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On fitting the bi-Weibull competing risks distribution with application

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Abstract

Survival analysis examines how long it takes an interesting event to occur. The lifespan of industrial systems, which are susceptible to various causes of failure, is described by competing risk models. Furthermore, when estimating the cause-specific mortality rate in the presence of other causes, mortality rate research took into account a variety of event types. One of the competing risks models that accurately describes this phenomenon is the bi-weibull distribution, which is the minimum between two Weibull distributions. In this work, after estimating the unknown parameters by several methods, we propose to construct a modified chi-square statistic to fit this model. This statistic can be used not only to recover information lost during data regrouping, but also to differentiate this distribution from its alternatives. A thorough simulation analysis and an application using actual data are provided to demonstrate the viability of the suggested test.

Keywords: bi-Weibull distribution, competing risks models, estimation methods, modified chi-square test.

1. Introduction

Patients involved in medical research typically have a variety of health issues, and there can be a wide range of reasons why they pass away. System components show multiple failure causes in reliability studies, depending on the various operating environments. To describe these lifetimes, practitioners have to use the appropriated models, which are known as competing risks models. These models are frequently based on the Weibull distribution found in the statistical literature. We can cite Ishioka and Nonaka (1991) who studied the maximum likelihood estimation of Weibull parameters for two independent competing risk, Berger and Sun (1993) concerned the Bayesian analysis for the poly-Weibull distribution. Later Bousquet et al., (2006) introduced an alternative competing risk model to the Weibull distribution called Bertholon model, Chouia and Seddik-Ameur (2014) proposed a goodness-of-fit test for a Competing Risk Model of Bertholon. Also, Agnes and Howard (2016) studied the estimation of a competing risks Weibull model, Freels et al., (2019) investigated the maximum likelihood estimation for the poly-Weibull distribution. Some authors have proposed the mixture of two Weibull distributions for modeling data related to these lifetimes. Our work, concerned the competing risks model called the bi-Weibull distribution which is the minimum between two Weibull distributions. Firstly, we use different estimation methods (maximum likelihood, Kolmogorov-Smirnov, Anderson-Darling and Cramer von-Mises) to evaluate the unknown parameters, then, we construct an efficient modified chi-square statistic capable to fit this model in a satisfactory manner. Based on maximum likelihood estimators on initial data, this statistic recovers the information lost while regrouping data and can be used to distinguish between this model and its alternatives. The practicability of the proposed test is shown by an intensive simulation study and an example of medical dataset.

2.Competing risks bi-Weibull distribution

Suppose that a random variable $T_j$follows a Weibull distribution with shape and scale parameters α_j and β_j the cumulative density function (cfd) is

$$\mathrm{F}(\mathrm{t};\alpha_{\mathrm{j}},\beta_{\mathrm{j}}) = 1 - \exp\left\{ -\left(\frac{t}{\beta_j}\right)^{\alpha_j}\right\}$$

Berger and Sun (1993) introduced a competing risks model named the poly-Weibull model which represents the distribution of the minimum between J=(1,2,...J)Weibull distributions. The expression of the cumulative $F(t;\alpha_j,\beta_j)$and probability density $f(t;\alpha_j,\beta_j)$ functions are expressed as:

$$F(t;\alpha_j,\beta_j) = 1 - \exp\left\{-\left[\sum_{j=1}^{J}\left(\frac{t}{\beta_j}\right)^{\alpha_j}\right]\right\}$$

$$f(t;\alpha_j,\beta_j) = \left[\sum_{j=1}^{J}\frac{\alpha_j}{\beta_j}\left(\frac{t}{\beta_j}\right)^{\alpha_j-1}\right]\exp\left\{-\left[\sum_{j=1}^{J}\left(\frac{t}{\beta_j}\right)^{\alpha_j}\right]\right\}$$

Let $T^{(0)}$and $T^{(1)}$be two independant random variables Weibull distributed with cdf $F_i(t), i = 0,1$.The minimum $T = \min\{T^{(0)},T^{(1)}\}$is said to have a bi-Weibull distribution and noted $BW(\alpha_0,\alpha_1,\beta_0,\beta_1)$where $\alpha_0,\alpha_1$are the shape parameters and β_0,β_1 the scale ones. Th corresponding cdf and pdf are given by:

$$F(t;\alpha_0,\alpha_1,\beta_0,\beta_1) = 1 - \exp\left\{-\left[\left(\frac{t}{\beta_0}\right)^{\alpha_0} + \left(\frac{t}{\beta_1}\right)^{\alpha_1}\right]\right\}$$

$$f(t) = \left[\frac{\alpha_1}{\beta_1}\left(\frac{t}{\beta_1}\right)^{\alpha_1-1} + \frac{\alpha_0}{\beta_0}\left(\frac{t}{\beta_0}\right)^{\alpha_0-1}\right]\exp\left\{-\left[\left(\frac{t}{\beta_0}\right)^{\alpha_0} + \left(\frac{t}{\beta_1}\right)^{\alpha_1}\right]\right\}$$

and Its hazard function is:

$$\lambda(t) = \frac{\alpha_1}{\beta_1}\left(\frac{t}{\beta_1}\right)^{\alpha_1-1} + \frac{\alpha_0}{\beta_0}\left(\frac{t}{\beta_0}\right)^{\alpha_0-1}$$

3.Parameter estimation methods

Let us consider a random sample of observations $(t_1, t_2, \dots, t_n)$from $BW(\alpha_0, \alpha_1, \beta_0, \beta_1)$with unknown parameters. In this section, we propose different estimation methods to evaluate their values.

3.1. Maximum likelihood estimation method

Because of their nice properties namely the consistency and the asymptomatic normality, the maximum likelihood estimators are generally required. The likelihood function is:

$$L(t,i) = \prod_{j=1}^{n} f(t_j, i_j)$$

$$= \prod_{j=1}^{n} \left[\frac{\alpha_0}{\beta_0}\left(\frac{t_j}{\beta_0}\right)^{\alpha_0-1}\right]^{1-i_j} \left[\frac{\alpha_1}{\beta_1}\left(\frac{t_j}{\beta_1}\right)^{\alpha_1-1}\right]^{i_j} \left[\exp\left\{-\left(\frac{t_j}{\beta_0}\right)^{\alpha_0}\right\}\right]\left[\exp\left\{-\left(\frac{t_j}{\beta_1}\right)^{\alpha_1}\right\}\right]$$

$$= \prod_{j=1}^{n} \left[\frac{\alpha_0}{\beta_0}\left(\frac{t_j}{\beta_0}\right)^{\alpha_0-1}\right]^{1-i_j} \left[\frac{\alpha_1}{\beta_1}\left(\frac{t_j}{\beta_1}\right)^{\alpha_1-1}\right]^{i_j} \exp\left\{-\left[\left(\frac{t_j}{\beta_0}\right)^{\alpha_0} + \left(\frac{t_j}{\beta_1}\right)^{\alpha_1}\right]\right\}$$

with i=0,1 and j=1,2,3,...,n

The ML estimators $\alpha_0, \alpha_1, \beta_0, \beta_1$are obtained by canceling the following score functions:

$$\frac{\partial \ln L\,(t,i)}{\partial \beta_0} = -\frac{n(1+\alpha_0)}{\beta_0} + \frac{\alpha_0}{\beta_0}\sum_{j=1}^{n} i_j + \frac{\alpha_0}{\beta_0} \sum_{j=1}^{n} \left(\frac{t_j}{\beta_0}\right)^{\alpha_0}$$

$$\frac{\partial \ln L(t,i)}{\partial \beta_1} = -\frac{\alpha_1}{\beta_1}\sum_{j=1}^{n} i_j + \frac{\alpha_1}{\beta_1}\sum_{j=0}^{n} \left(\frac{t_j}{\beta_1}\right)^{\alpha_1}$$

$$\frac{\partial \ln L\,(t,i)}{\partial \alpha_0} = \sum_{j=0}^{n}(1-i_j)\left[\frac{1}{\alpha_0} + \ln\left(\frac{t_j}{\beta_0}\right)\right] - \sum_{j=0}^{n}\left(\frac{t_j}{\beta_0}\right)^{\alpha_0} \ln\left(\frac{t_j}{\beta_0}\right)$$

$$\frac{\partial \ln L(t,i)}{\partial \alpha_1} = \sum_{j=0}^{n} i_j \left[\frac{1}{\alpha_1} + \ln\left(\frac{t_j}{\beta_1}\right)\right] - \sum_{j=0}^{n}\left(\frac{t_j}{\beta_1}\right)^{\alpha_1} \ln\left(\frac{t_j}{\beta_1}\right)$$

3.2. Kolmogorov-Smirnov estimation method

Maximizing the Kolmogorov-Smirnov statistic $D_n$with respect to the unknown parameters is another traditional method for calculating the unknown parameters.

$$D_n = max[D^+, D^-]$$

$$D^+ = \max_{1\le i\le n}\left[\left(\frac{i}{n}\right) - F\left(t_{(i)}\right)\right] \quad ; \quad D^- = \max_{1\le i\le n}\left[F\left(t_{(i)}\right) - \left(\frac{i-1}{n}\right)\right]$$

$$D^+ = Max_{\{1\le i\le n\}}\left[\left(\frac{i}{n}\right) - 1 + \exp\left(-\left(\left(\frac{t_i}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} + \left(\frac{t_i}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)\right)\right]$$

$$D^- = Max_{\{1\le i\le n\}}\left[1 - \exp\left(-\left(\left(\frac{t_i}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} + \left(\frac{t_i}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)\right) - \left(\frac{i-1}{n}\right)\right]$$

3.3. Anderson-Darling estimation

This method is based on the well-known Anderson-Darling statistic A_n^2. These estimators are derived from the minimum of A_n^2 with respect to the unknown parameters.

$$A_n^2 = -n - \frac{1}{n}\left\{\sum_{\mathrm{i}=0}^{\mathrm{n}}(2i-1)\left[\ln F\left(x_{(i)}\right) + \ln\left(1 - F\left(x_{(n+1-i)}\right)\right)\right]\right\}$$

$$A_n^2 = -n - \frac{1}{n}\left\{\sum_{\mathrm{i}=0}^{\mathrm{n}}(2i-1)\left[\ln\left(1 - \exp\left(-\left(\left(\frac{t_i}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} + \left(\frac{t_i}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)\right)\right]\right.\right.$$

$$\left.\left. - \left(\left(\frac{t_{n+1-i}}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} + \left(\frac{t_{n+1-i}}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)\right]\right\}$$

3.4. Cramer-Von-Mises estimation method

The Cramer-Von-Mises CVM method estimation consists in minimizing the following function with respect to the unknown parameters. This method is shown to give the smaller bias estimators than the other minimum distance estimators.

$$W_n^2 = \sum_{i=1}^{n}\left[F\left(x_{(i)}\right) - \frac{2i-1}{2n}\right]^2 + \frac{1}{12n}$$

$$W_n^2 = \sum_{i=0}^{n}\left[1-\exp\left(-\left(\left(\frac{t_i}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}}+\left(\frac{t_i}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)\right)-\frac{2i-1}{2n}\right]^2+\frac{1}{12n}$$

As all the analytical forms of the estimators cannot be given, so we use numerical methods to calculate the corresponding values.

4.Modified statistic test for bi-Weibull distribution

In this section, we propose to construct a new chi-square statistic test $Y^2$for fitting the bi-Weibull distribution using the approach of Nikulin, Rao and Robson (1973, 1974). Based on initial data, this statistic recovers the information lost in regrouping data. This statistic permit to practitioners to check the validity of this distribution without regarding the possible competitors.

4.1. Criteria test

For testing, the null hypothesis H_0that the sample $T = (t_1, \dots, t_2)^T$ of n independent and identically distribution random variables comes from a population with $F(\theta)$distribution with unknown parameters vector $\theta = (\theta_1, \theta_2, .., \theta s)^T$the authors proposed a modified Pearson statistic Y^2 defined by a sum of the Pearson statistic $X_n^2$and a quadratic form:

$$Y_n^2(\hat{\theta}) = X_n^2(\hat{\theta}) + \frac{1}{n}L^T(\hat{\theta})\left(I(\hat{\theta}) - J(\hat{\theta})\right)^{-1}L(\hat{\theta})$$

We grouped data into r classes $\Delta_j (j = 1, \dots, r)$such as $\Delta_j \cap \Delta_i = \phi for$ any $j \neq i$. The vector $\upsilon = (\upsilon_1, \dots, \upsilon_r)^T$reprents the frequencies of these classes, s the number of estimated parameters, $I(\theta)$and $J(\theta)$re the estimated information matrices on non-grouped and grouped data, and:

$$X_n(\theta)\left(\frac{\upsilon_1 - np_1(\theta)}{\sqrt{np_1(\theta)}}, \dots, \frac{\upsilon_r - np_r(\theta)}{\sqrt{np_r(\theta)}}\right)^T \quad ;$$

$$L(\theta) = (L_1, \dots, L_i)^T \qquad ; L_i(\theta) = \textstyle\sum_{i=0}^{n} \frac{\upsilon_j}{P_j} . \frac{\partial P_j(\theta)}{\partial \theta_i}; \qquad \text{i=1,......,s}$$

and

$$P(\theta) = \left(P_1(\theta), P_2(\theta), \dots, P_r(\theta)\right)^T$$

Where $$P_j(\theta) = \int_{\Delta_j} f(t;\theta)dt \quad j = 1, \dots, r$$

$$J(\theta) = B(\theta)^T . B(\theta)$$

With $b_{ji}(\theta) = \frac{1}{\sqrt{P_j}} . \frac{\partial P_j(\theta)}{\partial \theta_i}; \quad j = 1, \dots, r; \;\; i = 1, \dots, s$

Under $H_0$the statistic Y^2follows the chi-square distribution with r-1degrees of freedom.

4.2. Construction of statistic test for the bi-Weibull distribution

To verify if the sample $T = (t_1, \dots, t_n)^T$belongs to the bi-Weibull distribution $F_{BW}(t;\theta)$:

$$P\left(T_j \le t \mid H_0\right) = F_{BW}(t;\theta),\; t \ge 0, \quad \text{with} \quad \theta = (\alpha_0, \alpha_1, \beta_0, \beta_1)^T$$

Firstly, we calculate the ML estimators of the unknwon parameters. For this statistic, the r grouped classes $\Delta_j$must be equiprobable which means

$P_1 = P_2 = \dots = P_r = \frac{1}{r}$ *.In* this case the limits $a_j$of the classes

$\Delta_j = [a_{j-1}, a_j[$ are obtained by:

$$a_j = F_{BW}^{-1}\left(\frac{j}{r}\right) = \left(-\left(\frac{t}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} - \left(\frac{t}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right) - \ln\left(1 - \frac{j}{r}\right) \qquad j = 1, \dots, r-1$$

And

$$P_j(\hat{\theta}) = -\exp\left(-\left(\frac{a_j}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} - \left(\frac{a_j}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right) + \exp\left(-\left(\frac{a_{j-1}}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} - \left(\frac{a_{j-1}}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)$$

To provide the criteria statistic $Y^2$for fitting the$F_{BW}(t;\theta)$distribution, we need both of the estimated information estimation matrices $J(\hat{\theta})$and $I(\hat{\theta})$for grouped and non-grouped data.

4.2.1. Estimated information matrix $J(\hat{\theta})$

the components of the estimated symmetric matrix $J(\hat{\theta})$for the grouped data

$$J_{11}(\hat{\theta}) = \sum_j \frac{1}{P_j}\left(\frac{\partial P_j}{\partial \widehat{\beta_0}}\right)^2 \qquad ; \qquad J_{22}(\hat{\theta}) = \sum_j \frac{1}{P_j}\left(\frac{\partial P_j}{\partial \widehat{\beta_1}}\right)^2$$

$$J_{33}(\hat{\theta}) = \sum_j \frac{1}{P_j}\left(\frac{\partial P_j}{\partial \widehat{\alpha_0}}\right)^2 \qquad ; \qquad J_{44}(\hat{\theta}) = \sum_j \frac{1}{P_j}\left(\frac{\partial P_j}{\partial \widehat{\alpha_1}}\right)^2$$

$$J_{12} = J_{21} = \sum_j \frac{1}{P_j}\left(\frac{\partial P_j}{\partial \widehat{\beta_0}}\right)\left(\frac{\partial P_j}{\partial \widehat{\beta_1}}\right) \qquad ; \qquad J_{13} = J_{31} = \sum_j \frac{1}{P_j}\left(\frac{\partial P_j}{\partial \widehat{\beta_0}}\right)\left(\frac{\partial P_j}{\partial \widehat{\alpha_0}}\right)$$

$$J_{14} = J_{41} = \sum_j \frac{1}{P_j}\left(\frac{\partial P_j}{\partial \widehat{\beta_0}}\right)\left(\frac{\partial P_j}{\partial \widehat{\alpha_1}}\right) \qquad ; \quad J_{23} = J_{32} = \sum_j \frac{1}{P_j}\left(\frac{\partial P_j}{\partial \widehat{\beta_1}}\right)\left(\frac{\partial P_j}{\partial \widehat{\alpha_0}}\right)$$

$$J_{24} = J_{42} = \sum_j \frac{1}{P_j}\left(\frac{\partial P_j}{\partial \widehat{\beta_1}}\right)\left(\frac{\partial P_j}{\partial \widehat{\alpha_1}}\right) \qquad ; \quad J_{34} = J_{43} = \sum_j \frac{1}{P_j}\left(\frac{\partial P_j}{\partial \widehat{\alpha_0}}\right)\left(\frac{\partial P_j}{\partial \widehat{\alpha_1}}\right)$$

can be derived from the following partial derivatives

$$\frac{\partial P_j(\hat{\theta})}{\partial \widehat{\beta_0}} = -\left(\frac{\widehat{\alpha_0}}{\widehat{\beta_0}}\right)\left(\frac{a_j}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} \exp\left(-\left(\frac{a_j}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} - \left(\frac{a_j}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)$$

$$+\left(\frac{\widehat{\alpha_0}}{\widehat{\beta_0}}\right)\left(\frac{a_{j-1}}{\widehat{\beta_0}}\right)^{\alpha_0} \exp\left(-\left(\frac{a_{j-1}}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} - \left(\frac{a_{j-1}}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)$$

$$\frac{\partial P_j(\hat{\theta})}{\partial \widehat{\beta_1}} = -\left(\frac{\widehat{\alpha_1}}{\widehat{\beta_1}}\right)\left(\frac{a_j}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}} \exp\left(-\left(\frac{a_j}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} - \left(\frac{a_j}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)$$

$$+\left(\frac{\widehat{\alpha_1}}{\widehat{\beta_1}}\right)\left(\frac{a_{j-1}}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}} \exp\left(-\left(\frac{a_{j-1}}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} - \left(\frac{a_{j-1}}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)$$

$$\frac{\partial P_j(\hat{\theta})}{\partial \widehat{\alpha_0}} = \left(\frac{a_j}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} \ln\left(\frac{a_j}{\widehat{\beta_0}}\right) \exp\left(-\left(\frac{a_j}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} - \left(\frac{a_j}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)$$

$$-\left(\frac{a_{j-1}}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} \ln\left(\frac{a_{j-1}}{\widehat{\beta_0}}\right) \exp\left(-\left(\frac{a_{j-1}}{\widehat{\beta_0}}\right)^{\widehat{\alpha_0}} - \left(\frac{a_{j-1}}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)$$

$$\frac{\partial P_j(\hat{\theta})}{\partial \widehat{\alpha_1}} = \left(\frac{a_j}{\beta_1}\right)^{\widehat{\alpha_1}} \ln\left(\frac{a_j}{\beta_1}\right) \exp\left(-\left(\frac{a_j}{\widehat{\beta_0}}\right)^{\alpha_0} - \left(\frac{a_j}{\beta_1}\right)^{\widehat{\alpha_1}}\right)$$

$$-\left(\frac{a_{j-1}}{\widehat{\beta_1}}\right)^{\alpha_1} \ln\left(\frac{a_{j-1}}{\widehat{\beta_1}}\right) \exp\left(-\left(\frac{a_{j-1}}{\widehat{\beta_0}}\right)^{\alpha_0} - \left(\frac{a_{j-1}}{\widehat{\beta_1}}\right)^{\widehat{\alpha_1}}\right)$$

and the vector $L(\hat{\theta}) = (L_1, L_2, L_3, L_4)^T$is given by:

$$L_1(\hat{\theta}) = \sum_{i=1}^{r} \frac{\upsilon_j}{P_j} \frac{\partial Pj}{\partial \widehat{\beta_0}}(\hat{\theta}) \qquad ; \qquad L_2(\hat{\theta}) = \sum_{i=1}^{r} \frac{\upsilon_j}{P_j} \frac{\partial Pj}{\partial \widehat{\beta_1}}(\hat{\theta})$$

$$L_3(\hat{\theta}) = \sum_{i=1}^{r} \frac{\upsilon_j}{P_j} \frac{\partial Pj}{\partial \widehat{\alpha_0}}(\hat{\theta}) \qquad ; \qquad L_4(\hat{\theta}) = \sum_{i=1}^{r} \frac{\upsilon_j}{P_j} \frac{\partial Pj}{\partial \widehat{\alpha_1}}(\hat{\theta})$$

4.2.2. Fisher information matrix $I(\hat{\theta})$

The elements of the Fisher's information matrix on initial data, $I(\hat{\theta})_{4\times 4}$ for the bi-Weibull distribution are obtained as follows:

$$I_{11} = -\frac{\widehat{\alpha_0}}{\widehat{\beta_0^2}} \left[\sum_{j=1}^{n} (1 - i_j) + (\widehat{\alpha_0} + 1) \sum_{j=1}^{n} \left(\frac{t_j}{\widehat{\beta_0}} \right)^{\widehat{\alpha_0}} \right]$$

$$I_{22} = -\frac{\widehat{\alpha_1}}{\widehat{\beta_1^2}} \left[\sum_{j=1}^{n} i_j - (\widehat{\alpha_1} + 1) \sum_{j=1}^{n} \left(\frac{t_j}{\widehat{\beta_1}} \right)^{\widehat{\alpha_1}} \right]$$

$$I_{33} = \frac{1}{\widehat{\alpha_0^2}} (1 - i_j) + \sum_{j=1}^{n} \left(\frac{t_j}{\widehat{\beta_0}} \right)^{\widehat{\alpha_0}} \left[\ln \left(\frac{t_j}{\widehat{\beta_0}} \right) \right]^2$$

$$I_{44} = \frac{1}{\widehat{\alpha_1^2}} \sum_{j=1}^{n} i_j - \sum_{j=1}^{n} \left(\frac{t_j}{\widehat{\beta_1}} \right)^{\widehat{\alpha_1}} \left[\ln \left(\frac{t_j}{\widehat{\beta_1}} \right) \right]^2$$

$$I_{13} = I_{31} = \frac{1}{\widehat{\beta_0}} \left[\sum_{j=1}^{n} (1 - i_j) + \sum_{j=1}^{n} \left(\frac{t_j}{\widehat{\beta_0}} \right)^{\widehat{\alpha_0}} \left(1 - \alpha_0 \widehat{\ln \left(\frac{t_j}{\widehat{\beta_0}} \right)} \right) \right]$$

$$I_{24} = I_{42} = \frac{1}{\widehat{\beta_1}} \left[\sum_{j=1}^{n} (1 - i_j) + \sum_{j=1}^{n} \left(\frac{t_j}{\widehat{\beta_1}} \right)^{\widehat{\alpha_1}} \left(1 - \alpha_1 \widehat{\ln \left(\frac{t_j}{\widehat{\beta_1}} \right)} \right) \right]$$

$$I_{14} = I_{41} = I_{23} = I_{32} = 0$$

$$I_{34} = I_{43} = I_{12} = I_{21} = 0$$

Therefore, we can deduce the value of the test statistic Y^2 For α level of significance, $H_0$is accepted if $Y_n^2(\hat{\theta}) \leq \chi^2_{\alpha,r-1}$

5.Simulations and application

5.1. Parameters estimation

To assess the performance of the estimation methods used, we generated N=10,000 samples of different sizes (n=15,n=50,n=100,n=200)with the parameter values $\widehat{\alpha_0} = 1.5, \widehat{\beta_0} = 0.5, \widehat{\alpha_1} = 2, \widehat{\beta_1} = 1$, from the bi-Weibull distribution. Using R software, we compute the values of the maximum likelihood estimates (Table1), Kolmogorov-Smirnov estimates (Table 2), the Anderson-Darling estimates (Table 3), the Cramer von-Mises estimates (Table 4) and their corresponding mean squared errors.

	N=10,000	α_0	β_0	α_1	β_1
n=15	AEs	1.42172	0.32922	1.80297	0.70655
SME	0.01032	0.08924	0.03489	0.09123	
n=50	AEs	1.48036	0.45988	1.84011	0.76410
	SME	0.00343	0.02263	0.02263	0.03811
n=100	AEs	1.49068	0.47950	1.92406	0.89660
	SME	0.00244	0.00850	0.00819	0.03083
n=200	AEs	1.50730	0.50945	1.98636	0.98772
	SME	0.00039	0.00436	0.00435	0.00955

Table1:*ML estimators and their mean squared errors*

	N=10000	α_0	β_0	α_1	β_1
n=15	AEs	1.42135	0.48970	1.87589	0.86034
	SME	0.00630	0.07554	0.03280	0.07637
n=50	AEs	1.45627	0.33257	1.80787	0.89540
	SME	0.00720	0.03272	0.04266	0.04848
n=100	AEs	1.47742	0.54131	1.93220	0.90235
	SME	0.00381	0.00960	0.02312	0.03244
n=200	AEs	1.50325	0.53493	1.97310	0.93285
	SME	0.00107	0.00401	0.00687	0.00982

Table2: KS estimators for parameters and their mean square-errors

N=10000		α_0	β_0	α_1	β_1	
n=15		AEs	1.43752	0.38650	1.66156	0.86928
		SME	0.00489	0.02367	0.03638	0.02823
n=50		AEs	1.39721	0.53427	1.77915	0.84509
		SME	0.00376	0.02239	0.03067	0.02573
n=100		AEs	1.42639	0.42487	1.87606	0.87307
		SME	0.00226	0.00650	0.02165	0.02035
n=200		AEs	1.47239	0.44975	1.90849	0.98152
		SME	0.00050	0.00323	0.00663	0.00508

Table3: CVM estimators for parameters and their mean square-errors

N=10000	α_0	β_0	α_1	β_1	
n=15	AEs	1.45930	0.35685	1.92943	0.81630

	SME	0.03238	0.08264	0.06279	0.07526
n=50	AEs	1.42024	0.42163	1.82683	0.89053
	SME	0.00751	0.03485	0.03400	0.05339
n=100	AEs	1.49315	0.42759	1.96315	0.93049
	SME	0.00408	0.00897	0.05421	0.02738
n=200	AEs	1.49511	0.49382	1.98239	0.97203
	SME	0.00099	0.00591	0.00964	0.00866

Table4:AD estimators for parameters and their mean square-errors

From numerical experiments, it is observed that the obtained estimators are consistent and the average values of the estimates (AEs) and the average mean squared errors (SME) decrease as the sample size increases.

5.1.1. Criteria test

To show the tractability of the statistic test provided in this work, we test the null hypothesis H_0that samples are drawing from the bi-Weibull distribution. At this end, the theoretical levels of significance $\alpha = (0.01,0.05,0.1)$ are compared to those corresponding to empirical levels of significance for 10,000simulated samples from different sizes (n=15, 50,100,200) .The results are summarized in Table 5.

N	α=0.01	α=0.1	α=0.05
15	0.01697	0.1230	0.03879
50	0.01513	0.1080	0.04918
100	0.01222	0.1022	0.05204

Table5: critical chi-square and their corresponding empirical values

As expected, the obtained empirical values are very close to their corresponding theoretical ones, which implies the feasibility of the proposed goodness-of-fit test to validate the bi-Weibull distribution.

5.2. Application

An example of real data sets taken from the site (https://www.kaggle.com/datasets/marshalpatel3558/diabetes-prediction-dataset-legit-dataset) is used to confirm the theoretical results obtained in this work. The observations are related to diabetes and contains various biomedical measurements and patient characteristics. We selected the following data concerning the cholesterol level in the blood (measured in mg/dL or mmol/L). This typically refers to total cholesterol (Chol(4.2, 3.7, 4.2, 4.2, 0.9, 2.9, 3.6, 2.9, 3.8, 3.8, 3.6, 4.0, 4.9, 4.2, 4.0, 3.6, 5.3, 4.9, 0.5, 4.4, 0.5, 6.2, 4.2, 4.8, 4.6, 4.9, 0.8 2.8, 3.7, 5.6, 3.2, 4.6, 3.6, 0..2, 6.5, 4.0, 3.0, 3.7, 3.7, 4.4, 3.8, 9.5, 3.7, 4.7, 9.5, 0.7, 4.2, 5.2, 0.3, 4.7)) and Triglycerides level in the blood (measured in mg/dL or mmol/L). Triglycerides are a type of fat in the blood (TG(0.9, 1.4, 0.9, 0.9, 1.0, 1.0, 1.3, 0.8, 0.9, 2.0, 0.7, 1.1, 1.3, 1.7, 1.5, 1.1,0.8, 1.3, 1.3, 0.9, 1.9, 1.0, 1.5, 1.0, 1.3, 1.3, 1.8, 2.9, 1.3, 1.4, 1.8, 0.8,0.7, 1.0, 1.5, 0.6, 1.2, 1.3, 1.5, 1.6, 0.7, 1.7, 1.3, 1.9, 1.4, 1.1, 2.1, 1.8, 1.7, 1.6)).

For testing the hypothesis H_0that these data follow the bi-Weibull distribution, generally practitioners use classical selection criteria models to choose the appropriate model between the possible alternatives. In this case, we compare the fits of the BW distribution and the competing risks model of Bertholon $B(\lambda,\beta,\eta)$defined by the following pdf:

$$f(t/\lambda,\eta,\beta)=\lambda\exp(-\lambda t)+\frac{\beta}{\eta}\left(\frac{t-T}{\eta}\right)^{\beta-1}\exp\left(-\left(\frac{t-T}{\eta}\right)^{\beta}\right)$$

The results given in Table 6 show that the preferred model is the bi-Weibull.

Model	LL	AIC	BIC	KS-p value
Bi-Weibull	-29.90045	63.80091	67.62495	0.712648
Bertholon	-90.68764	18537528	189.19932	0.035918

Table 6: classical selection criteria models for bi-Weibull and Betholon

On another hand, we use the test statistic provided above to verify if the cited data can be model by BW distribution. We grouped the observations into r= 8 intervals. The values of the components of the test statistic Y^2 are given as follow:

L(0.001849,-0.0053089 ,0.00293371,0.0013059)

$$I_n(\hat{\theta})=\begin{pmatrix}1.155761 & 0.680780 & 0.450603 & 0.961035\\ 0.680780 & 2.221326 & -0.838179 & 0.677483\\ 0.450603 & -0.838179 & 1.312617 & 0.2817554\\ 0.961035 & 1.312617 & 0.2817554 & 1.902589\end{pmatrix}$$

$$J(\hat{\theta})=\begin{pmatrix}0.722107 & 0.735629 & 0.9193473 & 0.203987\\ 0.735629 & 0.058435 & 0.958696 & 0.283743\\ 0.9193473 & 0.958696 & 0.5153546 & 0.305858\\ 0.203987 & 0.283743 & 0.305858 & 0.879862\end{pmatrix}$$

We obtain $Y^2=3.962869$, so for the significance level $\alpha=0.05$, the critical chi-square value is $X_{r-1}^2=14{,}06714$. As $Y^2<X_{r-1}^2$,the null hypothesis H_0 cannot be rejected which confirms that the bi-Weibull distribution describe suitebly these observations. This result is clear if we compare the plots of BW and B distributions relating to these observations (see fig. 1).

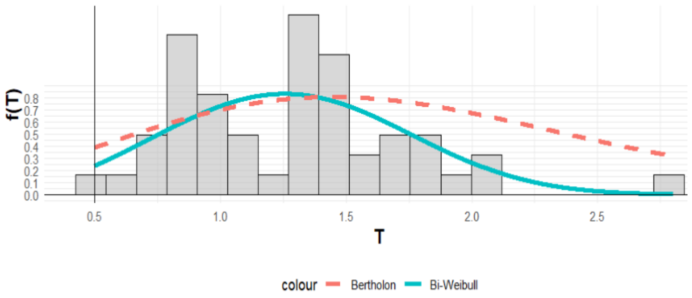

Fig.1: pdf plots of Bi-weibull and Bertholon model for total cholesterol levels dataset

Conclusion

The study examines a competing risks model that is based on the smaller of two Weibull distributions. Various estimation techniques were applied to ascertain the unknown parameters. A novel set of criteria that can effectively validate this model was developed. Without taking into consideration the potential alternatives, this statistic can be used to fit this distribution. The effectiveness of the methods employed and the viability of this test were demonstrated using thousands of simulated samples as well as an example of actual medical data.

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