Utilitas Mathematica

ISSN 0315-3681 Volume 122, 2025

Subclass of Bi-Univalent Functions Associated With q-Lommel Polynomials

Renu P. Pathak
Dept. of Mathematics, Sandip University, Nashik (MH), India.

Balasaheb B. Gadekar[1]
Research Scholar, Dept. of Mathematics, Sandip University, Nashik (MH), India.

Mallikarjun G. Shrigan[2*]
Dept. of Mathematics, School of Computational Sciences, JSPM University Pune (MH), India.

Email: [1]gbalasaheb033@gmail.com, [2*]mgshrigan@gmail.com

Abstract

This study uses the notion of the q-differential operator associated with q-Lommel polynomials to explain new subclass of bi-univalent functions. Additionally, we determine constraints for the initial Taylor-Maclaurin coefficients and initial Fekete-Sezgö inequalities for this recently specified subclass.

Keywords: Bi-Univalent function; starlike function; convex function; q-differential operator; q-Lommel polynomials.

1. Introduction

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} = \{l \in \mathbb{C}: |l| < 1\}$ be open unit disk in $\mathbb{C}$. Also, let $\mathcal{A}$ denote the class of all analytic functions of the form

$$f(l) = l + a_2 l^2 + a_3 l^3 + \cdots = l + \sum_{n=2}^{\infty} a_n l^n;\ l \in \mathbb{D} \tag{1}$$

normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{D}$. The class S is a subclass of $\mathcal{A}$, said to be univalent functions in $\mathbb{D}$. In the domain $\mathbb{D}$, function $f(l)$ is univalent if it satisfies one-one mapping such that for $l_1 \neq l_2$ in $\mathbb{D}, f(l_1) \neq f(l_2)$ exists.

For two functions $f_1(l)$ and $f_2(l) \in \mathbb{D}$, we say $f_1(l)$ is subordinate to $f_2(l)$ if there Schwarz function w exists in $\mathbb{D}$ which is analytic with $|w(z)| < 1$ and $w(0) = 0$, such that $f_1(l) = f_2\big(w(l)\big)$. This subordination is defined by $f_1 \prec f_2$ or $f_1(l) \prec f_2(l), l \in \mathbb{D}$. It is commonly known that, if the $f_2(l)$ is univalent in $\mathbb{D}$, then (see [10])

$$f_1(l) \prec f_2(l), l \in \mathbb{D} \Leftrightarrow f_1(0) = f_2(0) \text{ and } f_1(\mathbb{D}) \subset f_2(\mathbb{D}).$$

A domain is considered starlike with respect to a point $w_0 \in \mathbb{D}$ if the line segment connecting any two points in $\mathbb{D}$ to w_0 lies within $\mathbb{D}$, whereas if the line segment connecting any two points in $\mathbb{D}$ lies wholly in $\mathbb{D}$ then $\mathbb{D} \subset \mathbb{C}$ is convex. If $f(\mathbb{D})$ is a starlike domain with regard to the origin, then a function $f \in \mathcal{A}$ is starlike; if $f(\mathbb{D})$ is convex, then it is convex.

In analytical term, $f \in \mathcal{A}$ is classified as starlike if and only if

$$\mathrm{Re}\left(\frac{lf'(l)}{f(l)}\right) > 0\ (l \in \mathbb{D}).$$

The class of starlike functions, commonly referred to as $\mathcal{ST}$, can be described through the concept of subordination as follows:

$$\mathcal{ST} = \left\{f \in \mathcal{A}: \frac{lf'(l)}{f(l)} \prec \frac{1+l}{1-l}\right\},$$

whereas $f \in \mathcal{A}$ is convex if and only if and only if

$$\mathrm{Re}\left(1 + \frac{lf''(l)}{f'(l)}\right) > 0\ (l \in \mathbb{D}).$$

A familiar class of convex functions denoted as $\mathcal{CV}$. This subordination can be expressed as follows:

$$\mathcal{CV} = \left\{f \in \mathcal{A}: 1 + \frac{lf''(l)}{f'(l)} \prec \frac{1+l}{1-l}\right\}.$$

Also, Ma and Minda [9] introduced new function $\psi(l)$ by using the concept of subordination will belong to the class $\mathcal{P}$ if

$$\psi(l) = 1 + \sum_{n=1}^{\infty} a_n l^n, (l \in \mathbb{D}),$$

where $\mathrm{Re}(\psi(l)) > 0$ with $\psi(0) = 1, \psi'(0) > 0$ and for $n \in \mathbb{N}$. They defined $\mathcal{ST}(\psi)$ and $\mathcal{CV}(\psi)$ in the following way:

$$f(l) \in \mathcal{ST}(\psi) \Leftrightarrow \frac{lf'(l)}{f(l)} \prec \psi(l)$$

and

$$f(l) \in \mathcal{CV}(\psi) \Leftrightarrow 1 + \frac{lf''(l)}{f'(l)} \prec \psi(l).$$

Every $f \in \mathcal{ST}$ function guarantees that the image of the unit disk consists a disk of radius 1/4 as stated by/in the Koebe One-Quarter Theorem [3]. Consequently, there is an inverse f^{-1} for every univalent function, its inverse is described as

$$f^{-1}(f(l)) = l, (l \in \mathbb{D})$$

and

$$f(f^{-1}(w)) = w, \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right).$$

The inverse function $g = f^{-1}$ has the form

$$g = f^{-1} = w - (a_2)w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \qquad (2)$$

A function $f \in \mathcal{A}$ is considered bi-univalent in domain $\mathbb{D}$ if both functions f and f^{-1} are univalent in $\mathbb{D}$. The class of bi-univalent functions in $\mathbb{D}$ is denoted by Σ. The functions f_1, f_2 and f_3 are the examples of the families of Σ which are defined as

$$f_1(l) = l(1-l)^{-1}, f_2(l) = -\log(1-l) \text{ and } f_3(l) = \frac{1}{2}\log\frac{1+l}{1-l},$$

with inverse functions

$$g_1(w) = w(1+w)^{-1}, g_2(w) = \frac{e^w-1}{e^w} \text{ and } g_3(w) = \frac{e^{2w}-1}{e^{2w}+1}.$$

However, the famous Koebe function is not a member of Σ. Also $l - \frac{l^2}{2}$ and $\frac{l}{1-l^2}$ not the member of Σ. Lewin [7] introduced the bi-univalent class Σ and proved that $|a_2| < 1.51$. After that Brannan and Clunie [2] showed that $|a_2| \le \sqrt{2}$. Subsequently, Netanyahu [10] proved that $\max|a_2| = 4/3$. The exploration of the bi-univalent function class and analytic has actually been greatly enhanced and revitalized by [13] in their pioneering work (also see, [5], [6], [12], [14]).

In class $\mathcal{ST}$ a function's nth coefficient is constrained by n, and the coefficients bounds provide insights into geometric properties of the function. The well-known problem Fekete-Sezgö [4] is to resolute the greatest possible value for the coefficient functional $|a_3 - \mu a_2^2|$ through the class $\mathcal{ST}$ where $\mu \in [0,1]$, solved using the Loewner approach and is presented as follows:

$$|a_3 - \mu a_2^2| \le \begin{cases} 3 - 4\mu & \text{if } \mu \le 0 \\ 1 + 2\exp\left(\dfrac{2\mu}{\mu - 1}\right) & \text{if } 0 \le \mu < 1 \\ 4\mu - 3 & \text{if } \mu \ge 1. \end{cases}$$

Numerous scholars have recently carefully examined a number of noteworthy findings pertaining to the subclass of analytic functions and q operators ([8],[11],[14],[15]).

Definition 1.1. For $0 < q < 1$, the q-derivative of a $f(l)$ is defined as

$$D_q(l) = \begin{cases} \dfrac{f(ql) - f(l)}{(q-1)l} & \text{for } l \ne 0 \\ f'(l) & \text{for } l \ne 0. \end{cases}$$

Remember that $\lim\limits_{q\to 1} D_q(l) = f'(l)$. From (1) we deduce that,

$$D_q f(l) = 1 + \sum_{k=2}^{\infty} [k]_q a^k l^{k-1} \ , \tag{3}$$

where

$$\lim_{q\to 1}[k]_q = \frac{1-q^k}{1-q} = 1 + q + \cdots. + q^k \to k.$$

Definition 1.2. The expansion defined in the form of

$$(x+a)_q^n=\sum_{n=2}^{\infty} q^{\frac{j(j-1)}{2}}\binom{n}{j} a^j x^{n-j}\,, \tag{4}$$

is known as q-binomial expansion, where $\binom{n}{j}$ is q-binomial coefficient

$$\binom{n}{j}=\frac{[n]_q!}{[j]_q!\,[n-j]_q!}. \tag{5}$$

Al-Salam and Ismail [1] used the q-Lommel polynomials and give a new q-polynomials as follows

Definition 1.3. [1] The polynomial

$$K_n(x,\alpha,\xi)=\sum_{j=0}^{\frac{n}{2}} \frac{(-\alpha{:}\,q)\cdot(q{:}\,q)_{n-j}x^{n-2j}(-\xi)^j}{(-\alpha;q)\cdot(q;q)_j(q;q)_{n-j}}q^{j(j-1)}, \tag{6}$$

is known as generalized q-Lommel polynomials.

Theorem 1.1. [1] The generalized q-Lommel polynomials satisfy

$$k_{n+1}(x;\alpha,\xi)=k_n(x;\alpha,\xi)(1+\alpha q^n)x-k_{n-1}(x;\alpha,\xi)\xi q^{n-1}, \tag{7}$$

with initial condition

$$k_0(x;\alpha,\xi)=1 \text{ and } k_1(x;\alpha,\xi)=(1+\alpha)x.$$

Using the generalized q-Lommel polynomials, we establish the following:

Definition 1.4.: Let $\psi(x,l,\alpha,\xi,q)$ be defined as follows

$$\psi(x,l,\alpha,\xi,q)=1+\sum_{j=1}^{\infty} k_j(x,l,\alpha,\xi,q)l^j. \tag{8}$$

Nowadays, q-orthogonal polynomials are used extensively in geometric function theory. We present following subclass of bi-univalent functions and analytic using the q-starlike function and the subordination principle:

Definition 1.5. The subclass $\mathcal{F}_{\Sigma}^{q}(x,\alpha,\xi)$ contains a function $f(l)$ of the form (1), if

$$\frac{l\partial_q f(l)}{f(l)}+\xi\frac{l^2\partial_q\big(\partial_q f(l)\big)}{f(l)}\prec\psi(x,l,\alpha,\xi,q)\ \left(\frac{1}{2}<x<1{,}0<q<1,l\in\mathbb{D}\right)$$

and

$$\frac{\omega\partial_q f(w)}{f(w)}+\xi\frac{w^2\partial_q\big(\partial_q f(w)\big)}{f(w)}\prec\psi(x,w,\alpha,\xi,q)\left(\frac{1}{2}<x<1{,}0<q<1,w\in\mathbb{D}\right). \tag{9}$$

By using (7), we found the following relations:

$$k_1(x\colon \alpha,\xi) = (1+\alpha)x; k_2(x\colon \alpha,\xi) = x^2(1+\alpha q)(x+\alpha) - \xi. \tag{10}$$

We note from (8) that

$$\psi(l,x,\alpha,\xi,q) = 1 + k_1(l,x,\alpha,\xi,q)z + k_2(l,x,\alpha,\xi,q)z^2 + k_3(l,x,\alpha,\xi,q)z^3 + . \tag{11}$$

We need the following result to support our main finding.

Lemma 1.1. [9] If $\Phi(l) = 1 + \sum_{n=1}^{\infty} c_n l^n = 1 + c_1 l + c_2 l^2 + c_3 l^3 + \cdots$,

where $\Phi(l)$ belongs to the class $\mathcal{P}$ of functions having positive real root.

Then $|c_n| \leq 2, (n \in \mathbb{N})$. This inequality is sharp for each n.

2. Coefficients Bounds

Theorem 2.1. Let f given by (1) be in the class $\mathcal{F}_{\Sigma}^{q}(x,\alpha,\xi)$. Then

$$|a_2| \leq \left\{ \frac{x(1+\alpha)(\sqrt{x(1+\alpha)})}{\sqrt{x^2(1+\alpha)^2[\eta_1-\eta_2] - [\eta_2-1]^2[x^2(1+\alpha q)(1+\alpha) - \xi - x(1+\alpha)]}} \right\}$$

and

$$|a_3| \leq \frac{x^2(1+\alpha)^2}{\left[(1+\xi)[2]_q - 1\right]^2} + \frac{x(1+\alpha)}{\left[[3]_q(1+\xi[2]q) - 1\right]}, \tag{12}$$

where $\eta_1 = [3]_q\left(1+\xi[2]_q\right), \eta_2 = (1+\xi)[2]_q$.

Proof. Since $f(l) \in \mathcal{F}_{\Sigma}^{q}(x,\alpha,\xi)$, we have

$$\frac{l\partial_q f(l)}{f(l)} + \xi \frac{l^2 \partial_q \left(\partial_q f(l)\right)}{f(l)} = \psi(m(l),x;\alpha,\xi,q) \tag{13}$$

and

$$\frac{\omega\partial_q f(w)}{f(w)} + \xi \frac{w^2 \partial_q\left(\partial_q f(w)\right)}{f(w)} = \psi(n(w),x;\alpha,\xi,q) \tag{14}$$

Let $m,n \in S$, we have

$$m(l) = \frac{1+\tau(l)}{1-\tau(l)} = 1 + m_1 l + m_2 l^2 + m_3 l^3 + \cdots \tag{15}$$

and

$$\tau(l) = \frac{m(l)-1}{m(l)+1}, l \in \mathbb{D}.$$

For $n(w)$, we have

$$n(w) = \frac{1 + \varepsilon(w)}{1 - \varepsilon(w)} = 1 + n_1 w + n_2 w^2 + n_3 w^3 + \cdots, \tag{16}$$

where

$$\varepsilon(\omega) = \frac{n(w)-1}{n(w)+1}, \omega \in \mathbb{D}.$$

These forms suggest that $m(l)$ and $n(w)$ are power series expansions around $l = 0$ and $w = 0$, respectively. From (15) and (16), we get

$$\tau(l) = \frac{m(l) - 1}{m(l) + 1} = \frac{1}{2}\left[m_1 l + \left(m_2 - \frac{m_1^2}{2}\right) l^2 + \left(m_3 - m_1 m_2 + \frac{m_1^3}{4}\right) l^3 + \cdots\right] \tag{17}$$

and

$$\varepsilon(w) = \frac{n(w) - 1}{n(w) + 1} = \frac{1}{2}\left[n_1 w + \left(n_2 - \frac{n_1^2}{2}\right) w^2 + \left(n_3 - n_1 n_2 + \frac{n_1^3}{4}\right) w^3 + \cdots\right]. \tag{18}$$

From (17) and (18), applying $\psi(m(l), x; \alpha, \xi, q), \psi(n(\omega), x; \alpha, \xi, q)$ given by q-Lommel polynomials, we get

$$\begin{aligned}\psi(\tau(l), x; \alpha, \xi, q) = 1 &+ \frac{k_1(x, \alpha, \xi, q)}{2} m_1 l \\ &+ \left[\frac{k_1(x, \alpha, \xi, q)}{2}\left(m_2 - \frac{m_1^2}{2}\right) + \frac{k_2(x, \alpha, \xi, q)}{4} m_1^2\right] l^2 \\ &+ \left\{\frac{k_1(x, \alpha, \xi, q)}{2}\left(m_3 - m_1 m_2 + \frac{m_1^3}{4}\right)\right\} l^3 + \cdots \end{aligned} \tag{19}$$

and

$$\begin{aligned}\psi(\varepsilon(\omega), x; \alpha, \xi, q) = 1 &+ \frac{k_1(x, \alpha, \xi, q)}{2} n_1 w \\ &+ \left[\frac{k_1(x, \alpha, \xi, q)}{2}\left(n_2 - \frac{n_1^2}{2}\right) + \frac{k_2(x, \alpha, \xi, q)}{4} n_1^2\right] w^2 \\ &+ \left\{\frac{k_1(x, \alpha, \xi, q)}{2}\left(n_3 - n_1 n_2 + \frac{n_1^3}{4}\right) + \frac{k_2(x, \alpha, \xi, q)}{2} n_1 \left(n_2 - \frac{n_1^2}{2}\right)\right\} w^3 + \cdots. \end{aligned} \tag{20}$$

From (13), (19), (14) and (20), we have

$$\left[(1 + \xi)[2]_q - 1\right] a_2 = \frac{k_1(x, \alpha, \xi, q)}{2} m_1, \tag{21}$$

$$[3]q(1 + \xi[2]q) - 1a_3 - \left[(1 + \xi)[2]_q - 1\right] a_2^2 = \frac{k_1(x,\alpha,\xi,q)}{2}\left(m_2 - \frac{m_1^2}{2}\right) + \frac{k_2(x,\alpha,\xi,q)}{4} m_1^2, \tag{22}$$

$$-\left[(1+\xi)[2]_q-1\right]a_2=\frac{k_1(x,\alpha,\xi,q)}{2}n_1 \tag{23}$$

and

$$\left[[3]_q\left(1+\xi[2]_q\right)-1\right](2a_2^2-a_3)-\left[(1+\xi)[2]_q-1\right]a_2^2=\frac{k_1(x,\alpha,\xi,q)}{2}\left(n_2-\frac{n_1^2}{2}\right)+\frac{k_2(x,\alpha,\xi,q)}{4}n_1^2. \tag{24}$$

Using (21) and (23), we get

$$m_1=-n_1;\ m_1^2=n_1^2;\ m_1^3=-n_1^3 \tag{25}$$

and

$$a_2^2=\frac{k_1^2(x,\alpha,\xi,q)}{8\left[(1+\xi)[2]_q-1\right]^2}(m_1^2+n_1^2). \tag{26}$$

By using equation (25) and (26), we obtain

$$\frac{2a_2^2\left[(1+\xi)[2]_q-1\right]^2}{k_1^2(x,\alpha,\xi,q)}=\frac{n_1^2}{2}. \tag{27}$$

By adding (22), (24) and using (25), (26) we have

$$a_2^2=\frac{k_1^3(x,\alpha,\xi,q)(m_2+n_2)}{4k_1^2(x,\alpha,\xi,q)\left[[3]q(1+\xi[2]q)-(1+\xi)[2]_q\right]-4(k_2(x,\alpha,\xi,q)-k_1(x,\alpha,\xi,q))\left[(1+\xi)[2]_q-1\right]^2}. \tag{28}$$

By using Lemma 1.1, $|m_i|\le 2$ and $|n_i|\le 2$ and by result of q-Lommel Polynomials, we have

$$|a_2|\le\left\{\frac{x(1+\alpha)\left(\sqrt{x(1+\alpha)}\right)}{\sqrt{x^2(1+\alpha)^2[\eta_1-\eta_2]-[\eta_2-1]^2[x^2(1+\alpha q)(1+\alpha)-\xi-x(1+\alpha)]}}\right\}, \tag{29}$$

where $\eta_1=[3]_q\left(1+\xi[2]_q\right),\eta_2=(1+\xi)[2]_q$.
Now by subtracting (24) from (22) and using (25), we get

$$2a_3\left[[3]_q\left(1+\xi[2]_q\right)-1\right]-\left[[3]_q\left(1+\xi[2]_q\right)-1\right]2a_2^2=\frac{k_1(x,\alpha,\xi,q)}{2}(m_2-n_2). \tag{30}$$

Using (26) and Lemma 1.1, $|m_i|\le 2$ and $|n_i|\le 2$, we have

$$|a_3|\le\frac{x^2(1+\alpha)^2}{\left[(1+\xi)[2]_q-1\right]^2}+\frac{x(1+\alpha)}{\left[[3]_q\left(1+\xi[2]_q\right)-1\right]}. \tag{31}$$

3. Fekete-Sezgö Inequality

We introduce the $|a_3 - \mu a_2^2|$ for the function $f(l) \in \mathcal{A}$ in the class $\mathcal{F}_{\Sigma}^{q}(x, \alpha, \xi)$ associated with q-Lommel polynomials.

Theorem 3.1. Let $0 < q < 1$ and if the function $f(l)$ given by (1) belongs to $\mathcal{F}_{\Sigma}^{q}(x, \alpha, \xi)$ then for any complex μ

$$|a_3 - \mu a_2^2| = k_1(x, \alpha, \xi, q)[((1-\mu)\varpi(\xi) + \Xi(\xi))m_2 + ((1-\mu)\varpi(\xi) - \Xi(\xi))n_2].$$

where

$$\varpi(\xi) = \frac{k_1^2(x, \alpha, \xi, q)}{\Delta}$$

and

$$\Xi(\xi) = \frac{1}{4[[3]q(1+\xi[2]q)-1]}.$$

Proof. Using (28), equation (30) becomes,

$$a_3 = A_1(l)(m_2 + n_2) + \frac{k_1(x, \alpha, \xi, q)(m_2 - n_2)}{4[\eta_1 - 1]}, \tag{32}$$

where $\eta_1 = [3]_q\big(1 + \xi[2]_q\big)$, $\eta_2 = (1 + \xi)[2]_q$

and

$$A_1(l) = \frac{K_1^3(x, \alpha, \xi, q)}{\Delta}$$

and $\Delta = 4K_1^2(x, \alpha, \xi, q)[\eta_1 - \eta_2] - 4(K_2(x, \alpha, \xi, q) - K_1(x, \alpha, \xi, q))[\eta_2 - 1]^2$.

By using (28) and (32), we have

$$a_3 - \mu a_2^2 = [A_1(l) + A_2(l) - A_3(l)]m_2 + [A_1(l) - A_2(l) - A_3(l)]n_2 \tag{33}$$

where

$$A_2(l) = \frac{K_1(x, \alpha, \xi, q)}{4\big[[3]q(1 + [2]q) - 1\big]} \;\; ; \;\; A_3(l) = \frac{\mu K_1^3(x, \alpha, \xi, q)}{\Delta}$$

From (33), we have

$$a_3 - \mu a_2^2 = [(1-\mu)\, A_1(l) + A_2(l)]\, m_2 + [(1-\mu)\, A_1(l) - A_2(l)]n_2,$$

hence

$$a_3 - \mu a_2^2 = K_1(x,\alpha,\xi,q)\,[\,((1-\mu)\varpi(\xi) + \Xi(\xi)\,]\;m_2 + [\,((1-\mu)\varpi(\xi) - \Xi(\xi)\,]n_2,$$

where

$$\varpi(\xi) = \frac{k_1^2(x,\alpha,\xi,q)}{\Delta}$$

and

$$\Xi(\xi) = \frac{1}{4[[3]q(1+\xi[2]q)-1]}.$$

4. Conclusions

We introduce interesting subclass $\mathcal{F}_\Sigma^q(x,\alpha,\xi)$ of bi-univalent function by q-Lommel polynomials. Furthermore, for the recently described subclass, we obtain the Fekete-Szegö functional and initial coefficient bounds.

5. References

[1] W. Al-Salam and M. Ismail, Orthogonal polynomials associated with the Rogers-Ramanujan continued fraction, Pacific J. Math., 104, (1983), 269-283.

[2] D. A. Brannan and J. G. Clunie, Aspects of Contemporary Complex Analysis. Proceedings of the NATO Advanced Study Institute Held at the University of Durham, Durham; July 1-20, 1979. Academic Press, London 1980.

[3] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften; Springer: New York, NY, USA; Berlin/Hiedelberg, Germany; Tokyo, Japan, 1983.

[4] M. Fekete and G. Szegö, Eine Bemerkung Uber Ungerade Schlichte Funktionen, J. Lon. Math. Soc., 8 (2), (1933), 85-89.

[5] P. N. Kamble and M. G. Shrigan, Initial coefficient estimates for bi-univalent functions, Far East J. Math., 105 (2), (2018), 271-282.

[6] P. N. Kamble and M. G. Shrigan, Coefficient estimates for a subclass of bi-univalent functions defined by Sălăgean type q-calculus operator, 58(4), (2018), 677-688.

[7] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc., 18 (1967), 63-68.

[8] S. S. Miller and P. T. Mocanu, Differential Subordinations, Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, 225 (2000), Marcel Dekker Incorporated, New York and Basel.

[9] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in: Proceedings of the Conference on Complex Analysis, Tianjin, 1992, in: Conf. Proc. Lecture Notes Anal., vol. I, Int. Press, Cambridge, MA, 1994, pp. 157-169.

[10] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Ration. Mech. Anal., 32 (1969), 100-112.

[11] M. G. Shrigan, Second Hankel determinant for bi-univalent functions associated with q-differential operator, J. Sib. Fed. Univ. Math. Phys., 15 (5) (2022), 663-671.

[12] M. G. Shrigan, Coefficient estimates for certain class of bi-univalent functions associated with k-fibonacci numbers, Gulf J. Math., 14(1), (2023) ,192-202.

[13] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23 (10) (2010), 1188-1192.

[14] H. M. Srivastava, S. S. Eker and R. M. Ali, Coefficient bounds for a certain class of analytic and bi-univalent functions, Filomat, 29(8) (2015), 1839-1845.

[15] T. M. Seoudy and M. K. Auof, Coefficient estimation of new classes of q-starlike and q-convex functions of complex order, J. Math. Inequal., 10(1) (2016), 135-14.