A Mathematical Analysis of the Stability of the Logistic Growth Model for Asian Elephant Population Ecology in India.

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Abstract:

The logistic growth model which describes the population of Asian elephants in India, is influenced by factors such as the birth rate, death rate, immigration, migration, and other external factors that play a vital role in the population model's stability. We intend to study the logistic population model's ability to return to equilibrium after experiencing disturbance or change. The model's stability enhances the prediction of the change in the population of Asian elephants in India and helps develop strategies to ensure ecosystem sustainability. We conducted a sensitivity study by varying the input parameter to see how perturbed the system is as well as to test stability and compare the model's predictions with real-world data.

Keywords: Logistic growth model, Population dynamics, Mathematical Ecology, Evolution and Social Sciences, Asian elephant population, Stability of population model.

Introduction:

A clear understanding of a population model's stability is crucial for making accurate predictions and informed decisions. A thorough examination of this attribute will ensure reliable insights and effective management strategies. The capacity of a population to sustain a relatively constant size over time, minimizing the likelihood of significant fluctuations or declines, can be defined as the stability of a population.

Several factors impact the stability of a population model. One key element is the availability of essentials for the population to survive. It is often believed that a population can grow indefinitely if the food supply is abundant. However, in real-world scenarios, this growth is usually not sustainable because the population can only thrive if sufficient resources, such as food, water, and shelter, are available to support its members, and these resources are often quite limited. If resources become scarce or limited, the population may experience a decline in numbers or even a collapse leading to possible extinction. The presence of predators and competitors constitutes a significant factor influencing population stability. Predators can diminish the size of a population by preying on its members. At the same time, competitors can also restrict resource availability by competing for the same food sources. In both scenarios, populations may experience challenges in maintaining their size and stability.

Environmental factors like climate change and natural disasters can also significantly impact population stability. Extreme weather events like droughts, floods, and hurricanes can disrupt ecosystems and lead to population declines. Climate change, in particular, can alter species distribution and disrupt the delicate balance of ecosystems, making it difficult for populations to adapt and survive (Kristie L Ebi 1 2022).

Human activities, such as habitat destruction, pollution, and overexploitation of resources, can also threaten the stability of populations (Crooks May 2021). Humans can push populations to the brink of collapse by altering natural habitats and depleting resources. Conservation efforts and sustainable management practices are essential to ensure the long-term stability of populations and prevent species from becoming endangered or extinct.

To assess the stability of a population model, scientists use mathematical models and simulations to predict how populations will respond to different environmental conditions and disturbances. These models consider factors such as birth rates, death rates, immigration, emigration, and carrying capacity to estimate population growth and stability over time.

Overall, the stability of a population model is a complex and multifaceted issue that requires careful consideration of various factors and variables. By understanding the factors that influence population stability and using advanced modeling techniques, scientists can make more accurate predictions and informed decisions to ensure the long-term survival of populations in the face of environmental challenges.

1. The Asian elephant population in India

India is home to the largest population of Asian elephants, estimated to be above 26000 hence accounting for nearly 60 percent of the species' total population (N. Baskaran 2011). Although elephants are native only to Africa and Asia, they hold significant cultural and symbolic importance worldwide. Unfortunately, human expansion into their habitats, coupled with agricultural development and infrastructure projects such as roads, canals, and fences, has severely disrupted their lives. Additionally, elephants are losing access to their historic migration routes.

Currently, Asian elephants face many threats, particularly from wildlife crime, including poaching for the illegal ivory trade. They are also highly vulnerable to habitat loss due to human encroachment. By 2015, Asian elephants had lost over 60 percent of their habitat in India, and today, their known range is even smaller (Shermin de Silva 2020). The exact number of elephants remaining and their specific locations are unclear.

Collecting this data is crucial for focusing conservation efforts on regions where elephants are in danger. Given the limited resources available for assessing the population of Asian elephants in India, we can, to some extent, develop a mathematical model to predict their population and estimate whether their numbers are increasing or decreasing.

The official data employed in this study was collected over nearly 40 years, with censuses conducted at four- to five-year intervals, focusing exclusively on the population of Asian elephants within the Indian subcontinent. According to the International Zoo Yearbook, the estimated population of Asian elephants in 2019 was between 48,323 and 51,680. Table 1 shows the population of Asian elephants in India between 1980 and 2021 (Elephant population (AfESG &AsESG n.d.)

Table 1: Population of Asian elephants in India between 1980-2021

Year	Population
1980	15627
1985	18975
1989	20862
1993	15604
1997	25877
2002	26413
2007	27694
2012	29391
2017	27312
2021	29964

2. Logistic growth model for Asian elephants in India.

The population at time t is represented by P(t), and the change in population size is denoted by dP/dt or P′(t). For this analysis, we will assume that the population size primarily determines the population growth rate. This assumption simplifies the modeling process; however, it is essential to acknowledge that it may also introduce certain inaccuracies in our estimations.

This focus on population growth is particularly relevant to the endangered Asian elephant, where understanding population dynamics is crucial for effective conservation strategies. Therefore, our population must accurately reflect the actual size of the population in the wild.

Consider the Verhulst-Pearl logistic growth model, where P is the population of Asian elephants in India.

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

Using the available data for the population of Asian elephants, we have derived the basic essential population growth of the Asian elephants in the Indian subcontinent, as shown below:

$$\frac{dP}{dt} = 0.711P\left(1 - \frac{P}{23700}\right)$$
(1)

The above differential equation demonstrates asymptotic stability, signifying that the population has a carrying capacity of 23700. This suggests that, under optimal conditions, the population can sustain itself without significant decline or growth beyond this threshold. Considering the initial assumptions underlying the population model, it is vital to consider various external factors that may influence its accuracy. Factors such as climate change, which can alter habitat conditions, habitat loss due to human development, human-wildlife conflict stemming from overlapping habitats, and poaching practices have likely introduced substantial deviations in the model's predictions.

Furthermore, the data utilized in our study spans from 1980 to 2021, providing a comprehensive overview of the population dynamics over four decades. Notably, Ramesh K's team's evaluation of population stability occurred more recently in 2024, which suggests that their findings are based on a relatively short timeframe of stability assessment (Ramesh K. Pandey 2024). This emphasizes the dynamism of the population and the ongoing changes that could be affecting its trajectory.

Given these insights, we can reasonably suggest that the Asian elephant population is exhibiting promising signs of moving toward stability, as indicated by the estimates generated by our model. The combination of historical data and recent evaluations provides a nuanced understanding of the factors at play, allowing us to assess the potential for long-term stability in the face of ongoing environmental challenges.
A particular solution to the Logistic Model described in equation (1) is given below.

$$P(t) = \frac{23700}{1 + 0.5165e^{-0.711t}}$$
(2)

3. Stability of the Logistic model by comparison and change in the initial population.

Using the solution (2) derived from the logistic model's differential equation, we could estimate the population of Asian elephants in India. Table 2 compares the actual population with the estimated one to find the error difference in the population estimation.

year	number of years since 1980	Actual	Estimated	Difference
1980	0	15627	15629.07	2.0729641

1985	5	18975	23356.69	4381.6911
1989	9	20862	23681.15	2819.1458
1993	13	15604	23700.3	8096.3044
1997	17	25877	23701.42	2175.5798
2002	22	26413	23701.49	2711.5128
2007	27	27694	23701.49	3992.5109
2012	32	29391	23701.49	5689.5109
2017	37	27312	23701.49	3610.5109
2021	41	29964	23701.49	6262.5109

Table 2: Comparison of Asian elephants' population between actual and estimated value

The differential equation described above illustrates the concept of asymptotic stability, which signifies that the population has a specific carrying capacity of 23700. Given the available resources and optimal conditions, this capacity represents the maximum population size that the environment can sustain. The data from Table 2 indicates that under ideal circumstances, the population can stabilize at this level without significant fluctuations, avoiding substantial decline and unchecked growth beyond this threshold.

However, it is crucial to consider a range of external factors that can influence the accuracy of this population model. For instance, climate change can significantly alter habitat conditions, potentially reducing critical resources such as food and water availability. Additionally, habitat loss due to human development, urbanization, agriculture, and infrastructure expansion can further diminish the natural environments necessary for population stability.

Moreover, human-wildlife conflict, which often arises when human activities encroach upon natural habitats, can increase competition for resources and contribute to population declines. Poaching practices also pose a significant threat, as illegal hunting can drastically reduce population numbers and disrupt social structures within species (Annika Mozer 2023). These external factors likely introduce substantial deviations from the model's predictions, highlighting the importance of continuous monitoring and adaptation in population management strategies. Understanding these influences will be essential for ensuring the long-term sustainability and health of the population of Asian elephants in India.

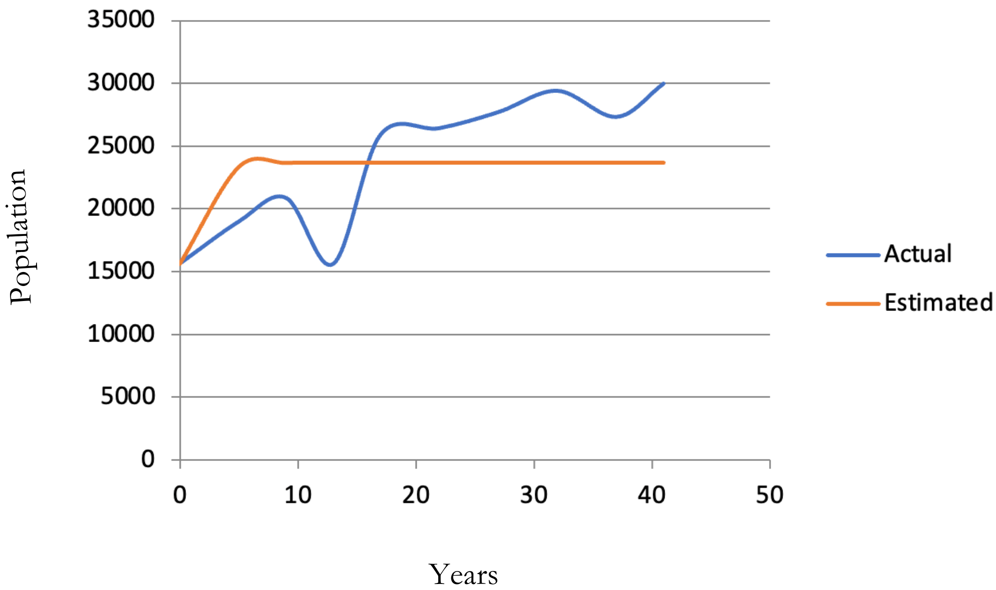

Figure 1: Graph comparing the actual and the estimated population from Table 2

In Figure 1, we can see a clear trend demonstrating that both the actual and estimated populations of Asian elephants in India stabilize at roughly 25,000 individuals. This stabilization indicates that this number likely represents the population's carrying capacity, which is the maximum size that the environment can sustain over time, irrespective of the initial conditions or population densities.

The data suggest that initial fluctuations, whether due to environmental disturbances, ecological factors, or human activities, do not have a lasting effect on the long-term stability of the elephant population. Instead, these majestic creatures tend to rebound to this approximate population size even when starting figures vary significantly. This phenomenon speaks to the remarkable adaptability of Asian elephants within their natural habitats, allowing them to thrive despite changes in conditions.

Furthermore, the insights gained from this data highlight the critical need for effective conservation strategies. By understanding the factors that contribute to this population stabilization, we can implement targeted measures to support and enhance the resilience of Asian elephants. These strategies may include habitat protection, anti-poaching efforts, and promoting human-elephant coexistence, all of which are essential for preserving the long-term health and stability of this iconic species in India.

4. Stability is shown by enveloping with a linear fractional function.

The difference equation below represents the differential equation for the population model (1).

$P_{n+1} = r P_n (1 - P_n)$

(3)

To prove the stability of such a setup, we need to define the equilibrium point of the population model, which is defined as follows.

Definitions:

A population model defined by the difference equation

$$x_{t+1} = f(x_t)$$

where f is a continuous function, and there is a positive number $\bar{x}$, the equilibrium point, such that the population model is globally stable if and only if for all x_0 such that $f(x_0) > 0$ we have

$$\lim_{t\to\infty} x_t = \bar{x}$$

where $\bar{x}$ is the unique equilibrium point of the population model.

A population model is locally stable if and only if for every small enough neighborhood, if x_0 is in this neighborhood for all t and

$$\lim_{t\to\infty} x_t = \bar{x}$$

Let's delve into the population growth model to gain a comprehensive understanding of how a mathematical function can effectively encompass another function within a defined domain by considering the population growth model.

$$x_{t+1} = x_t e^{2(1-x_t)}$$

(4)

Analyzing the graph presented in Figure 2, we can clearly identify the local stability line, described by the equation y = 2 - x. This line acts as a boundary that indicates the conditions under which the population remains stable over time. The trends illustrated by the graph suggest that the population model, represented by equation (4), exhibits local stability, meaning that small changes in population size will not lead to significant fluctuations. This observation reinforces the model's reliability in predicting population behavior within the given parameters.

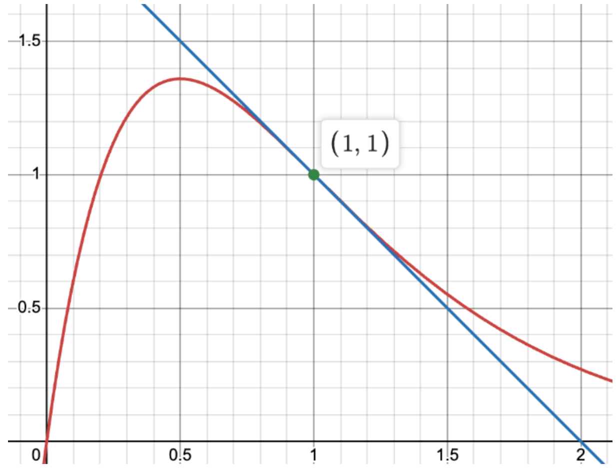

Figure 2: Graph $\emptyset(x) = 2 - x$ envlopes f(x)

This leads to the definition of an enveloping function for f(x). A function $\emptyset(x)$ envelopes a function f(x) if and only if

- $\emptyset(x) > f(x)$for $x \in (0,1)$
- $\emptyset(x) < f(x)$for x >1 such that $\emptyset(x) > 0$ and $(x) > 0$

A general enveloping function can be constructed by taking the ratio of two linear functions with a ratio of 1 and having its derivative as -1 when x =1.

Consider a linear fractional function $\emptyset(x) = \frac{1-\alpha x}{\alpha-(2\alpha-1)x}$ where $\alpha \in [0,1)$ and having the following properties. Figure 3: shows the various forms of the linear fractional function

- $\emptyset(1) = 1$
- $\acute{\emptyset}(1) = -1$
- $\emptyset\big(\emptyset(x)\big) = x$
- $\acute{\emptyset}(x) < 0$

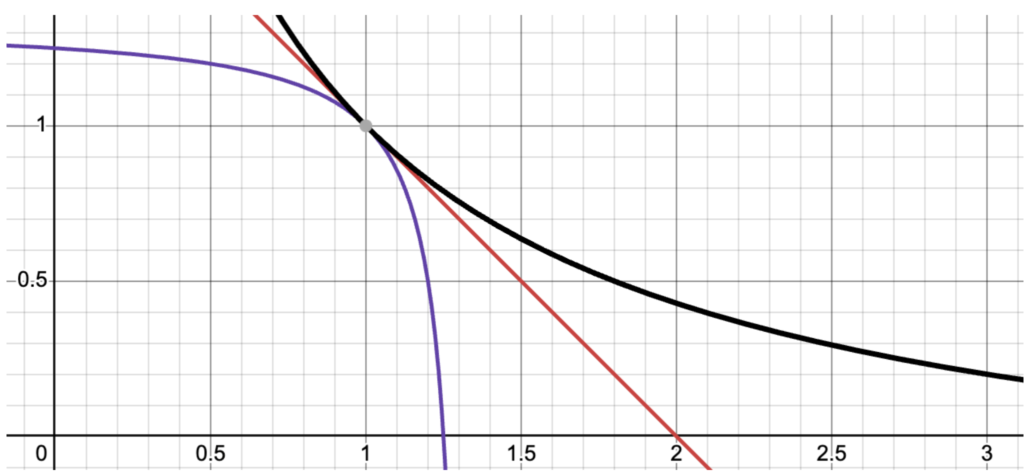

Figure 3 Linear fractional function for $\alpha = 0.5$ $\alpha = 0.2$ $\alpha = 0.8$

To prove the iterative Pn+1- Pn = Pn r (1- Pn) model's stability, we list down the following Theorems and definitions.

Theorem: Let $x_{t+1} = f(x_t)$ where $f(x) = xh(1-x)$ and h(z) is doubly positive, then f(x) is enveloped by the linear fractional function

$$\emptyset(x) = \frac{1-\alpha x}{\alpha - (2\alpha - 1)x}$$

where $\alpha = \frac{3-h_2}{4-h_2} \geq \frac{1}{2}$ and the model $x_{t+1} = f(x_t)$ is globally stable.

Also, we need to take note of the definition of doubly positive functions:

A function h(z) is doubly positive if and only if

1. h(z) has a power series h(z) = $\sum_{k=1}^{\infty} h_k z^k$
2. $h_0 = 1$ and $h_1 = 2$
3. For all $n \geq 1$, $h_n \geq h_{n+1}$
4. For all $n \geq 2$, $h_n - 2h_{n+1} + h_{n+2} \geq 0$

The initial iterative equation, expressed as Pn+1- Pn = Pn r (1- Pn) , can be interpreted as a simplified version of equation (4). In the context of equation (4), a critical condition for achieving local stability is established when the parameter r falls within the range of 0 to 2. This range is significant because it dictates the behavior of the system; specifically, values of r that are too high or too low could lead to instability (Cull and Chaffee 2000).

Similarly, our model retains this condition for local stability, encompassing the iterative equation for all r values where 0 < r < 2. Notably, it also considers the boundary case where r = 2, which provides insight into the transitional behaviors and potential bifurcations in the system. Understanding these parameters is essential for predicting the dynamics of the modeled phenomena and ensuring that the system behaves stably and predictably.

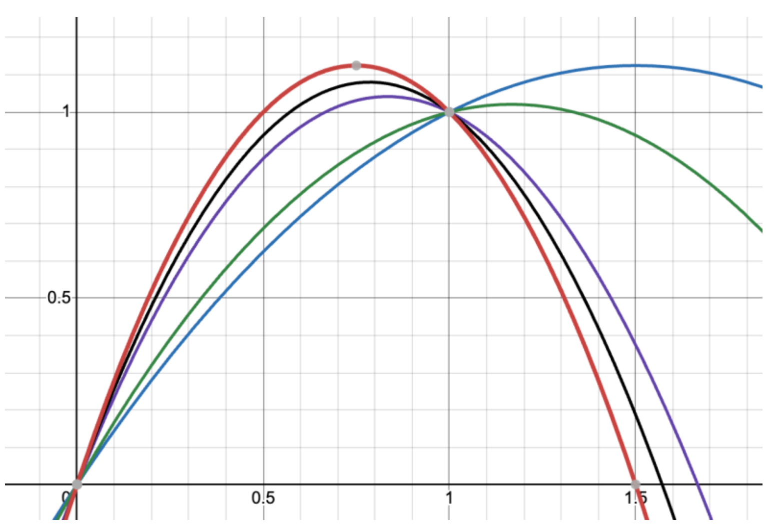

Figure 4 shows that with r =2 enveloping the model, with all r values ranging between 0<r<2

Unlike the growth model in (4), this model is not enveloped by a straight line, but instead we will be using the idea of a doubly positive function h(z), which specifically in this case, is taken as h(z) = 1+ 2z; therefore, we have

$$h_n - h_{n+1} = \begin{cases} 2, & n = 1 \\ 0, & n > 1 \end{cases} \geq 0$$

and

$$h_n - 2h_{n+1} + h_{n+2} = \begin{cases} 2, & n = 1 \\ 0, & n > 1 \end{cases} \geq 0$$

Since $h_2 = 0$ the enveloping function has $\alpha = \frac{3}{4}$ and the function is

$$\emptyset(x) = \frac{4 - 3x}{3 - 2x}$$

To demonstrate the stability of the model, we first assess the criteria for a doubly positive function. This involves verifying that the parameters involved adhere to the necessary mathematical specifications. Additionally, we need to ensure that a linear fractional function adequately envelops the iterative equation defined as

Pn+1 - Pn = Pn r (1 - Pn).

And it can be established, based on the theorem below, that we can conclude Pn+1 =Pn + Pn r (1 - Pn) is globally stable.

Theorem: If f(x) is enveloped by a linear fractional function, then f(x) is globally stable.

By establishing these conditions, we can confirm that the model behaves predictably and remains stable throughout its iterations. (Wherry 2007) This dual approach not only strengthens the reliability of our model but also provides insights into its underlying dynamics.

Conclusion

Initially, we conducted an in-depth analysis of the stability of the logistic growth model which represents the population dynamics of Asian elephants in India. By disturbing the model's parameters, we found that the elephant population tends to stabilize over time, independent of the initial population size or condition. This observation suggests that the logistic model is robust in predicting long-term population trends.

Furthermore, we introduced the concept of an all-encompassing function to outline specific conditions under which the logistic model can achieve a stable equilibrium. This approach allows us to establish parameters that ensure the model's predictions remain valid, thus providing insights into the management and conservation strategies for Asian elephants. By understanding these dynamics, we can better monitor and support these vulnerable species within their natural habitat.

Analysts are keen to explore the system's behavior and make future predictions. While there are established analytical techniques for linear systems, the tools available for nonlinear systems are limited. In this paper, we present an approach used by P Cull and J Chaffee that demonstrates how limiting the dimensionality of a system can lead to effective analytical methods for assessing the stability of population models (Cull and Chaffee 2000).

We specifically focus on one-dimensional difference equations, where the right-hand side can be represented by a linear fractional function. Our findings indicate that such configurations are globally stable, meaning they will tend toward a stable equilibrium regardless of initial conditions. Moreover, we illustrate that for certain biological models, the ability to create this enveloping linear fractional function aligns perfectly with instances where the model exhibits local stability.

To illustrate this concept, we examine the logistic population model, a well-known example in population dynamics. This model shows a smooth and predictable behavior along a specific curve, indicating that it is conducive to analysis. Consequently, we are also able to construct a linear fractional curve that effectively envelops the behavior of the logistic model. This relationship not only highlights the stability of the population growth predictions but also enhances our understanding of the dynamics involved in biological systems. By integrating these insights, we pave the way for more robust analytical frameworks for nonlinear systems in future research.

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