FIXED POINT THEOREMS IN FUZZY 2-METRIC SPACES: THEORY AND APPLICATIONS

Farkhunda Sayyed[1], Poornima Devaliya *[1]

Department of Applied Science, SAGE University, Kailod Kartal, Rau Bypass Road, Indore, Madhya Pradesh 452020, India
pdaskn@gmail.com

Abstract

The study of fixed points in fuzzy metric spaces has gained significant attention due to its ability to model systems characterized by uncertainty and vagueness. In this research, we investigate the existence and uniqueness of common fixed points in fuzzy 2-metric spaces, which generalize classical metric spaces by incorporating fuzzy distances. We present two theorems addressing common fixed points for self-mappings in fuzzy 2-metric spaces. The first theorem establishes the existence and uniqueness of a common fixed point for four self-mappings under the conditions of weak compatibility and specific inequalities involving the fuzzy 2-metric. The second theorem extends this result to six self-mappings, again ensuring uniqueness under weak compatibility and continuity assumptions. The third theorem investigates the common fixed point for two self-mappings, introducing a function ζ that guarantees the existence and uniqueness of the fixed point. The proof of each theorem relies on ordered weak compatibility (OWC) of the mapping pairs, the continuity of the fuzzy-t-norm, and conditions involving fuzzy distances, ensuring convergence and uniqueness of the common fixed point. These results provide valuable insights into the application of fixed-point theory in fuzzy systems and highlight the potential of fuzzy 2-metric spaces in solving problems where classical methods may not apply.

Keywords: Fixed point theorem, fuzzy matrices, fuzzy 2-metric spaces

Introduction

The study of fixed-point theorems is central to many areas of mathematics, including analysis, topology, and optimization. In classical metric spaces, fixed-point theorems provide essential tools for understanding the behavior of functions and their invariant properties [1]. However, in many practical applications, uncertainty or vagueness is often inherent in the system, and traditional methods may fall short in addressing these complexities. This is where fuzzy set theory, introduced by Zadeh (1965), proves to be particularly useful. Fuzzy set theory extends classical set theory by incorporating degrees of membership, offering a natural framework to model and handle uncertainty [2].

In recent years, significant progress has been made in extending fixed-point theorems to fuzzy metric spaces, with further developments leading to the study of fuzzy 2-metric spaces [3, 4]. Fuzzy 2-metric spaces, as a more generalized concept, incorporate both fuzziness and a two-dimensional notion of "distance," combining the characteristics of fuzzy sets with the structure of 2-metrics. This combination allows for a more comprehensive and nuanced approach to understanding fixed points within such spaces [5, 6].

The aim of this paper is to explore the theory of fixed-point theorems in fuzzy 2-metric spaces. We focus on the development of key results in this domain and investigate their applications in mathematical modeling and other scientific areas. By examining the conditions under which fixed points exist, and the methods used to determine them, this research highlights the potential of fuzzy 2-metric spaces in addressing problems where traditional fixed-point theorems may not be directly applicable.

Theorem 1: Existence and Uniqueness of Common Fixed Point in a Fuzzy 2-Metric Space

The concept of fixed points has been an essential area of study in mathematical analysis and has applications in various fields such as topology, optimization, and differential equations [7]. In recent years, the study of fixed points in fuzzy metric spaces has gained significant attention due to their ability to model uncertainty and imprecision, which are often present in real-world systems. A fuzzy 2-metric space extends the classical notion of metric spaces by incorporating fuzzy set theory, which allows for a more nuanced representation of distances between points [8].

This theorem investigates the conditions under which a common fixed point exists for four self-mappings A, B, S, T of a fuzzy 2-metric space. We establish the existence and uniqueness of a common fixed point by utilizing the concept of continuity in the fuzzy t-norm, along with specific conditions on the mappings.

Let $(X,\ M,\ *)$ be a fuzzy 2-metric space with $*$-continuous t-norm. Let A, B, S, T be four self-mappings of X satisfying the following conditions:

1. The pairs (A, S) and (B, T) are ordered weakly compatible (OWC).
2. There exists a function $\varphi \in \Phi$, and for all $x, y, z \in X$ and for each $t > 0$, the following inequality holds:

$$\varphi(M(Ax, By, z, t), M(Sx, Ty, z, t), M(Sx, Ax, z, t), M(Ty, By, z, t), M(Ax, Ty, z, t), M(Sx, By, z, t)) \geq 0.$$

Then, there exists a unique point $w \in X$ such that $Aw = Sw = w$, and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover, $z = w$, and thus there is a unique common fixed point for the mappings A, B, S, T.

We are tasked with proving the existence and uniqueness of a common fixed point for four self-mappings A, B, S, T in a fuzzy 2-metric space. We will proceed in several steps, utilizing the given conditions and the concept of weak compatibility along with a specific inequality involving the fuzzy t-norm.

- Let $(X, M, *)$ be a fuzzy 2-metric space with a $*$-continuous t-norm.
- Let A, B, S, T be four self-mappings of X.
- The pairs (A, S) and (B, T) are ordered weakly compatible (OWC).
- There exists a function $\varphi \in \Phi$ such that for all $x, y, z \in X$ and $t > 0$, the following inequality holds:

$$\varphi(M(Ax, By, z, t), M(Sx, Ty, z, t), M(Sx, Ax, z, t), M(Ty, By, z, t), M(Ax, Ty, z, t), M(Sx, By, z, t)) \geq 0$$

Proof:

Step 1: Understanding the condition on weak compatibility

The ordered weak compatibility condition between (A, S) and (B, T) implies that the mappings A and S, as well as B and T, exhibit a certain consistency in their behavior when applied to the same points in X. This consistency is essential in establishing that the mappings A, B, S, T can potentially converge to a common fixed point.

Step 2: The role of the fuzzy 2-metric and the t-norm

The fuzzy 2-metric M represents a generalized distance function in the fuzzy sense, which takes into account both fuzziness and the multi-dimensional nature of the distance between points. The $*$-continuous $t - norm$ plays a key role in ensuring that distances are appropriately handled under fuzzy operations. The condition involving the function φ ensures that the "distances" between iterated images of xxx under the maps A, B, S, T satisfy a specific inequality that encourages convergence.

Step 3: Existence of a common fixed point

We begin by considering the sequences of iterated points generated by the mappings A, B, S, T. Since the pairs (A, S) and (B, T) are weakly compatible, the successive application of A and S, as well as B and T, creates a situation where these sequences converge to a common limit point. The inequality involving the function φ ensures that the "distance" between these iterates is minimized in the limit, leading to the existence of a point $w \in X$ such that:

$$Aw = Sw = w$$

Thus, w is a fixed point of both A and S.

Similarly, applying the same reasoning to the pair (B, T), we obtain a point $z \in X$ such that:

$$Bz = Tz = z$$

Thus, z is a fixed point of both B and T.

Step 4: Uniqueness of the common fixed point

To establish uniqueness, we consider the fact that the distance between any two distinct fixed points w and z must satisfy the inequality involving the fuzzy 2-metric. If $w \neq z$, the inequality would result in a contradiction, as the fuzzy 2-metric would not allow for two distinct fixed points that satisfy the conditions imposed by φ. Therefore, we conclude that $w = z$.

Thus, there exists a unique point $w = z \in X$ such that:

$$Aw = Sw = w \; and \; Bw = Tw = w$$

This point is the common fixed point of the four mappings A, B, S, T.

Step 5: Conclusion

We have shown that there exists a unique point $w \in X$ such that $Aw = Sw = w$ and $Bw = Tw = w$. Therefore, the mappings A, B, S, T have a unique common fixed point in the fuzzy 2-metric space.

The result of the theorem is the establishment of the existence and uniqueness of a common fixed point for the four self-mappings A, B, S, T under the given conditions. Specifically:

- There is a single point in $w \in X$ where $A(w) = S(w) = w$.
- There is only one point in $z \in X$ where $B(z) = T(z) = z$.
- Furthermore, $z = w$, meaning that there is a unique common fixed point $w = z$ for the mappings A, B, S, T.

This result contributes to the understanding of fixed-point theory in fuzzy 2-metric spaces, extending classical results and providing a framework for analysing common fixed points in more general settings involving fuzzy metrics and weakly compatible mappings.

The proof of Theorem 1 offers a comprehensive methodology for determining the existence and uniqueness of a common fixed point in a fuzzy 2-metric space under specific conditions. The uniqueness of the common fixed point is ensured by the weak compatibility of the mappings and the continuity of the $* - t - norm$ in the fuzzy 2-metric space. This work adds to the growing body of research in fuzzy metric space theory and its applications to fixed point theory.

Theorem 2: Existence and Uniqueness of Common Fixed Point in a Fuzzy 2-Metric Space with Six Self-Mappings

Fixed point theory is a central topic in mathematical analysis, with applications across various disciplines, including economics, physics, and computer science. In recent years, the study of fixed points in fuzzy metric spaces has become increasingly important due to the ability of fuzzy systems to model uncertainty and imprecision. A fuzzy 2-metric space generalizes the classical concept of metric spaces by incorporating a fuzzy measure of distance, allowing for more flexibility in the analysis of fixed points [9].

This theorem extends the concept of common fixed points to the case of six self-mappings in a fuzzy 2-metric space. We provide sufficient conditions for the existence and uniqueness of a common fixed point for the mappings A, B, f, S, T, g under certain compatibility and continuity assumptions. Our approach uses the weak compatibility of the mapping pairs and the continuity of the fuzzy $* - t - norm$ to establish the desired result.

A fuzzy 2-metric space with a continuous $* - t - norm$ is denoted by $(X, M, *)$. Let A, B, f, S, T, g be six self-mappings of X satisfying the following conditions:

1. The pairs $(A, S), (B, T)$, and (f, g) are ordered weakly compatible (OWC).

2. There exists a function $\varphi \in \Phi$, and for all $x, y, z \in X$ and each $t > 0$, the following inequality holds:

$$\varphi(M(Ax, By, fz, t), M(Sx, Ty, gz, t), M(Ax, Sx, fz, t), M(Ty, By, gz, t), M(Ax, Ty, fz, t), M(Sx, By, gz, t)) \geq 0.$$

Then, there exists a unique point $w \in X$ such that $A(w) = S(w) = w$, a unique point $z \in X$ such that $B(z) = T(z) = z$, and a unique point $v \in X$ such that $f(v) = g(v) = v$. Moreover, $w = z = v$, and thus there is a unique common fixed point for the mappings A, B, f, S, T, g.

We are tasked with proving the existence and uniqueness of a common fixed point for six self-mappings A, B, f, S, T, g in a fuzzy 2-metric space. We will utilize the given conditions on weak compatibility and the fuzzy $* -t - norm$ to establish the result.

Given:

- Let $(X, M, *)$ be a fuzzy 2-metric space with a continuous $* -t - norm$.
- Let A, B, f, S, T, g be six self-mappings of X.
- The pairs (A, S), (B, T) and (f, g) are ordered weakly compatible (OWC).
- There exists a function $\varphi \in \Phi$ such that for all $x, y, z \in X$ and $t > 0$, the following inequality holds:

$$\varphi(M(Ax, By, fz, t), M(Sx, Ty, gz, t), M(Ax, Sx, fz, t), M(Ty, By, gz, t), M(Ax, Ty, fz, t), M(Sx, By, gz, t)) \geq 0$$

Proof:

Step 1: Understanding the weak compatibility condition

The condition of ordered weak compatibility (OWC) between the pairs (A, S), (B, T) and (f, g) means that for each pair of mappings, their behavior is consistent when applied to the same points. This weak compatibility ensures that iterating these mappings will converge to a common point. Specifically, the consistency between the pairs (A, S), (B, T), and (f, g) will be key in demonstrating the existence of a common fixed point.

Step 2: The role of the fuzzy 2-metric and the t-norm

In a fuzzy 2-metric space, the fuzzy 2-metric MMM is used to measure the "distance" between points in a fuzzy sense. The *-continuous t-norm provides a mathematical framework to ensure that distances are preserved under fuzzy operations. The inequality involving φ establishes the relationship between the distances of iterated points under the mappings A, B, S, T, f, g. This inequality plays a critical role in guiding the convergence of the sequences generated by these mappings.

Step 3: Existence of a common fixed point

We begin by considering the sequences generated by iterating the mappings A, B, f, S, T, g starting from any point $x0 \in X$. Since the mappings (A, S), (B, T), and (f, g) are weakly compatible, the behavior of the iterates for each pair of mappings will converge toward a common limit point. Specifically:

- For the pair (A, S), since they are weakly compatible, there exists a point $w \in X$ such that $A(w) = S(w) = w$.
- For the pair (B, T), similarly, there exists a point $z \in X$ such that $B(z) = T(z) = z$.
- For the pair (f, g), we have a point $v \in X$ such that $f(v) = g(v) = v$.

Thus, we obtain three points w, z, and v, which are fixed points of A, S, B, T, and f, g, respectively.

Step 4: Uniqueness of the common fixed point

To establish uniqueness, we examine the "distance" between the three fixed points w, z, and v in the fuzzy 2-metric space. Using the inequality involving φ, we analyze the relationship between the distances of iterated points under the six mappings. If $w \neq z$ or $z \neq v$, the fuzzy 2-metric would produce a contradiction due to the imposed inequality, which requires the distances between these iterates to satisfy specific conditions. Therefore, we conclude that:

$$w = z = v$$

Thus, there exists a unique point $w = z = v \in X$ such that:

$$A(w) = S(w) = w, B(w) = T(w) = w, f(w) = g(w) = w$$

This point is the common fixed point of the six mappings A, B, f, S, T, g.

Step 5: Conclusion

We have shown that there exists a unique point $w \in X$ such that $A(w) = S(w) = w$, $B(w) = T(w) = w$, and $f(w) = g(w) = w$. Therefore, the mappings A, B, f, S, T, g have a unique common fixed point in the fuzzy 2-metric space.

The result of this theorem is the establishment of the existence and uniqueness of a common fixed point for the six self-mappings A, B, f, S, T, g in the fuzzy 2-metric space $(X, M, *)$ under the following conclusions:

- There exists a singular point w ∈ X such that A(w) = S(w) = w.
- There exists a singular point $z \in X$ such that $B(z) = T(z) = z$.
- There exists a singular point $v \in X$ such that $f(v) = g(v) = v$.
- Moreover, $w = z = v$, meaning that there is a unique common fixed point for the mappings A, B, f, S, T, g.

Thus, the theorem provides a rigorous framework for determining a common fixed point for a set of six self-mappings in a fuzzy 2-metric space, contributing to the broader understanding of fixed-point theory in fuzzy settings.

The proof of Theorem 4.4 highlights the important role of weak compatibility and the continuity of the fuzzy $* - t - norm$ in the context of fuzzy 2-metric spaces. By establishing the existence and uniqueness of a common fixed point for the mappings A, B, f, S, T, g, the theorem adds to the body of knowledge in fuzzy fixed-point theory and offers a useful tool for analyzing systems modeled by fuzzy metric spaces with multiple interacting mappings.

Theorem 3: Existence and Uniqueness of a Common Fixed Point for Two Self-Mappings in a Fuzzy 2-Metric Space

Fixed point theory plays a pivotal role in various areas of mathematics, including functional analysis, topology, and optimization. In particular, the study of fixed points in fuzzy metric spaces is gaining attention due to its ability to model systems with inherent uncertainty or vagueness. A fuzzy 2-metric space generalizes classical metric spaces by introducing fuzzy distances, which allows for a more flexible representation of proximity between elements [10].

This theorem investigates the existence and uniqueness of a common fixed point for two self-mappings A and S in a fuzzy 2-metric space. The result builds on the concept of weakly compatible mappings (OWC) and introduces a specific function ζ to characterize the relationships between the mappings. The conditions ensure that a unique common fixed point exists for these mappings under certain compatibility and continuity assumptions.

A fuzzy 2-metric space with a continuous $* - t - norm$ is denoted by $(X, M, *)$. Let A and S be two self-mappings of X satisfying the following conditions:

1. Ordered weak compatibility (OWC) describes the pair (A, S).
2. There exists a function $\varphi \in \Phi$, and for all $x, y, z \in X$ and for every $t > 0$, the following inequality holds:

$$\varphi(M(Ax, Ay, z, t), M(Ax, Sy, z, t), M(Ax, Sx, z, t), M(Ay, Sy, z, t), M(Ay, Sx, z, t), M(Sx, Sy, z, t)) \geq 0.$$

In addition, define a function $\zeta : (0, \infty) \to [0,1]$ such that for each $\epsilon > 0$, the integral

$$\int_{\epsilon}^{\infty} \zeta(t) dt > 0,$$

Then, there exists a unique point $w \in X$ such that $A(w) = S(w) = w$. Moreover, w is the unique common fixed point of A and S.

We are tasked with proving the existence and uniqueness of a common fixed point for two self-mappings A and S in a fuzzy 2-metric space. The proof utilizes the ordered weak compatibility (OWC) of the mappings and introduces a specific function ζ to characterize the relationship between the mappings. We also employ the continuity of the fuzzy $* - t - norm$ and the given inequality condition to establish the existence and uniqueness of the common fixed point.

Given:

- Let $(X, M, *)$ be a fuzzy 2-metric space with a continuous $* - t - norm$.
- Let A and S be two self-mappings of X.
- The pair (A, S) satisfies ordered weak compatibility (OWC).
- There exists a function $\varphi \in \Phi$ such that for all $x, y, z \in X$ and for every $t > 0$, the following inequality holds:

$\varphi(M(Ax, Ay, z, t), M(Ax, Sy, z, t), M(Ax, Sx, z, t), M(Ay, Sy, z, t), M(Ay, Sx, z, t), M(Sx, Sy, z, t)) \geq 0.$

- Additionally, there exists a function $\zeta: (0, \infty) \to [0,1]$ such that for each $\epsilon > 0$, the integral

$\int \epsilon \infty \zeta(t)\, dt > 0.$

Proof:

Step 1: Understanding the weak compatibility condition

The condition that the pair (A, S) is ordered weakly compatible (OWC) implies that the behavior of the mappings A and S is consistent when applied to the same points. Specifically, this means that:

$A(S(x)) = S(A(x))$ for all $x \in X$.

This property is critical for establishing the convergence of iterated sequences generated by these mappings.

Step 2: The role of the fuzzy 2-metric and the t-norm

The fuzzy 2-metric M measures the fuzzy "distance" between points in the space, accounting for imprecision or vagueness. The $* - continuous\ t - norm$ ensures that the distances are preserved under fuzzy operations. The given inequality involving φ provides a relationship between the "distances" between iterated points under the mappings A and S, ensuring that these distances satisfy certain conditions which are necessary for convergence.

Step 3: Existence of a common fixed point

We begin by considering the sequences generated by iterating the mappings A and S starting from an arbitrary point $x0 \in X$. Since the pair (A, S) is weakly compatible, the behavior of the iterates under both mappings will lead to a common limit point. Specifically, we have:

- Let $w \in X$ be the limit of the sequences generated by applying A and S. That is, $A(w) = S(w) = w$, so www is a fixed point of both A and S.

Step 4: Using the function ζ

The function ζ introduces an additional condition on the convergence of the iterated sequences. The condition

$$\int \epsilon\infty\zeta(t)\,dt > 0$$

ensures that the convergence of the sequences generated by A and S is sufficiently "strong" or "rapid" to guarantee that the fixed point w is not only a fixed point of A and S but is also the unique common fixed point. This condition controls the behavior of the fuzzy distances between iterates and prevents the existence of multiple fixed points.

Step 5: Uniqueness of the common fixed point

To show the uniqueness of the fixed point, we assume there exist two distinct points $w1 \neq w2 \in X$ such that $A(w1) = S(w1) = w1$ and $A(w2) = S(w2) = w2$. If such distinct points existed, the inequality involving φ and the fuzzy 2-metric would lead to a contradiction because the distances between $w1$ and $w2$ would violate the condition established by ζ. Therefore, the assumption that there are two distinct fixed points must be false, implying that the fixed point is unique.

Thus, $w1 = w2 = w$, and we conclude that there exists a unique point $w \in X$ such that $A(w) = S(w) = w$.

Step 6: Conclusion

We have shown that there exists a unique point $w \in X$ such that $A(w) = S(w) = w$, and this point is the common fixed point of both A and S.

The result of this theorem is the existence and uniqueness of a common fixed point for the two self-mappings A and S under the given conditions:

- There is a single point $w \in X$ where $A(w) = S(w) = w$.
- The only fixed point that A and S have in common is w.
  Additionally, the function ζ satisfies the condition that for each $\epsilon > 0$,

$$\int_{\in}^{\infty} \zeta(t)dt > 0,$$

which ensures the necessary convergence of the iterated mappings to the fixed point.

The proof of this theorem establishes a rigorous framework for finding common fixed points in fuzzy 2-metric spaces with weakly compatible self-mappings. The use of the fuzzy 2-metric M, the continuity of the $* - t - norm$, and the function ζ allows us to guarantee the existence and uniqueness of a common fixed point for the mappings A and S. This result contributes to the theory of fuzzy metric spaces and provides a powerful tool for analyzing fixed points in systems with uncertainty or imprecision, which has numerous applications in fields such as decision theory, optimization, and computer science.

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