VOLTERRA INTEGRODIFFERENTIAL EQUATIONS AND RUNGE-KUTTA METHODS

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Abstract : The following Volterra Integro-Differential Equations are examined for their nonlinear stability and convergence includes using RK methods of order 4.

$$y'(t) = f\left(t, y(t), y(t-\tau), \int_{t-\tau}^{t} g\left(t, V, y(V)\right) dV\right), t \in [0,T],$$

$$y(t) = \varphi(t),\ t \in [-\tau, 0] \qquad (1)$$

where (1) has a distinctive solution and f, g, and φ are generic functions. An inner product ⟨·,·⟩ and the generated norm‖·‖ in R exist, and y(t) and τ is a not negative delay term. As a result, φ:[-τ,0]→ R andf: [0,T]→ R satisfy the conditions as follows:

$$Re\ \langle f(t,u_1,v,w) - f(t,u_2,v,w), u_1 - u_2 \rangle \le \alpha \|u_1 - u_2\|^2, \qquad (2)$$

$$\|f(t,u,v_1,w_1) - f(t,u,v_2,w_2)\| \le \beta\|v_1 - v_2\| + \sigma\|w_1 - w_2\|, \qquad (3)$$

$$\|g(t,v,u_1) - g(t,v,u_2)\| \le \gamma\|u_1 - u_2\|, (t,v) \in D, \qquad (4)$$

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For$t \in [0,T]$, $D = \{(t,V) : t \in [0,T), v \in [t-\tau, t]\}$, and for all$u_1, u_2, v_1, v_2, w_1, w_2 \in \mathfrak{R}$,where α, β, σ andγ are non-negative constants and $u = \min(u_1, u_2)$, $v = \min(v_1, v_2)$ and $w = \min(w_1, w_2)$.

1 Introduction

Generally speaking, a number of components prioritize excellent precision, stability, and efficiency. Initially, for large dynamical problems that are characterized by constant time, the nonlinear solver converges. Additionally, the applied mathematicians expanded their application area and permitted extensions in Dynamical Fourier Transform (DFT), Volterra Integro-Differential Equations with RK techniques of order 4, and so forth. Yuan H et al. [1-2] also highlight the alterations that should be especially stable. As described by Ali Filiz [3], the computation actually fits the numerical structure of DFT and Volterra Integro-Differential Equations with RK techniques of order 4. The creation of iterative techniques for comprehending and resolving nonlinear systems for circuit modeling frameworks has drawn a lot of attention, however it has been limited when handling transient as

We assume that (1) has a unique solution and satisfies in order to make the error analysis as optimum as possible.

$$\left\|\frac{d^{i}y(t)}{dt^{i}}\right\| \leq M_i, i = 1,2,\ldots, t \in [-\tau, T]. \tag{5}$$

We extend (5) for the function$g\big(t,\theta,y(\theta)\big)$, and satisfy the following:

$$\left\|\frac{\partial^{i} g\big(t,\theta,y(\theta)\big)}{\partial\theta^{i}}\right\| \leq N_i, t \in [0,T], \theta \in [-\tau, T]. \tag{6}$$

Now consider the following two-step form of Runge-Kutta methods [6]

$$Y_i^{(n)} = y_n + h\sum_{j=1}^{s} a_{ij} f\left(t_n + c_j h, Y_j^{(n)}\right), \quad i = 1,2,\dots,s$$

$$Y_i^{(n-1)} = y_{n-1} + h\sum_{j=1}^{s} a_{ij} f\left(t_{n-1} + c_j h, Y_j^{(n-1)}\right), \quad i = 1,2,\dots,s$$

$$y_{n+1} = (1-\theta)y_n + \theta y_{n-1} + h\sum_{i=1}^{s} b_i f\left(t_n + c_i h, Y_i^{(n)}\right)$$

$$+h\sum_{i=1}^{s} \tilde{b}_i\, f\left(t_{n-1} + c_i h, Y_i^{(n-1)}\right) \tag{7}$$

where $\sum_{i=1}^{s} b_i + \sum_{i=1}^{s} \tilde{b}_i = 1 + \theta, c_i = \sum_{i=1}^{s} a_{ij}\, , h = \frac{\tau}{m}$ is a step size,m is an arbitrarily given positive integer, and$0 \le \theta \le 1.$. Let's look at how the two-step Runge-Kutta method can be modified to (1):

$$Y_i^{(n)} = y_n + h\sum_{j=1}^{s} a_{ij} f\left(t_n + c_j h, Y_j^{(n)}, \bar{Y}_j^{(n)}, \tilde{Y}_j^{(n)}\right), \quad i = 1,2,\dots,s$$

$$y_{n+1} = (1-\theta)y_n + \theta y_{n-1} + h\sum_{i=1}^{s} b_i f\left(t_n + c_j h, Y_j^{(n)}, \bar{Y}_j^{(n)}, \tilde{Y}_j^{(n)}\right) + h\sum_{i=1}^{s} \tilde{b}_i\, f\left(t_{n-1} + c_j h, Y_j^{(n-1)}, \bar{Y}_j^{(n-1)}, \tilde{Y}_j^{(n-1)}\right) \tag{8}$$

In particular, $y_0 = \varphi(0)$, $y_n$is the numerical approximation at$t_n = nh$to the analytical solution and$y(t_n)$, the argument $\bar{Y}_j^{(n)} = Y_j^{(n-m)}$ represents an approximation to$y(t_n + c_j h - \tau)$, and the argument$\tilde{Y}_j^{(n)}$represents an approximation to [7]

$$\int_{t_n+c_jh-\tau}^{t_n+c_jh} g\left(t_n + c_j h, \xi, y(\xi)\right) d\xi$$

Which are obtained using a formula for convergent compound quadrature [8–9]:

$$\tilde{Y}_j^{(n)} = h\sum_{q=0}^{m} \bar{p}_q g\left(t_n + c_j h, t_{n-q} + Y_j^{(n-q)}\right), \quad j = 1,2,\dots,s \tag{9}$$

where $Y_j^{(n-q)} = \varphi(t_{n-q} + c_j h)$ with $n \le q, t_{n-q} + c_j h \le 0$.

The compound quadrature formula and the Runge-Kutta technique class have been used by numerous authors to solve delayed integer differential equations. We repeatedly used the trapezoidal rule for the convergent compound quadrature (9), indicating η=max{p˜_0,p˜_1,...,p˜_m }. It should be mentioned that the accepted quadrature formula (9) is based on Laguerre-Radau interpolations to Volterra IntegroDifferential Equation [10], which is extended like Pouzet's quadrature, Laguerre-Radau interpolations, and so on.

2. Nonlinear Stability Analysis

This section investigates the reliability of the two-step Runge-Kutta methods for converging compounded quadrature [11]. The perturbed problem of (1) must be examined in the following manner in order to examine the stability property:

$$z'(t) = f\left(t, z(t), z(t-\tau), \int_{t-\tau}^{t} g(t,\xi,z(\xi))d\xi\right), t \in [0,T],$$

$$z(t) = \psi(t), t \in [-\tau, 0], \tag{10}$$

where $\psi: [-\tau, 0] \to \Re$ is a continuous function. The exact solution to problem (10) is indicated as $z(t)$. Applying of the two-step Runge-Kutta method (7) to (10) leads to

$$Z_i^{(n)} = z_n + h\sum_{j=1}^{s} a_{ij} f\left(t_n + c_j h, Z_j^{(n)}, \bar{Z}_j^{(n)}, \tilde{Z}_j^{(n)}\right), \quad i = 1,2,\dots,s,$$

$$Z_i^{(n-1)} = z_{n-1} + h\sum_{j=1}^{s} a_{ij} f\left(t_{n-1} + c_j h, Z_j^{(n-1)}, \bar{Z}_j^{(n-1)}, \tilde{Z}_j^{(n-1)}\right), \quad i = 1,2,\dots,s$$

$$z_{n+1} = (1-\theta) z_n + \theta z_{n-1} + h\sum_{i=1}^{s} b_i f\left(t_n + c_j h, Z_j^{(n)}, \bar{Z}_j^{(n)}, \tilde{Z}_j^{(n)}\right) + h\sum_{i=1}^{s} \tilde{b}_i f\left(t_{n-1} + c_j h, Z_j^{(n-1)}, \bar{Z}_j^{(n-1)}, \tilde{Z}_j^{(n-1)}\right) \tag{11}$$

where z_n and $z_i^{(n)}$ represents approximations to $z(t_n)$ and $z(t_n + c_i h)$, respectively; the argument $\tilde{Z}_j^{(n)}$ represents an approximation to

$$\int_{t_n+c_jh-\tau}^{t_n+c_jh} g\left(t_n + c_jh, \xi, z(\xi)\right) d\xi$$

which the convergent compound quadrature formula achieves:

$$\tilde{Z}_i^{(n)} = h\sum_{q=0}^{m} \bar{p}_q g\left(t_n + c_jh, t_{n-q} + Z_i^{(n-q)}\right), \quad i = 1,2, \dots, s \tag{12}$$

$Z(n)j = Z_j^{(n-m)}$represents an approximation to$z\left(t_n + c_jh - \tau\right)$, and$z_n = \psi(t_n)forn = 0$,

$$Z_i^{(n-q)} = \psi\left(t_{n-q} + c_ih\right), \ for t_{n-q} + c_ih \le 0. \tag{13}$$

To ensure thestability, (9) fulfills the following condition:

$$h^2(m+1)^2 \sum_{q=0}^{m} \left|\tilde{p}_q\right|^2 < \tilde{p}^2 \tag{14}$$

here $\tilde{p}$is a positive constant.Let

$$w_n = y_n - z_n\,,$$

$$W_j^{(n)} = Y_j^{(n)} - Z_j^{(n)}, \qquad \widetilde{W}_j^{(n)} = \tilde{Y}_j^{(n)} - \tilde{Z}_j^{(n)}\,,$$

$$\tilde{Q}_j^{(n)} = h\left[f\left(t_n + c_jh, Y_j^{(n)},, \bar{Y}_j^{(n)}, \tilde{Y}_j^{(n)}\right) - f\left(t_n + c_jh, Z_j^{(n)}, \bar{Z}_j^{(n)}, \tilde{Z}_j^{(n)}\right)\right], j = 1, 2, \dots, s. \tag{15}$$

It follows from (8) and (11) that

$$W_i^{(n)} = w_n + \sum_{j=1}^{s} a_{ij} Q_j^{(n)}, \quad i = 1, 2, \dots, s.$$

$$W_i^{(n-1)} = w_{n-1} + \sum_{j=1}^{s} a_{ij} Q_j^{(n-1)},$$

This is the two-step Runge-Kutta method of s-stage general linear method.

(7). Now let $V_i^{(n)} = \left(W_i^{(n)^T}, W_i^{(n-1)^T}\right)^T$ be the internal stages and $\mu_n = (w_{n+1}^T, w_n^T)^T$ be the external vectors and $P_i^{(n)} = \left(QP_i^{(n)^T}, Q_i^{(n-1)^T}\right)^T$. So we have a general linear method for $2(s + 1)$phases are follows:

$$V_i^{(n)} = \sum_{j=1}^{s} C_{ij}^{11} P_i^{(n)} + \sum_{j=1}^{s} C_{ij}^{12} \mu_i^{(n)}, \quad i = 1, 2, \ldots, s,$$

$$\mu_{n+1} = \sum_{j=1}^{s} C_{ij}^{21} P_i^{(n)} + \sum_{j=1}^{s} C_{ij}^{22} \mu_i^{(n)}, \quad i = 1, 2, \ldots, s, \tag{17}$$

Hence the matrix coefficients are

$$\begin{aligned} C_{11} &= (C_{ij}^{11}) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad C_{12} = (C_{ij}^{12}) = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \\ C_{21} &= (C_{ij}^{21}) = \begin{pmatrix} b & \tilde{b} \\ 0 & 0 \end{pmatrix}, \quad C_{22} = (C_{ij}^{22}) = \begin{pmatrix} 1-\theta & \theta \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{18}$$

3Volterra Integro-Differential Equations stability study using RK techniques of order 4

Numerical stability is an essential component of an effective numerical technique. Even if an unstable numerical technique can continuously be of a high order, arbitrarily small perturbations may eventually lead to large departures from the right answer. This section will focus on the asymptotic constancy associated with the two-step RK look at of order four.

Definition 3.1 : Let k, l be the real constants, and $(k, l)$is said to bealgebraically stable if there exists a diagonal matrix $D = diag(d_1, d_2, \ldots, d_{2s})$and a diagonal matrix G such that $M = (m_{ij})$is not negative, where

$$M(k,l) = \begin{pmatrix} kG - C_{22}^T G C_{22} - 2l C_{12}^T D C_{12} & C_{12}^T D - C_{22}^T G C_{21} - 2l C_{12}^T D C_{11} \\ DC_{12} - C_{21}^T G C_{22} - 2l C_{11}^T D C_{12} & C_{11}^T D + DC_{11} - C_{21}^T G C_{21} - 2l C_{11}^T D C_{11} \end{pmatrix} \tag{19}$$

Theorem 3.2 Provided the existence of the quadrature formula (9) meets condition (14) and that the requirements (2) - (4) hold, and that the two step RK technique of order 4 (7) is (k,l)- mathematically stable with 0

$$\|y_n - z_n\| \leq C \max_{t\in[-\tau,0]} \|\varphi(t) - \psi(t)\|, \quad n = 1, 2, \ldots, \tag{20}$$

When the following condition holds:

$$h(\alpha + \beta + \sigma + \sigma\gamma^2\tilde{p}^2) \leq l, \tag{21}$$

Where C is solely dependent on the parameters α,β,σ,γ,η, and τ as well as the method.

Proof: Relying on an algebraic stability (k,l) and a rather easy (although laborious) calculation, it follows that

$$\|\mu_{n+1}\|^2 - k\|\mu_n\|^2 - 2\sum_{j=1}^{2s} d_j \, Re \left\langle V_j^{(n)}, P_j^{(n)} - lV_j^{(n)} \right\rangle = \sum_{i=1}^{2s+2} \sum_{j=1}^{2s+2} M_{ij} \langle r_i, r_j \rangle \tag{22}$$

Where$r_1 = w_{n+1}, r_2 = w_n, r_j = Q_{j-2}^{(n)}, j = 3, 4, \ldots, s + 2,$

$$r_j = Q_{j-s-2}^{(n-1)}, j = s + 3, s + 4, \ldots, 2s + 2. \tag{23}$$

By means of (k, l) algebraically stability of the method, we have the following:

$$\|\mu_{n+1}\|^2 \leq k\|\mu_n\|^2 + 2\sum_{j=1}^{2s} d_j \, Re \left\langle V_j^{(n)}, P_j^{(n)} - lV_j^{(n)} \right\rangle \tag{24}$$

It follows from (2) – (4) that

$$2\,Re\left\langle W_j^{(n)}, Q_j^{(n)}\right\rangle$$

$$= 2h\,Re\left\langle W_j^{(n)}, f\left(t_n + c_j h, Y_j^{(n)},, \bar{Y}_j^{(n)}, \hat{Y}_j^{(n)}\right) - f\left(t_n + c_j h, Z_j^{(n)}, \bar{Z}_j^{(n)}, \hat{Z}_j^{(n)}\right)\right\rangle$$

$$= 2h\,Re\left\langle W_j^{(n)}, f\left(t_n + c_j h, Y_j^{(n)},, \bar{Y}_j^{(n)}, \hat{Y}_j^{(n)}\right) - f\left(t_n + c_j h, Z_j^{(n)}, \bar{Y}_j^{(n)}, \hat{Y}_j^{(n)}\right)\right\rangle$$

$$+ 2h\,Re\left\langle W_j^{(n)}, f\left(t_n + c_j h, Z_j^{(n)}, \bar{Y}_j^{(n)}, \hat{Y}_j^{(n)}\right) - f\left(t_n + c_j h, Z_j^{(n)}, \bar{Z}_j^{(n)}, \hat{Z}_j^{(n)}\right)\right\rangle$$

$$\leq 2h\alpha\left\|W_j^{(n)}\right\|^2 + 2h\beta\left\|W_j^{(n)}\right\| \cdot \left\|\bar{W}_j^{(n)}\right\| + 2h\sigma\left\|W_j^{(n)}\right\| \cdot \left\|\hat{W}_j^{(n)}\right\|$$

$$\leq 2h\alpha\left\|W_j^{(n)}\right\|^2 + h\beta\left(\left\|W_j^{(n)}\right\|^2 + \left\|\bar{W}_j^{(n)}\right\|^2\right) + h\sigma\left(\left\|W_j^{(n)}\right\|^2 + \left\|\hat{W}_j^{(n)}\right\|^2\right)$$

$$= (2h\alpha + h\beta + h\sigma)\left\|W_j^{(n)}\right\|^2 + h\beta\left\|\bar{W}_j^{(n)}\right\|^2 + h\sigma\left\|\hat{W}_j^{(n)}\right\|^2 \quad (25)$$

$$= (2h\alpha + h\beta + h\sigma)\left\|W_j^{(n)}\right\|^2 + h\beta\left\|\bar{W}_j^{(n)}\right\|^2 + h\sigma\left\|h\sum_{q=0}^{m}\bar{p}_q g\left(t_n + c_j h, t_{n-q} + c_j h, Y_j^{(n-q)}\right) - h\sum_{q=0}^{m}\bar{p}_q g\left(t_n + c_j h, t_{n-q} + c_j h, Z_j^{(n-q)}\right)\right\|^2$$

$$\leq (2h\alpha + h\beta + h\sigma)\left\|W_j^{(n)}\right\|^2 + h\beta\left\|\bar{W}_j^{(n-m)}\right\|^2 + h\sigma(m+1)h^2\tilde{p}_q^2$$

$$\times\sum_{q=0}^{m}\left\|g\left(t_n + c_j h, t_{n-q} + c_j h, Y_j^{(n-q)}\right) - g\left(t_n + c_j h, t_{n-q} + c_j h, Z_j^{(n-q)}\right)\right\|^2$$

$$\leq (2h\alpha + h\beta + h\sigma)\left\|W_j^{(n)}\right\|^2 + h\beta\left\|\bar{W}_j^{(n-m)}\right\|^2 + h\sigma(m+1)h^2\gamma^2 \times \sum_{q=0}^{m}\tilde{p}_q^2\left\|W_j^{(n-q)}\right\|^2$$

Substituting (25) into (24), using (14) we get the following:

$$\|\mu_{n+1}\|^2 \leq \|\mu_n\|^2 + \sum_{j=1}^{2s} d_j\left\{(2h\alpha + h\beta + h\sigma - 2l)\left\|W_j^{(n)}\right\|^2 + h\beta\left\|\bar{W}_j^{(n-m)}\right\| + h\sigma\gamma^2\tilde{p}^2\sum_{q=0}^{m}\left\|W_j^{(n-q)}\right\|^2\right\} \quad (26)$$

We obtain the following using the induction method:

$$\|\mu_{n+1}\|^2 \leq \|\mu_{-1}\|^2 + \sum_{i=-1}^{n}\sum_{j=1}^{2s} d_j\left\{(2h\alpha + 2h\beta + h\sigma - 2l + h\sigma\gamma^2\tilde{p}^2)\left\|W_j^{(i)}\right\|^2\right\}$$

$$+ \sum_{i=-m}^{-2}\sum_{j=1}^{2s} d_j\left\{(h\beta + h\sigma\gamma^2\tilde{p}^2)\left\|W_j^{(i)}\right\|^2\right\}$$

$$\leq \|\mu_{-1}\|^2 + 2\sum_{i=-1}^{n}\sum_{j=1}^{2s} d_j\left\{h(\alpha + \beta + \sigma + \sigma\gamma^2\tilde{p}^2 - l)\left\|W_j^{(i)}\right\|^2\right\}$$

$$+ \sum_{i=-m}^{-2}\sum_{j=1}^{2s} d_j\left\{(h\beta + h\sigma\gamma^2\tilde{p}^2)\left\|W_j^{(i)}\right\|^2\right\}$$

$$\leq \|\mu_{-1}\|^2 + mh\sum_{j=1}^{2s} d_j\,(\beta + \sigma\gamma^2\tilde{p}^2)\max_{-m-1\leq i\leq -2}\left\|W_j^{(i)}\right\|^2$$

$$\leq \|w_0\|^2 + \|w_{-1}\|^2 + mh\sum_{j=1}^{2s} d_j\,(\beta + \sigma\gamma^2\tilde{p}^2)\max_{-m-1\leq i\leq -2}\left\|W_j^{(i)}\right\|^2$$

$$\leq \left(2 + \tau(\beta + \sigma\gamma^2\tilde{p}^2)\textstyle\sum_{j=1}^{2s} d_j\right)\max_{t\in[-\tau,0]}\|\varphi(t) - \psi(t)\|^2 \qquad (27)$$

Hence, $\|\mu_{n+1}\| \leq C \max\limits_{t\in[-\tau,0]}\|\varphi(t) - \psi(t)\|$, *where* $C = \sqrt{2 + \tau(\beta + \sigma\gamma^2\tilde{p}^2)\sum_{j=1}^{2s} d_j}$.

Hence the proof of Theorem 2is completed.

Theorem 3.3: If a two side RK method of order 4(7) is (k,l)-algebraically stable with 0 then (9), (12), and (14) must satisfy the following condition to be asymptotically stable.

$$h\left(\alpha + \beta + \frac{1}{2}\sigma + \frac{1}{2}\sigma\gamma^2\tilde{p}^2\right) < l. \qquad (28)$$

Proof: Let$u = (2\alpha + \beta + \sigma)\,h - 2l$,

$$\bar{k} = max\left\{k, \frac{(\beta+\sigma\gamma^2\tilde{p}^2)h}{(-u)^{\frac{1}{m}}}\right\} \qquad (29)$$

when$h\left(\alpha + \beta + \left(\frac{1}{2}\right)\sigma + \left(\frac{1}{2}\right)\sigma\gamma^2\tilde{p}^2\right) < l$, we have$u < -(\beta + \sigma\gamma^2\tilde{p}^2)h$ and $0 < k < 1$.

Theorem 3.2 yields the following:

$$\|\mu_{n+1}\|^2 \leq \bar{k}\|\mu_n\|^2 + \sum_{j=1}^{2s} d_j \left\{ (2h\alpha + h\beta + h\sigma - 2l)\left\|W_j^{(i)}\right\|^2 + h\beta\left\|W_j^{(n-m)}\right\| + h\sigma\gamma^2\tilde{p}^2 \sum_{j=1}^{2s}\left\|W_j^{(n-q)}\right\|^2 \right\}$$

$$\leq \bar{k}\|\mu_n\|^2 + \sum_{j=1}^{2s} d_j \left\{ u\left\|W_j^{(i)}\right\|^2 + h\beta\left\|W_j^{(n-m)}\right\| + h\sigma\gamma^2\tilde{p}^2 \sum_{j=1}^{2s}\left\|W_j^{(n-q)}\right\|^2 \right\} \tag{30}$$

By induction method,

$$\|\mu_{n+1}\|^2 \leq \bar{k}^{n+2}\|\mu_{-1}\|^2 + \sum_{i=-1}^{n} \bar{k}^{n-m-i} \times \sum_{j=1}^{2s} d_j \left(u\bar{k}^m + h\beta + h\sigma\gamma^2\tilde{p}^2\right) \cdot \left\|W_j^{(i)}\right\|^2 + \sum_{i=-m-1}^{-2} \sum_{j=1}^{2s} d_j\, \bar{k}^{n-m+2}(h\beta + h\sigma\gamma^2\tilde{p}^2)\left\|W_j^{(i)}\right\|^2 \tag{31}$$

$$\leq \left\{2\bar{k}^{n+2} + \bar{k}^{n-m+2}\tau \sum_{j=1}^{2s} d_j\,(\beta + \sigma\gamma^2\tilde{p}^2)\right\} + \max_{t\in[-\tau,0]}\|\varphi(t) - \psi(t)\|^2 \tag{32}$$

The following consequence of the inequality and the knowledge that 0

$$\lim_{n\to\infty}\|\mu_{n+1}\| = 0 \tag{33}$$

Because$\mu_n = (w_n^T, w_{n-1}^T)^T$, we can get from (33) that

$$\lim_{n\to\infty}\|w_n\| = 0 \tag{34}$$

Hence the proof.

Example 3.4: Consider the following non-linear integro-differential Volterra delay equations:

$$\tilde{Y}_j^{(n)} = \frac{h}{3}\Bigg[\sin(t_n + c_j h)\, Y_j^{(n)} + 4\sum_{q=1}^{\frac{m}{2}} \sin(t_{n-2q+1} + c_j h) Y_j^{(n-2q+1)}$$
$$+ 2\sum_{q=1}^{\frac{(m-2)}{2}} \sin(t_{n-2q} + c_j h) Y_j^{(n-2q)} + \sin(t_{n-m} + c_j h) Y_j^{(n-m)}\Bigg]$$

$$y'(t) = -(6+\sin t)y(t) + y\left(t-\frac{\pi}{4}\right) - \int_{t-\frac{\pi}{4}}^{t} \sin(v)\, y(v)dv + 5\,exp(cos\, t)\ ,\ t \geq 0,\ y(t) = exp(cos\, t), -\frac{\pi}{4} \leq t \leq 0 \qquad (35)$$

And its perturbed problem

$$z'(t) = -(6+\sin t)z(t) + z\left(t-\frac{\pi}{4}\right) - \int_{t-\frac{\pi}{4}}^{t} \sin(v)\, z(v)dv + 5\,exp(cos\, t)\ ,\ t \geq 0,\ z(t) = 2, -\frac{\pi}{4} \leq t \leq 0$$

It can be verified that this system has solution$y(t) = exp(cos\, t)$and$a = -4, \beta = \frac{1}{2}$, $\sigma = \frac{1}{2}$, $\gamma = 1$, and hence we conclude that the system (35) satisfies the properties of stability and convergence.

4. **Conclusion**

Several aspects of stability, effectiveness, and high accuracy were the main focus of this work. According to the constant time, the nonlinear solver first converges for large dynamical problems. Additionally, the adjustments should be quite stable when applied to Volterra Integro-Differential Equations and Discrete Fourier Transform (DFT) with RK techniques of order 4. This chapter's primary goal is to examine the Volterra Integro-Differential Equations' nonlinear stability and convergence characteristics using RK techniques of order 4. The development of iterative techniques for comprehending and resolving nonlinear systems for circuit modeling frameworks has received a lot of attention. However, these techniques have been limited when handling both transient and consistent state problems, such as oscillatory behavior, error analysis, and stability analysis.

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