq r\}$ SIDS of P_m with r - elements. Thus $\gamma_0^s(P_m) \leq r = \lceil m/3 \rceil$.

Sub case 2: Suppose $m = 3r + 1$ or $m = 3r + 2$ for $r \geq 2$.

Since the set $D = \{w_{3i-1} : 1 \leq i \leq r\}$ is a SIDS of P_m with $r + 1$ - elements. Thus $\gamma_0^s(P_m) \leq r + 1 = \lceil m/3 \rceil$.

Theorem 4.2 : For a cycle C_m of order ($m > 5$), we have $\gamma_0^s(C_m) = \lceil m/3 \rceil$.

Proof 13: Let $V(C_m) = \{z_1, z_2, \dots, z_m\}$ and D be a SIDS of C_m . By Theorem 2.1, $\lceil m/3 \rceil = \gamma(C_m) \leq \gamma_0^s(C_m)$.

Case 1: When $m = 3r$ for $r \geq 1$.

Since the set $D = \{z_{3i-2} : 1 \leq i \leq r\}$ is a SIDS of C_m with r elements. Thus $\gamma_0^s(C_m) \leq r = \lceil m/3 \rceil$.

Case 2: Suppose $m = 3r + 1$ or $m = 3r + 2$ for $r \geq 1$.

Since the set $D = \{z_{3i-2} : 1 \leq i \leq r\} \cup \{z_{m-2}\}$ is a SIDS of C_m with $r+1$ elements. Thus $\gamma_0^s(C_m) \leq r + 1 = \lceil m/3 \rceil$

Remark 4.1: For the graph , $\{v_1\}$ is a SIDS and $\gamma_0^s(C_3) = 1$.

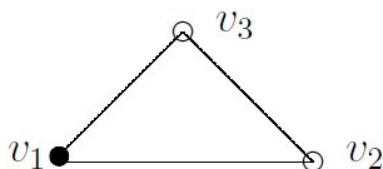


Figure 4.1: C_3

Remark 4.2: The graph C_n ($n = 4,5$) does not admit SIDS.

Proof 14: Since $e(v) = 2$ for any vertex v of C_4 and C_5 , C_4 and C_5 does not admit SIDS.

Theorem 4.3 : Let G^+ be an order $m \geq 1$ graph. Then $\gamma_0^s(G^+) = 1$ if and only if $\Delta(G) = m - 1$.

Proof 15: Assume $\gamma_0^s(G^+) = 1$ and let $F = \{z\}$ be a minimum SIDS of G^+ . If G^+ is a singleton graph, after that we are finished. Suppose G^+ is not singleton graph. Obviously, each vertex in $V(G^+) - F$ is adjacent to z . Since $|V(G^+)| = m$, $degG(z) = m - 1$. Consequently, $\Delta(G^+) = m - 1$. The converse is true.

The next corollary follows straight from the above theorem.

Corollary 4.1 Let S_{n-1} , K_n and W_{n-1} be star graph, complete graph and wheel graph of $n \geq 2$ vertices, sequentially. Then $\gamma_0^s(S_{n-1}) = \gamma_0^s(K_n) = \gamma_0^s(W_{n-1}) = 1$.

Theorem 4.4 : The complete bipartite graph $K_{p,q} = (P,Q)$ admits SIDS if and only if $p = 1$ or $q = 1$.

Proof 16: Given $K_{p,q} = (P, Q)$ be a complete bipartite graph. Let $P = \{r_i : 1 \leq i \leq p\}$ and $Q = \{s_i : 1 \leq i \leq q\}$. Take D^* be a SIDS and x be the SI vertex in $\langle D^* \rangle$. Assume $K_{p,q}$ admits strongly isolated dominating set. Claim, $p = 1$ or $q = 1$. Assume $p \neq 1$ and $q \neq 1$. In any case assume that $x \in P$. As x is isolated in $\langle D^* \rangle$, D^* cannot have any vertex of Q . Consequently to dominate every vertex of P , D^* should necessarily add each and every vertices of P . Hence $P = D^*$. Here $N_2(x) \cap D^* \neq \phi$, a contradiction. Similarly, we get the same contradiction, when $x \in Q$. Therefore, $p = 1$ or $q = 1$.

Converse part is trivial.

Theorem 4.5: Let $b \geq 2$ be an integer and $G = K_{p_1, p_2, \dots, p_b} = (P_1, P_2, \dots, P_b)$ be a complete b -partite graph. If $p_j = 1$ for an integer j with $1 \leq j \leq b$ iff G admits SIDS.

Proof 17 : Given $G = K_{p_1, p_2, \dots, p_b} = (P_1, P_2, \dots, P_b)$ be a complete b -partite graph. Take a SIDS G and a SI vertex y . Claim, $p_j = 1$ for an integer j with $1 \leq j \leq b$. In any case, we may suppose that $y \in P_1$. As y is isolated in $\langle D \rangle$, $D \cap (P_2 \cup P_3 \cup \dots \cup P_b) = \emptyset$. Thus to dominate each and every vertex of P_1 , D necessarily take all the vertices of P_1 . Thus $D = P_1$. Here $N_2(y) \cap D \neq \emptyset$, a contradiction.

Converse part is trivial.

Theorem 4.6 : For any Helm graph H_m with $2m + 1$ vertices, we have $\gamma_0^s(H_m) = m$.

Proof 18 : Let H_m be a Helm graph. Let $V(H_m) = \{q_i : 1 \leq i \leq m\} \cup \{p_i : 1 \leq i \leq m\} \cup \{x\}$ and $E(H_m) = \{q_i p_i, x q_i : 1 \leq i \leq m\} \cup \{q_i q_{i+1} : 1 \leq i \leq m - 1\} \cup \{q_1 q_m\}$. By Theorem 2.1, $m \leq \gamma(H_m) \leq \gamma_0^s(H_m)$. Since the set $D = \{p_i : 1 \leq i \leq m - 1\} \cup \{p_m\}$ is a SIDS of H_m with m elements, $\gamma_0^s(H_m) \leq m$.

Corollary 4.2: The flower graph does not admit SIDS.

Theorem 4.7: For every Comb graph $P_n * k_1$ of order $n (\geq 3)$, $\gamma_0^s(P_n * k_1) = n$.

Proof 19: Let $P_n * k_1$ be a Comb graph and let $V(P_n * k_1) = C \cup D$, where $V(C) = \{c_i : 1 \leq i \leq n\}$, $V(D) = \{d_i : 1 \leq i \leq n\}$ and $E(P_n * k_1) = \{c_i c_{i+1} : 1 \leq i \leq (n - 1)\} \cup \{c_i d_i : 1 \leq i \leq n\}$. Since each d_i is pendent, either d_i or c_i must be in every dominating set and so $\gamma(G) \geq n$. Consequently by Theorem 2.1, $n \leq \gamma_0^s(P_n * k_1)$. Since the set $D = \{d_i : 1 \leq i \leq n\}$ is a SIDS of $P_n * k_1$ and so $\gamma_0^s(P_n * k_1) \leq n$.

Theorem 4.8: For every graph G^+ of order $m (\geq 1)$, $\gamma_0^s(G^+ * k_1) = m$.

Proof 20: Let $V(G^+) = \{u_1, u_2 \dots \dots, u_m\}$ for each $1 \leq i \leq m$. Let v_i be the pendent vertex such that $u_i v_i \in E(G^+ * k_1)$ and D be a SIDS of $G^+ * k_1$. To dominate the vertex v_i , either u_i or v_i must be in D . Thus $|D| \geq m$ and so $\gamma_0^s(G^+ * k_1) \geq m$. Since the set $D = \{v_1, v_2 \dots \dots, v_m\}$ is a SIDS of $G^+ * k_1$. Thus $\gamma_0^s(G^+ * k_1) \leq m$.

Theorem 4.9: For any firecracker graph $F(s, t)$ of order $st (s, t \geq 2)$, we have $\gamma_0^s(F(s, t)) = s$.

Proof 21 By Theorem 2.1, $s = \gamma(F(s, t)) \leq \gamma_0^s(F(s, t))$. Let $V(F(s, t)) = \{u_{k,r} : k = 1, 2, 3 \dots, t; r = 1, 2, 3 \dots, s\}$ and let $E(F(s, t)) = \{u_{1,r} u_{k,r} : k = 2, 3 \dots, t; r = 1, 2, 3 \dots, s\} \cup \{u_{b,r} u_{b,r+1} : k = 1, 2, 3 \dots, (s - 1)\}$. Since the set $D = \{u_{1,r} : r = 1, 2, 3 \dots, s\}$ is a SIDS of $F(s, t)$ and so $\gamma_0^s(F(s, t)) \leq s$.

Theorem 4.10: The sun graph $\text{sun}(s)$ with $2s$ vertices, admits SIDS with $\gamma_0^s(\text{sun}(s)) = s$.

Proof 22: By Theorem 2.1, $s = \gamma(\text{sun}(s)) \leq \gamma_0^s(\text{sun}(s))$. Let $V(\text{sun}(s)) = \{p_i : 1 \leq i \leq s\} \cup \{q_i : 1 \leq i \leq s\}$ and $E(\text{sun}(s)) = \{p_i q_i : 1 \leq i \leq s\}$. Let D be a SIDS of $\text{sun}(s)$. Since the set $D = \{p_i : 1 \leq i \leq s\}$ is a SIDS of $\text{sun}(s)$ with s elements so $\gamma_0^s(\text{sun}(s)) \leq s$.

Theorem 4.11: If G_m be the gear graph of order $2m + 1$, then $\gamma_0^s(G_m) = \lceil 2m/3 \rceil$.

Proof 23 : By Theorem 2.1, $\lceil 2m/3 \rceil = \gamma(G_m) \leq \gamma_0^s(G_m)$. Let $V(G_m) = \{p_i : 1 \leq i \leq m\} \cup \{q_i : 1 \leq i \leq m\}$. Let D be a SIDS of G_m .

Case 1: When $m = 3t, t \geq 1$.

Since the set $D = \{q_{3i-2} : 1 \leq i \leq t\} \cup \{p_{3i-1} : 1 \leq i \leq t\}$ is a SIDS of G_m with t elements. Thus $\gamma_0^s(G_m) \leq t = \lceil 2m/3 \rceil$.

Case 2: When $m = 3t + 1$ or $m = 3t + 2$ for $t \geq 1$. Since the set $D = \{q_i : 1 \leq i \leq t\} \cup \{p_{3i-1} : 1 \leq i \leq t\}$ is a SIDS of G_m with $t + 1$ - elements. Thus $\gamma_0^s(G_m) \leq t + 1 = \lceil 2m/3 \rceil$.

Theorem 4.12: The book graph $B_n, n \geq 2$ admits strongly isolate dominating set and $\gamma_0^s(B_n) = n$.

Proof 24: Let f and d be the centre vertices of two copies of S_{n+1} in the Cartesian product $S_{n+1} \circ P_2$. Let f_1, f_2, \dots, f_n be the vertices adjacent to f and d_1, d_2, \dots, d_n be the vertices adjacent to d . By Remark 2.1, f and d are not in SIDS of B_n . Thus for each $1 \leq i^* \leq n$, to dominate the vertex f_{i^*} , each f_{i^*} or d_{i^*} must be in any SIDS of B_n and so $\gamma_0^s(B_n) \geq n$. Since the set $\{f_1\} \cup \{f_2, f_3, \dots, f_n\}$ remains a SIDS with n vertices, $\gamma_0^s(B_n) \leq n$.

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