ADVANCED TRIGONOMETRIC APPROXIMATION IN WEIGHTED L_w^q SPACES VIA DEFERRED VORONOI–NÖRLUND AND RIESZ MEANS

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Abstract

The degree of approximation for functions in generalized Lipschitz classes can be established using deferred Voronoi–Nörlund and deferred Riesz transforms applied to the partial sums of their trigonometric Fourier series in weighted Lebesgue spaces.

Keywords: Fourier series; Deferred Voronoi–Nörlund transformation; Degree of approximation; Modulus of continuity; Lipschitz class.

Introduction

Xhevat Z. Krasniqi [17] introduced deferred Voronoi–Nörlund and Riesz means, enhancing classical summability methods by incorporating additional sequences. These advanced means provide greater flexibility and control over the convergence of sequences and series. The deferred Voronoi–Nörlund means use a double sequence of weights, refining the summability process, while the deferred Riesz means add a component to consider more factors in summation.

The degree of approximation of an integrable $2\pi -$ periodic function

$f(x) \in Lip(\delta, q)$ by $n - th$ partial sum

$$S_n(f;y) = \frac{u_0}{2} + \sum_{k=1}^{n} (u_k cosky + v_k sinky)$$

of its Fourier series (at the point y)

$$f(y) \sim \frac{u_0}{2} + \sum_{k=1}^{\infty} (u_k cosky + v_k sinky)$$

The study of L^q-norms ($q \in [1, \infty)$ has been significantly advanced by E. S. Quade [6], who analyzed the values of δ and q that determine when the degree of approximation is of order $O(n^{-\delta})$. His results serve as an excellent foundation for the continued publication of numerous generalizations that have recently been achieved by other researchers. For example, presented by Xh. Z. Krasniqi in [16, 17, 18], S. Z. Jafarov in [12, 13] and U. Dẽger in [14, 15]. Later on, more systematic results gives by Chandra in [10] by using Approximation by trigonometric polynomials, weighted norm [4], Cesàro means [15], Trigonometric approximation [3], [6] and [7]. In the same spirit, Guven [1] and [2] results.

We must recall the concept of the weighted Lebesgue space L_w^q. A measurable 2π-2periodic function $w: [0,2\pi] \to [0, \infty]$ is defined as a weight function if the Collection $\mathrm{w}^{-1}(0, \infty)$ has then the Lebesgue measure zero. The space $L_w^q = L_w^q[0,2\pi]$, where $q \in [1, \infty)$ and w is a weighted function, contains all measurable functions f that are with 2π-2periodic.

$$\|f\|_{q,w} = \left(\int_0^{2\pi} |f(y)|^q \, w(y) dx\right)^{\frac{1}{q}} < \infty$$

Let $q \in (1, \infty)$. a weight function w belongs to class $\mathcal{A}_q$ if there exists a contain $C > 0$, such that for every ball $\gamma \subset R^n$.

$$\left(\frac{1}{|\gamma|}\int_\gamma w(y) dy\right)\left(\frac{1}{|\gamma|}\int_\gamma |w(y)|^{1-q'} \, dy\right) < \infty$$

Where q' denotes the Hölder conjugate of q, defined $\frac{1}{q} + \frac{1}{q'} = 1$.

Assuming $q \in (1, \infty), w \in A_q$ and $f \in L_w^q$ the modulus of continuity $w(f, t)$ of the f can be Identified in terms of the weighted L^q norm. It measures how much the function f can change over small distance, for L_w^q the modulus of continuity $w(f, t)$ is defined by

$$w(f, t) = \sum_{|h| \leq t} \|f(.+h) - f(.)\|_{L_w^q},$$

Where $\|.\|_{L_w^q}$ is the weighted L^q norm, Identified by

$$\|g\|_{L_w^q} = \left(\int_R |g(y)|^q \, w(y)dy\right)^{\frac{1}{q}}$$

show that the Lipschitz class $Lip(\delta, q, w)$, where $0 < \delta < 1$, can be defined as

$$Lip(\delta, q, w) = \{f \in L_w^q : |f(.+h) - f(.)|_{L_w^q} \leq C|h|^{\delta} \forall |h| \leq \nu, \qquad when\ C > 0\ and\ \nu > 0\}$$

Before recalling Güven's [1] results, we need to first establish some preliminary concepts. Whenever necessary, we utilize the $\mathrm{n}-\mathrm{th}$ partial sums of the Fourier series of $\mathrm{f(y)}$ evaluated at the point x, expressed as follows:

$$S_n(f;y) = \sum_{k=0}^{n} A_k(f;y),$$

Where

$$A_0(f;y) = \frac{u_0}{2}, A_k(f;y) = u_k cosky + v_k sinky, (k = 1,2, \ldots).$$

Let $(q_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. We investigate two transformations, the so-called Nörlund and Riesz transforms, of the sums $S_n(f;y)$ specified by:

$$N_n(f;y) = \frac{1}{Q_n} \sum_{\mu=0}^{n} q_{n-\mu} \, S_{\mu}(f;y)$$

and

$$R_n(f;y) = \frac{1}{Q_n} \sum_{\mu=0}^{n} q_{\mu} \, S_{\mu}(f;y),$$

where, $Q_n = \sum_{\mu=0}^{n} q_{\mu}$ As established by Krasniqi [17],we utilize the same results for the case where $p_n = q_n$ as follows:

Theorem-1.1: Consider$1 < q < \infty, w \in A_q, 0 < \delta \leq 1$ and let $(q_n)_{n=0}^{\infty}$ be a monotonic sequence of positive real numbers such that

$$(n+1)q_n = O(Q_n)$$

then for any $f \in Lip(\delta, q, w)$, the evaluation

$$\|f - N_n(f)\|_{q,w} = O\left(n^{\delta}\right), n = 1,2, \ldots$$

holds.

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$$\|f - N_n(f)\|_{q,w} = O\left(n^{\delta}\right), n = 1,2, \ldots$$

holds.

Theorem-1.2: Consider$1 < q < \infty, w \in A_q, 0 < \delta \leq 1$ and let $(q_n)_{n=0}^{\infty}$ be a sequence of the positive real numbers adheres relation

$$\sum_{\mu=0}^{n-1} \left| \frac{Q_\mu}{\mu+1} - \frac{Q_{\mu+1}}{\mu+2} \right| = O\left(\frac{Q_n}{n+1}\right)$$

Then for every $\mathrm{f} \in \mathrm{Lip}(\delta, \mathrm{q}, \mathrm{w})$, the assessment

$$|f - R_n(f)|_{q,w} = O\left(n^{-\delta}\right), \quad n = 1,2, \ldots$$

is achieved. To clarify our intention, let's recall some notations and concepts.

Let $u = (u_n)$ and $v = (v_n)$ be sequences of non-negative integers that satisfy the following conditions:

Given

$$u_n \leq v_n, n = 1,2, \ldots \tag{1.1}$$

And

$$\lim_{n\to\infty} u_n \leq \lim_{n\to\infty} v_n = +\infty \tag{1.2}$$

The deferred Cesáro mean, established by sequences a and b, is specified as

$$D_n = D_u^{bv} = \frac{S_{u_{n+1}} + S_{u_{n+2}} + \cdots + S_{v_n}}{v_n - u_n}$$

When (S_μ) denotes a sequence of real or complex numbers. Given that each D_u^v with conditions (1.1) and (1.2) meets the Silverman–Toeplitz criteria, every D_u^v is regular.

$$D_u^v N_n(f;y) = \frac{1}{Q_0^{v_n-u_n-1}} \sum_{\mu=u_n+1}^{v_n} q^{v_n-\mu} S_\mu(f;y)$$

And

$$D_u^v R_n(f;y) = \frac{1}{Q_{u_n+1}^{v_n}} \sum_{\mu=u_n+1}^{v_n} q_\mu S_\mu(f;y),$$

Where

$$Q_0^{v_n-u_n-1} = \sum_{\mu=0}^{v_n-u_n-1} q_\mu \neq 0, Q_{u_n+1}^{v_n} = \sum_{\mu=u_n+1}^{v_n} q_\mu \neq 0$$

These two methods are referred to as the deferred Voronoi–Norlund means, $(D_u^v N, q)$, and the deferred Riesz means, $(D_u^v R, q)$, respectively.

$$D_u^v(f;y) = \frac{1}{v_n - u_n} \sum_{\mu=u_n+1}^{v_n} S_\mu(f;y)$$

Lemmas

The following statements are required for the proof of our results.

Lemma-2.1: [17] Let $1 < q < \infty$, $0 < \delta \leq 1$ and $w \in A_q$. Thus, for each $f \in Lip(\delta, q, w)$ the evaluation

$$\|f - S_n(f)\|_{q,w} = O(n^{-\delta}), \quad n = 1,2,\dots$$

hold.

Lemma-2.2: [17] Let $1 < q < \infty$, $0 < \delta \leq 1$ and $w \in A_q$. Thus, for every $f \in Lip(1, q, w)$ the evaluation

$$\|S_n(f) - \sigma_n(f)\|_{q,w} = O(n^{-1}), \quad n = 1,2,\dots$$

hold.

To denote these classes, we use $c \in AMDS$ and $c \in AMIS$ for almost monotone decreasing and increasing sequences, respectively. The following lemma is crucial in establishing our main results.

$$v_n q^{v_n - u_n - 1} = O\left(Q_0^{v_n - u_n - 1}\right) \qquad (2.1)$$

Then

$$\sum_{\mu=0}^{v_n - u_n - 1} q^{v_n - u_n - 1 - \mu}(u_n + 1 + \mu)^{-\delta} = O\left((v_n - u_n)^{-\delta} Q_0^{v_n - u_n - 1}\right)$$

$for\ 0 < \alpha < 1.$

Proof: Consider the sum

$$\sum_{\mu=0}^{v_n - u_n - 1} q^{v_n - u_n - 1 - \mu}(u_n + 1 + \mu)^{-\delta}$$

Let analyze the general term $q^{v_n - u_n - 1 - \mu}(u_n + 1 + \mu)^{-\delta}$.

Since $(q_n) \in AMDS$,

We have, $q^{v_n - u_n - 1} \leq K q^{v_n - u_n - 1}$

Using this in the sum,

$$\sum_{\mu=0}^{v_n - u_n - 1} q^{v_n - u_n - 1 - \mu}(u_n + 1 + \mu)^{-\delta} \leq q^{v_n - u_n - 1} \sum_{\mu=0}^{v_n - u_n - 1} K(u_n + 1 + \mu)^{-\delta}$$

Next, observe that the term $(u_n + 1 + \mu)^{-\delta}$ decreases as μ increases. Hence, we can bound the sum by integrating

$$\sum_{\mu=0}^{v_n - u_n - 1} (u_n + 1 + \mu)^{-\delta} \leq \int_0^{v_n - u_n} (u_n + 1 + y)^{-\delta}\, dy.$$

Substituting, $\eta = u_n + 1 + y$, $d\eta = dy$

$$\int_0^{v_n-u_n}(u_n+1+y)^{-\delta}\,dy=\int_{u_n+1}^{v_n+1}\eta^{-\delta}\,d\eta=\left[\frac{\eta^{1-\delta}}{1-\delta}\right]_{u_n+1}^{v_n+1}. \tag{2.2}$$

Evaluating the integral

$$\frac{(v_n+1)^{1-\delta}-(u_n+1)^{1-\delta}}{1-\delta}$$

For $0<\delta<1$, the dominant term is $(v_n+1)^{1-\delta}$, so

$$\int_{u_n+1}^{v_n+1}\eta^{-\delta}\,d\eta\ \approx\frac{(v_n+1)^{1-\delta}}{1-\delta}.$$

Thus,

$$\sum_{\mu=0}^{v_n-u_n-1}(u_n+1+\mu)^{-\delta}\le\frac{(v_n+1)^{1-\delta}}{1-\delta}.$$

Using this bound,

$$\sum_{\mu=0}^{v_n-u_n-1}q^{v_n-u_n-1-\mu}(u_n+1+\mu)^{-\delta}\le q^{v_n-u_n-1}K.\frac{(v_n+1)^{1-\delta}}{1-\delta}$$

Recall that, $v_nq^{v_n-u_n-1}\le CQ_0^{v_n-u_n-1}$.

$$q^{v_n-u_n-1}\le\frac{Q_0^{v_n-u_n-1}}{v_n}$$

Substituting,

$$\sum_{\mu=0}^{v_n-u_n-1}q^{v_n-u_n-1-\mu}(u_n+1+\mu)^{-\delta}\le\frac{Q_0^{v_n-u_n-1}}{v_n}K.\frac{(v_n+1)^{1-\delta}}{1-\delta}.$$

Simplifying, we get

$$\sum_{\mu=0}^{v_n-u_n-1} q^{v_n-u_n-1-\mu}(u_n+1+\mu)^{-\delta} = O\big((v_n-u_n)^{-\delta} Q_0^{v_n-u_n-1}\big)$$

Main Theorems:

Theorem-3.1: Let $q \in (1,\infty), w \in A_q$ and $f \in Lip(\delta,q,w)$. If at least one of the conditions

i. $0 < \delta < 1, and (q_n) \in AMDS$
ii. $0 < \delta < 1, (q_n) \in AMIS\ \ and\ (2.1)\ holds$
iii. $\delta = 1, \sum_{j=1}^{v_n-u_n-1} |\Delta(q_j)| = O(1),\ and\ (2.1)\ holds$
iv. $\delta = 1, \sum_{j=1}^{v_n-u_n-1} j|\Delta(q_j)| = O(1).$

is true, where $\Delta(q_j) = q_j - q_{(j+1)}$, the

$$\|f - D_u^v N_n(f)\|_{q,w} = O\big((v_n-u_n)^{1-\delta}\big), \qquad (n = 1,2,\dots)$$

Proof: Let $0 < \delta < 1$, Using Lemma 2.1 and Lemma 2.3, the cases (i) and (ii) can be proved simultaneously. Namely, since

$$f(y) = \frac{1}{Q_0^{v_n-u_n-1}} \sum_{\mu=0}^{v_n-u_n-1} q^{v_n-u_n-1-\mu} f(y)$$

we can write

$$f(y) - D_u^v N_n(f;y) = \frac{1}{Q_0^{v_n-u_n-1}} \sum_{\mu=0}^{v_n-u_n-1} q^{v_n-u_n-1-\mu}\big[f(y) - S_{u_n+1+\mu}(f;y)\big],$$

Therefore, by utilizing Lemma 2.1 and Lemma 2.3, we derive

$$\|f - D_u^v N_n(f)\|_{q,w} \le \frac{1}{Q_0^{v_n-u_n-1}} \sum_{\mu=0}^{v_n-u_n-1} q^{v_n-u_n-1-\mu}\, |f - S_{u_n+1+\mu}(f)|_{q,w}$$

$$\le \frac{1}{Q_0^{v_n-u_n-1}} \sum_{\mu=0}^{v_n-u_n-1} q^{v_n-u_n-1-\mu}\, |f - S_{u_n+1+\mu}(f)|_{q,w}(u_n+1+\mu)^{-\delta}$$

$$= \frac{1}{Q_0^{v_n-u_n-1}} O\left(\sum_{\mu=0}^{v_n-u_n-1} q^{v_n-u_n-1-\mu}(u_n+1+\mu)^{-\delta}\right)$$

$$= \frac{1}{Q_0^{v_n-u_n-1}} O\big(Q_0^{v_n-u_n-1}(v_n-u_n)^{1-\delta}\big)$$

$$= O\big((v_n-u_n)^{1-\delta}\big)$$

We need to specifically address the case where $\delta = 1$. Applying Abel's transformation results in the following,

$$D_u^v N_n(f;y) = \sum_{\mu=0}^{v_n-u_n-2} q^{v_n-u_n-1-\mu} S_{u_n+\mu+1}(f;y)$$

$$= \frac{1}{Q_0^{v_n-u_n-1}} \sum_{\mu=0}^{v_n-u_n-1} Q_0^{v_n-u_n-1} q^{v_n-u_n-1-\mu} S_{u_n+\mu+1}(f;y)$$

$$= \frac{1}{Q_0^{v_n-u_n-1}} \left\{ \sum_{\mu=0}^{v_n-u_n-1} Q_0^{v_n-u_n-1} \left(s_{u_n+\mu+1}(f;y) - S_{u_n}(f;y) \right) \sum_{\mu=0}^{v_n-u_n-1} Q_0^{v_n-u_n-1} S_{u_n}(f;y) \right\}$$

$$= \frac{1}{Q_0^{v_n-u_n-1}} \sum_{\mu=0}^{v_n-u_n-1} Q_0^{v_n-u_n-1} \left(s_{u_n+\mu+1}(f;y) - S_{u_n}(f;y) \right) + S_{u_n}(f;y)$$

$$= \frac{1}{Q_0^{v_n-u_n-1}} \sum_{\mu=0}^{v_n-u_n-1} Q_0^{v_n-u_n-1} A_{u_n+\mu+1}(f;y) + S_{u_n}(f;y).$$

We utilize the evident equality

$$S_{v_n}(f;y) = \frac{1}{Q_0^{v_n-u_n-1}} \sum_{\mu=0}^{v_n-u_n-1} Q_0^{v_n-u_n-1} A_{u_n+\mu+1}(f;y) + \sum_{\mu=0}^{u_n} A_\mu(f;y)$$

and Abel's transformation to derive

$$S_{v_n}(f;y) - D_u^v N_n(f;y) = \frac{1}{Q_0^{v_n-u_n-1}} \sum_{\mu=0}^{v_n-u_n-1} \frac{Q_0^{v_n-u_n-1} - Q_0^{v_n-u_n-1-\mu}}{\mu} \mu A_{u_n+1+\mu}(f;y)$$

$$= \frac{1}{Q_0^{v_n-u_n-1}} \left\{ \Delta\left(\frac{Q_0^{v_n-u_n-1} - Q_0^{v_n-u_n-1-\mu}}{\mu} \right) \sum_{j=1}^{\mu} j A_{u_n+1+\mu}(f;y) \right.$$
$$\left. + \frac{Q_0^{v_n-u_n-1} - Q_0^0}{v_n-u_n-1} \sum_{j=1}^{v_n-u_n-1} j A_{u_n+1+\mu}(f;y) \right\}$$

$$= \frac{1}{Q_0^{v_n-u_n-1}} \left\{ \Delta\left(\frac{Q_0^{v_n-u_n-1} - Q_0^{v_n-u_n-1-\mu}}{\mu} \right) \sum_{j=1}^{\mu} j A_{u_n+1+\mu}(f;y) \right.$$
$$\left. + \frac{Q_0^{v_n-u_n-1} - 1}{v_n-u_n-1} A_{u_n+1+\mu}(f;y) \right\}$$

Thus, Lemma 2.3 gives

$$\left\| \sum_{j=1}^{\mu} j A_{u_n+1+\mu}(f;y) \right\| \le \frac{1}{\mu(\mu+1)} = O(1)$$

and therefore

$$\left\|S_{v_n}(f)-D_u^v N_n(f)\right\|_{q,w} \le O\left(\frac{1}{Q_0^{v_n-u_n-1}}\sum_{\mu=0}^{v_n-u_n-1}\left|\Delta\left(\frac{Q_0^{v_n-u_n-1}-Q_0^{v_n-u_n-1-\mu}}{\mu}\right)\right|\right)$$

Moreover, the equality

$$\Delta\left(\frac{Q_0^{v_n-u_n-1}-Q_0^{v_n-u_n-1-\mu}}{\mu}\right)=-Q_0^{v_n-u_n-1}\left(\frac{1}{\mu(\mu+1)}\right)-\left(\frac{Q_0^{v_n-u_n-1-(\mu+1)}}{\mu+1}-\frac{Q_0^{v_n-u_n-1}}{\mu}\right)$$

$$=-\frac{Q_0^{v_n-u_n-2-\mu}-Q_0^{v_n-u_n-1}}{\mu+1}$$

holds true

we obtain

$$\left\|S_{v_n}(f)-D_u^v N_n(f)\right\|_{q,w}=O\left(\frac{1}{Q_0^{v_n-u_n-1}}\sum_{\mu=1}^{v_n-u_n-1}\left|\Delta\left(\frac{Q_0^{v_n-u_n-1}-Q_0^{v_n-u_n-1-\mu}}{\mu}\right)\right|\right)$$

$$=O\left(\frac{1}{Q_0^{v_n-u_n-1}}\sum_{\mu=1}^{v_n-u_n-1}\left|\frac{Q_0^{v_n-u_n-1}}{\mu(\mu+1)}-\frac{Q_0^{v_n-u_n-1-\mu}}{\mu}\right|\right)$$

$$=O\left(\frac{1}{Q_0^{v_n-u_n-1}}\sum_{m=1}^{v_n-u_n-1}\left|\frac{Q_0^{v_n-u_n-1}}{\mu+1}-\frac{Q_0^{v_n-u_n-1}-Q_0^{v_n-u_n-1-\mu}}{\mu}\right|\right)$$

$$=O\left(\frac{1}{Q_0^{v_n-u_n-1}}.\left(1-\frac{1}{v_n-u_n}\right)\right)$$

$$=O\left(\frac{1}{Q_0^{v_n-u_n-1}}\right)$$

$$=O(1)$$

To finalize the proof of our theorem, we need to address the remaining case (iv). First, we demonstrate that, under the assumption

$$\sum_{j=1}^{v_n-u_n-1} j\left|\Delta(q_j)\right|=O\left(Q_0^{v_n-u_n-1}\right)$$

the inequality

$$K_{u_n,v_n}=\sum_{\mu=1}^{v_n-u_n-1}\left|\Delta\left(\frac{Q_0^{v_n-u_n-1}-Q_0^{v_n-u_n-1-\mu}}{\mu}\right)\right|=O(1) \tag{3.1}$$

holds true.

So, we have proved that

$$\left\|S_{v_n}(f)-D_u^v N_n(f)\right\|_{q,w}=O(1)$$

and using Lemma 2.1, we obtain

$$\|f-D_u^v N_n(f)\|_{q,w}=\left\|f-S_{v_n}(f)\right\|_{q,w}+\left\|S_{v_n}(f)-D_u^v N_n(f)\right\|_{q,w}=O(1)$$

The proof is completed.

Theorem-3.2: Let $1<q<\infty$, $w\in A_q$, $0<\delta\leq 1$. Also, let $(q_n)_{n=0}^{\infty}$ be a sequence of the positive real numbers that fulfills the relation

$$\sum_{m=0}^{n-1}\left|\frac{Q_{u_n+1}^{\mu+u_n+1}}{u_n+2+\mu}-\frac{Q_{u_n+1}^{\mu+u_n+2}}{u_n+3+\mu}\right|=O\left(\frac{Q_{u_n+1}^{v_n-1}}{v_n}\right)\tag{3.2}$$

Then for every $f\in Lip(\delta,q,w)$, the estimate

$$\|f-R_n(f)\|_{q,w}=O\left(n^{-\delta}\right),\qquad n=1,2,\ldots$$

is satisfied.

Proof: Let $0<\delta<1$, First f all, we write

$$f(y)-D_u^v R_n(f;y)=\frac{1}{Q_{u_n+1}^{v_n}}\sum_{\mu=0}^{v_n-u_n-1}q_{u_n+1+\mu}\left[f(y)-S_{u_n+1+\mu}(f;y)\right]$$

Using Lemma 2.1,

$$\|f(y)-D_u^v R_n(f;y)\|_{q,w}\leq\frac{1}{Q_{u_n+1}^{v_n}}\sum_{\mu=0}^{v_n-u_n-1}q_{u_n+1+\mu}\,|f-S_{u_n+1+\mu}(f)|_{q,w}$$

$$\leq\frac{1}{Q_{u_n+1}^{v_n}}\sum_{\mu=0}^{v_n-u_n-1}q_{u_n+1+\mu}\,O\left((u_n+1+\mu)^{-\delta}\right)\tag{3.3}$$

We utilize the method of summation by parts to achieve $\left(Q_{u_n+1}^{0}=0\right)$:

$$\sum_{\mu=0}^{v_n-u_n-1}q_{u_n+1+\mu}(u_n+1+\mu)^{-\delta}=\sum_{v_n-u_n-2}^{\delta=0}\left[(u_n+1+\mu)^{-\delta}-(u_n+2+\mu)^{-\delta}\right]Q_{u_n+1}^{\mu+u_n+1}$$
$$+\,v_n^{-\delta}Q_{u_n+1}^{v_n}$$

$$=O(1)\sum_{\mu=0}^{v_n-u_n-2}(u_n+2+\mu)^{-1-\delta}\,Q_{u_n+1}^{\mu+u_n+1}+\frac{Q_{u_n+1}^{v_n}}{(v_n-u_n)^{\delta}}\tag{3.4}$$

By using summation by parts again and invoking condition (3.2), we derive

$$\sum_{\mu=0}^{v_n-u_n-2} (u_n+2+\mu)^{-\delta}\frac{Q_{u_n+1}^{\mu+u_n+1}}{u_n+2+\mu}$$

$$\leq \sum_{\mu=0}^{v_n-u_n-3}\left|\frac{Q_{u_n+1}^{\mu+u_n+1}}{u_n+2+\mu}-\frac{Q_{u_n+1}^{\mu+u_n+2}}{u_n+3+\mu}\right|\sum_{j=0}^{\mu}(u_n+2+j)^{-\delta}$$

$$+\frac{Q_{u_n+1}^{v_n-1}}{v_n}\sum_{j=0}^{v_n-u_n-2}(u_n+2+j)^{-\delta}$$

$$=\sum_{\mu=0}^{v_n-u_n-3}\left|\frac{Q_{u_n+1}^{\mu+u_n+1}}{u_n+2+\mu}-\frac{Q_{u_n+1}^{\mu+u_n+2}}{u_n+3+\mu}\right|O\big((u_n+2+\mu)^{1-\delta}\big)+\frac{Q_{u_n+1}^{v_n-1}}{v_n}O\big(v_n^{1-\delta}\big)$$

$$=O\left(\frac{Q_{u_n+1}^{v_n-1}}{v_n}(v_n-1)^{1-\delta}\right)+O\left(\frac{Q_{u_n+1}^{v_n-1}}{v_n}\right)$$

$$=O\left(\frac{Q_{u_n+1}^{v_n-1}}{(v_n-u_n)^{\delta}}\right) \qquad (3.5)$$

Thus, from equations (3.3), (3.4), and (3.5), it follows that

$$\|f(y)-D_u^vR_n(f;y)\|_{q,w}=O\left(\frac{1}{(v_n-u_n)^{\delta}}\right)$$

Now, let's consider the case where $\delta=1$. Using summation by part, we obtain

$$D_u^vR_n(f;y)=\frac{1}{Q_{u_n+1}^{v_n}}\sum_{\mu=0}^{v_n-u_n-1} q_{u_n+1+\mu}S_{u_n+1+\mu}(f;y)$$

$$=\frac{1}{Q_{u_n+1}^{v_n}}\left[\sum_{\mu=0}^{v_n-u_n-2}\Big(S_{u_n+1+\mu}(f;y)-S_{u_n+2+\mu}(f;y)\Big)\sum_{j=0}^{\mu}q_{u_n+1+j}+S_{v_n(f;y)\sum_{j=0}^{v_n-u_n-1}q_{u_n+1+j}}\right]$$

$$=-\frac{1}{Q_{u_n+1}^{v_n}}\left[\sum_{\mu=0}^{v_n-u_n-2}A_{u_n+2+\mu}(f;y)Q_{u_n+1}^{u_n+1+\mu}-S_{v_n}(f;y)Q_{u_n+1}^{v_n}\right]$$

$$D_u^vR_n(f;y)-S_{v_n}(f;y)=-\frac{1}{Q_{u_n+1}^{v_n}}\sum_{\mu=0}^{v_n-u_n-2}Q_{u_n+1}^{u_n+1+\mu}A_{u_n+2+\mu}(f;y).$$

Applying summation by parts once more, we obtain

$$\sum_{\mu=0}^{v_n-u_n-2}Q_{u_n+1}^{u_n+1+\mu}A_{u_n+2+\mu}(f;y) = \sum_{\mu=0}^{v_n-u_n-2}\frac{Q_{u_n+1}^{u_n+1+\mu}}{u_n+2+\mu}(u_n+2+\mu)A_{u_n+2+\mu}(f;y)$$

$$= \sum_{\mu=0}^{v_n-u_n-2}\left(\frac{Q_{u_n+1}^{u_n+1+\mu}}{u_n+2+\mu}-\frac{Q_{u_n+1}^{u_n+2+\mu}}{u_n+3+\mu}\right)\sum_{j=0}^{\mu}(u_n+2+\mu)A_{u_n+2+\mu}(f;y)$$

$$+\frac{Q_{u_n+1}^{v_n-1}}{v_n}\sum_{j=0}^{v_n-u_n-2}(u_n+2+j)A_{u_n+2+\mu}(f;y)$$

Hence, using the equality

$$\sum_{j=0}^{\mu}(u_n+2+j)A_{u_n+2+\mu}(f;y)=\sum_{j=u_n+2}^{\mu+u_n+2}jA_j(f;y)$$

$$=\sum_{j=0}^{\mu+u_n+2}jA_j(f;y)-\sum_{j=0}^{u_n+1}jA_j(f;y)$$

$$=(\mu+u_n+3)\left(S_{\mu+u_n+2}(f;y)-\chi_{\mu+u_n+2}(f;y)\right)-(u_n+2)\left(S_{u_n+1}(f;y)-\chi_{u_n+1}(f;y)\right)$$

we get,

$$\left\|\sum_{\mu=0}^{v_n-u_n-2}Q_{u_n+1}^{u_n+1+\mu}A_{u_n+2+\mu}(f;y)\right\|_{q,w}$$

$$\leq \sum_{\mu=0}^{v_n-u_n-3}\left|\frac{Q_{u_n+1}^{u_n+1+\mu}}{u_n+2+\mu}-\frac{Q_{u_n+1}^{u_n+2+\mu}}{u_n+3+\mu}\right|\left\|\sum_{j=0}^{\mu}(u_n+2+\mu)A_{u_n+2+\mu}(f)\right\|_{q,w}$$

$$+\frac{Q_{u_n+1}^{v_n-1}}{v_n}\left\|\sum_{j=0}^{v_n-u_n-2}(u_n+2+j)A_{u_n+2+\mu}(f)\right\|_{q'w}$$

$$\leq \sum_{\mu=0}^{v_n-u_n-3}(\mu+u_n+3)\left|\frac{Q_{u_n+1}^{u_n+1+\mu}}{u_n+2+\mu}-\frac{Q_{u_n+1}^{u_n+2+\mu}}{u_n+3+\mu}\right|\left\|S_{\mu+u_n+2}(f)-\chi_{\mu+u_n+2}(f)\right\|_{q,w}$$

$$+\sum_{\mu=0}^{v_n-u_n-3}(u_n+2)\left|\frac{Q_{u_n+1}^{u_n+1+\mu}}{u_n+2+\mu}-\frac{Q_{u_n+1}^{u_n+2+\mu}}{u_n+3+\mu}\right|$$

$$\left\|S_{u_n+1}(f)-\chi_{u_n+1}(f)\right\|_{q,w}$$

$$+\frac{Q_{u_n+1}^{v_n-1}}{v_n}\Big[(v_n+1)\left\|S_{v_n}(f)-\chi_{v_n}(f)\right\|_{q,w}$$

$$+(u_n+2)\left\|S_{u_n+1}(f)-\chi_{u_n+1}(f)\right\|_{q,w}\Big]$$

Therefore, by applying Lemma 2.2 and condition (3.2), we obtain

$$\left\|D_u^vR_n(f)-S_{v_n}(f)\right\|_{q,w}=\frac{1}{Q_{u_n+1}^{v_n}}O\left(\sum_{\mu=0}^{v_n-u_n-3}\left|\frac{Q_{u_n+1}^{u_n+1+\mu}}{u_n+2+\mu}-\frac{Q_{u_n+1}^{u_n+2+\mu}}{u_n+3+\mu}\right|+\frac{Q_{u_n+1}^{v_n-1}}{v_n}\right)$$

$$= O\left(\frac{1}{v_n}\right) = O\left(\frac{1}{v_n - u_n}\right).$$

Finally, the latest estimate and Lemma 2.1 imply

$$\|f - D_u^v(f)\|_{q,w} \leq \left\|f - S_{v_n}(f)\right\|_{q,w} + \left\|S_{v_n}(f) - D_u^v R_n(f)\right\|_{q,w}$$

$$= O\left(\frac{1}{v_n - u_n}\right).$$

The proof is completed.

Remark-3.1: If we take $u_n = 0$ and $\mathrm{v_n = n, (n = 1,2, \dots)}$. in our theorems, then we obtain the results proved in [1].

Assume that F is a subset of N and represented as the range of a strictly increasing sequence of positive integers, denoted by $F = \left(\lambda(n)\right)_1^\infty$

The polynomials

$$N_n^\lambda(f; y) = \frac{1}{Q_{\lambda(n)}} \sum_{k=0}^{\lambda(n)} q_{\lambda(n)-k}\, s_k(f; y),$$

and

$$R_n^\lambda(f; y) = \frac{1}{Q_{\lambda(n)}} \sum_{k=0}^{\lambda(n)} q_k\, s_k(f; y),$$

introduced in [15], are the particular case (for $u_n = 0$ and $v_n = \lambda\big((n)\big)$) of the means $D_u^v N_n(f; y)$ and $D_u^v R_n(f; y)$, respectively. Therefore Theorem 3.1 implies.

Corollaries

[13] Let $q \in (1, \infty), w \in A_q$, and $f \in Lip(\delta, q, w)$. if any one of the conditions

i. $\mathbf{0 < \delta < 1}$, **and** $(\boldsymbol{q_n}) \in \boldsymbol{AMDS}$

ii. $\mathbf{0 < \delta < 1}, (\boldsymbol{q_n}) \in \boldsymbol{AMIS\ and}\ \ (\boldsymbol{\lambda(n) + 1})\boldsymbol{q_{\lambda(y)}} = \boldsymbol{Q_{\lambda(n)}}$ **holds**

iii. $\mathbf{\delta = 1}, \sum_{j=1}^{v_n - u_n - 1} \left|\Delta\left(q_j\right)\right| = \boldsymbol{O}(\mathbf{1})$, **and (2.1) holds**

iv. $\mathbf{\delta = 1}, \sum_{j=1}^{v_n - u_n - 1} j\left|\Delta\left(q_j\right)\right| = \boldsymbol{O}(\mathbf{1})$

is true, where $\Delta(q_j) = q_j - q_{j+1}$, then

$$\left\| f - N_n^{\lambda}(f) \right\|_{q,w} = O\left(\left(\lambda(n) \right)^{1-\delta} \right), \quad (n = 1,2, \dots).$$

Additionally, for $q_k = 1, \quad k = 0,1, \dots, \lambda(n)$, we derive the polynomials

$$C_n^{\lambda}(f; y) = \frac{1}{\lambda(n) + 1} \sum_{k=0}^{\lambda(n)} s_k(f; y),$$

which for $\lambda(n) = n$, reduce to the ordinary Cesàro mean. Thus, under suitable conditions, the deviation

$$\left\| f - C_n^{\lambda}(f) \right\|_{q,w} = O\left(\left(\lambda(n) \right)^{1-\delta} \right), \quad (n = 1,2, \dots).$$

is also implied from our results.

Finally, Theorem 3.2 implies the following.

Corollary 4.2: Let $1 < q < \infty, w \in A_q, 0 < \delta \leq 1$.Also, Let $(q_n)_{n=0}^{\infty}$ be a sequence of the positive real numbers that satisfies the relation

$$\sum_{\mu=0}^{\lambda(n)-3} \left| \frac{Q_1^{\mu+1}}{\mu + 2} - \frac{Q_1^{\mu+2}}{\mu + 3} \right| = O\left(\frac{Q_1^{\lambda(n)-1}}{\lambda(n)} \right)$$

Then for every $f \in Lip(\delta, q, w)$, the estimate

$$\left\| f - R_n^{\lambda}(f) \right\|_{q,w} = O\left(\left(\lambda(n) \right)^{-\delta} \right), \quad n = 1,2, \dots$$

is satisfied.

Application

a) The importance of Summability and Approximation Theory has been clearly illustrated in numerous fields, such as Fourier Analysis, Analytic Continuation, Quantum Mechanics, Probability Theory, and Signal Analysis.

b) Deferred methods in JavaScript are often associated with promises and async/await to manage asynchronous operations, such as network requests or file I/O. These methods delay execution until the asynchronous operation is complete.

Declarations

I hereby certify that the submission is completely my own work, written wholly in my own words, and that all sources used in researching it are fully recognized and all quotations properly idented.

Conclusions

This paper explores the trigonometric approximation of functions using deferred Voronoi–Nörlund and Riesz means in weighted function spaces. It presents convergence results and error estimates, highlighting the effectiveness of these methods within weighted norms. The findings enhance approximation theory and open avenues for broader applications of these techniques.

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References

Guven, A. (2009). Trigonometric approximation of functions in weighted L^p spaces. Sarajevo J. Math, 5(17), 99-108.

Güven, A., & İsrafilov, D. M. (2010). Trigonometric approximation in generalized Lebesgue spaces $L^p(x)$.

Avşar, A. H., & Yıldırır, Y. E. (2020). Trigonometric approximation in variable exponent weighted Lebesgue spaces using sub-matrix method. J. Math. Anal, 11(6), 1-16.

Muckenhoupt, B. (1972). Weighted norm inequalities for the Hardy maximal function. *Transactions of the American Mathematical Society*, *165*, 207-226.

Armitage, D. H., & Maddox, I. J. (1989). A new type of Cesaro mean. Analysis, 9(1-2), 195-204.

Quade, E. S. (1937). Trigonometric approximation in the mean.

Leindler, L. (2005). Trigonometric approximation in L^p-norm. Journal of Mathematical Analysis and Applications, 302(1), 129-136.

Ky, N. X. (1997). Moduli of mean smoothness and approximation with Ap-weights. In Annales Univ. Sci. Budapest (Vol. 40, pp. 37-48).

Kováik, O., & Rákosnik, J. On spaces $L^p(x)$ and $WL^p(x)$. Czech. Math. J, 41, 592-618.

Chandra, P. (2002). Trigonometric approximation of functions in L^p-norm. Journal of Mathematical Analysis and Applications, 275(1), 13-26.

Agnew, R. P. (1932). On deferred Cesàro means. Annals of Mathematics, 33(3), 413-421.

Jafarov, S. Z. (2017). Derivatives of a polynomial of best approximation and modulus of smoothness in generalized Lebesgue spaces with variable exponent. *Demonstratio Mathematica*, *50*(1), 245-251.

Jafarov, S. Z. (2017). Approximation by means of Fourier trigonometric series in weighted Lebesgue spaces. Sarajevo Journal of Mathematics, 13(2), 217-226.

Deger, U. (2018). On approximation by Nörlund and Riesz sub methods in variable exponent Lebesgue spaces. Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat, 67, 46-59.

Değer, U. (2018). On approximation by Nörlund and Riesz sub methods in variable exponent Lebesgue spaces. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 67(1), 46-59.

Krasniqi, X. Z. (2013). On some results on approximation of functions in weighted L^p spaces. Advances in Pure and Applied Mathematics, 4(4), 389-397.

Krasniqi, X. Z. (2024). Trigonometric Approximation by Deferred Voronoi–Norlund and by Deferred Riesz Means in the Weighted space L^p. Transactions of A. Razmadze Mathematical Institute, 178(1).

Krasniqi, X. Z. (2017). Trigonometric approximation of (signals) functions by Nörlund type means in the variable space L^p(x). Palestine Journal of Mathematics, 6(1), 84-93