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A Novel Approach to Kinetic Equations of Fractional Order Pertaining to Hyper-Bessel Function

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Abstract

Special functions included in fractional kinetic equations have been shown to be helpful in the explanation and resolution of numerous important mathematical and mathematical physics issues. Because arbitrary-order kinetic equations are so important, the goal of this study is to use the Sumudu Transform technique to solve a new fractional-order kinetic equation involving the Hyper Bessel function with their fractional derivatives. MATLAB-generated graphical representations used in our analysis to demonstrate the behavior of these solutions under various parametric values. The outcomes of the study are highly flexible and may lead to both confirmed and maybe undiscovered research findings in this field.

Key words: Generalized fractional kinetic equation, Hyper-Bessel function, Fractional order derivative, Sumudu Transform, Mittag-Leffler function.

2010 Mathematics Subject Classification: 26A33, 33E12, 33E20, 44A99

1. Introduction

Fractional calculus provided the mathematical foundation for solving fractional differential equations, which deals with fractional derivatives and integrals. Fractional differential equations are incredibly significant in the field of applied sciences, especially in dynamic systems, control systems, mathematical physics, and engineering, and have been used recently to develop mathematical models of a variety of physical phenomena. However, fractional derivatives are nonlocal, then the solving fractional differential equations can be challenging. Due to this reason, special functions and numerical techniques are frequently used to acquire approximate solutions for fractional differential equations. Researchers are always creating new methods and resources to improve comprehension and solutions. In recent decades, fractional order differential equations have proven to be highly relevant in many branches of engineering and physical science.

Many important applications of fractional kinetics equations have the crucial role in the fields of modeling complicated physical systems, engineering, management, physical science, and social sciences. Many researchers have looked at fractional kinetic equations (FKE) that include specific special functions and contributed many significant results in the bag of the literature of fractional calculus. Haubold and Mathai's [9] and Saxena and Kalla's [15] contributions were significant steps in this direction. Several scholars have studied FKE extensively to explore new applications. Agarwal and Bhargava [1] Agrawal et al. [2], Baricz, and Mehrez [3-4], Chaurasia and Pandey [6], Gupta and Parihar [7], Gupta et al. [8] Kumar et al. [11], Nisar et al. [13], Saichev and Zaslavsky [14], Saxena et al. [16], Singh et al. [22], Sharma and Bhargava [17-20], Suthar et al. [21], Jain and Bhargava [10], and Zaslavsky [25] etc. are few of them, researched on the extension of fractional kinetic equations related to special functions and produced several intriguing findings of significant importance.

Saxena and Kalla [15] defined generalized FKE as

$$N(t) - N_0 f(t) = -c^v \, {}_0D_t^{-v} \, N(t), Re(v) > 0 \tag{1}$$

$N(t)$: number density of species at the time t, $N_0 = N(t = 0)$, $c \neq 0$ is a constant, $f(t) \in L(0,\infty)$ and ${}_0D_t^{-v}$ is the fractional differential operator [12].

Due to the importance of Hyper-Bessel functions which appeared in the various field of applied and pure mathematical frameworks. Hyper-Bessel has several uses in science and engineering and is frequently used to solve fractional order differential equations. Here, in this paper, we consider the function f(t) as a Hyper-Bessel function and acquire the solution of FKE with utilizing the powerful Sumudu Transform [24] technique.

The Hyper-Bessel function [5] in generalized form defined as

$$\mathfrak{J}^{(\vartheta)}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}(\xi) = \left(\frac{\xi}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i} \widetilde{\mathfrak{J}}^{(\vartheta)}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}(\xi)$$

where

$$\widetilde{\mathfrak{J}}^{(\vartheta)}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}(\xi) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{\xi}{\vartheta+1}\right)^{\ell(\vartheta+1)} \tag{2}$$

where $\xi, \tau_i \in \mathbb{C}, \Re(\tau_i+1) > 0, |\xi| < \infty, i = 1,2,\ldots,\vartheta$. For more details, we refer [3, 4, 5].

The fractional derivative [12] of order λ of the function $f(t) = t^\beta$ is given by

$$D^\lambda t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\lambda+1)} t^{\beta-\lambda} \quad ;\Re(\beta) > -1, 0 < \Re(\lambda) < 1, t > 0 \tag{3}$$

So, in view of (2) and (3) we have

$$\begin{aligned} {}_0D_t^\lambda\left(\mathfrak{J}^{(\vartheta)}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}(\xi)\right) &= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)} \\ &\times \frac{\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)}{\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)-\lambda+1\right)}(\xi)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)-\lambda} \end{aligned} \tag{4}$$

The Sumudu Transform [24] is described as follows over the set 'Ω' of functions as

$$S[f(t)] = G(u) = \frac{1}{u}\int_0^\infty f(t)e^{-t/u}\,dt \qquad ;\ 0 < t < \infty, u \in (-\tau_1,\tau_2) \tag{5}$$

where $\Omega = \{f(t) |\ Me^{|t|/\tau_j}\ ; if\ t \in (-1)k \times [0,\infty)\}, M\ is$ a constant and $\tau_1, \tau_2 > 0$
We need the following to support our major findings:

Lemma 1:

$$S\left(\mathfrak{J}^{(\vartheta)}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}(\xi)\right) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)}(u)^{\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)} \tag{6}$$

Lemma 2:

$$\begin{aligned} &S\left[{}_0D_t^\lambda\left(\mathfrak{J}^{(\vartheta)}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}(\xi)\right)\right] \\ &= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)}(u)^{\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)-\lambda\right)} \end{aligned} \tag{7}$$

Proof: In view of definition (2), (4), (5) and doing simple calculations we can obtain (6) and (7) easily.

In this manuscript, we acquire the findings in terms of ML function [23], defined as

$$E_{\theta,\delta}(z)=\sum_{l=0}^{\infty}\frac{z^l}{\Gamma(l\theta+\delta)}\ \ ;\ \Re(\theta),\Re(\delta)>0,\theta,\delta\ \epsilon\ \mathbb{C} \tag{8}$$

2. Main Results

Theorem1: If $c>0, v>0, |t|<\infty, \Re(\tau_i+1)>0, \Re(\tau)>0, \Re(\zeta)>0, t,\tau_i,\zeta,\eta,\xi\in\mathbb{C}, r\in\mathbb{R}, \eta\in(0,1)\cup\mathbb{N},\ \Re(\xi)>0$, then the FKE

$$N(t)-N_0\left\{\mathfrak{I}^{(\vartheta)}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}(t)\right\}=-c^v\,{}_0D_t^{-v}\,N(t) \tag{9}$$

and its solution is

$$N(t)=N_0\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)t^{\ell(\vartheta+1)+\sum_{i=1}^{\vartheta}\tau_i}}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!\,(\vartheta+1)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)}}\times E_{v,\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1}(-c^v t^v) \tag{10}$$

Proof: Using (5) on (9),

$$N(u)=N_0\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{u}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)}(1+c^v u^v)^{-1}$$

$$=N_0\sum_{j=0}^{\infty}\frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{u}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)}\sum_{\alpha=0}^{\infty}(-c^v u^v)^{\alpha}$$

$$=N_0\sum_{j=0}^{\infty}\frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+1\right)}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)}\sum_{\alpha=0}^{\infty}(-c^v)^{\alpha}u^{\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)+v\alpha\right)} \tag{11}$$

Now using the definition of inverse Sumudu Transform and with simple evaluation we achieve the result (10).

Mathematical and Graphical Explanation

We obtain several values of N(t) for (10), by varying t while keeping v constant. These values are interpreted in table 1 and the 3D and 2D graphs 1(a), 1(b), which illustrate the behavior of the result for the kinetic equation (9).

Table 1: The values of $N(t)$ with fix v for (10)

t	N(t) at $v=0.1$	N(t) at $v=0.5$	N(t) at $v=0.9$	N(t) at $v=1.3$
0	0	0	0	0
0.08	2.40124E+15	-2.4822E+14	2.68905E+13	-2.91384E+12
0.16	3.63589E+15	-4.85009E+14	6.89933E+13	-9.85605E+12
0.24	4.73184E+15	-7.30177E+14	1.21619E+14	-2.04129E+13
0.32	5.77971E+15	-9.8714E+14	1.83711E+14	-3.45584E+13
0.4	6.81295E+15	-1.25741E+15	2.54859E+14	-5.23604E+13

0.48	7.84798E+15	-1.54188E+15	3.34905E+14	-7.39265E+13
0.56	8.89422E+15	-1.84112E+15	4.23812E+14	-9.93848E+13
0.64	9.95756E+15	-2.15559E+15	5.21612E+14	-1.28875E+14
0.72	1.1042E+16	-2.48563E+15	6.28378E+14	-1.62545E+14
0.8	1.21504E+16	-2.83156E+15	7.44211E+14	-2.00546E+14
0.88	1.32849E+16	-3.19364E+15	8.69228E+14	-2.43033E+14
0.96	1.44471E+16	-3.57213E+15	1.00356E+15	-2.90163E+14
1.04	1.56384E+16	-3.96726E+15	1.14736E+15	-3.42094E+14
1.12	1.68597E+16	-4.37925E+15	1.30076E+15	-3.98988E+14
1.2	1.81121E+16	-4.80835E+15	1.46393E+15	-4.61005E+14
1.28	1.93963E+16	-5.25475E+15	1.63702E+15	-5.28307E+14
1.36	2.0713E+16	-5.71868E+15	1.82019E+15	-6.01057E+14
1.44	2.20629E+16	-6.20036E+15	2.01362E+15	-6.79418E+14
1.52	2.34464E+16	-6.7E+15	2.21748E+15	-7.63555E+14
1.6	2.48642E+16	-7.21782E+15	2.43194E+15	-8.53632E+14
1.68	2.63167E+16	-7.75405E+15	2.65716E+15	-9.49816E+14
1.76	2.78045E+16	-8.30889E+15	2.89335E+15	-1.05227E+15
1.84	2.93279E+16	-8.88258E+15	3.14066E+15	-1.16117E+15
1.92	3.08875E+16	-9.47535E+15	3.39929E+15	-1.27667E+15
2	3.24837E+16	-1.00874E+16	3.66943E+15	-1.39896E+15

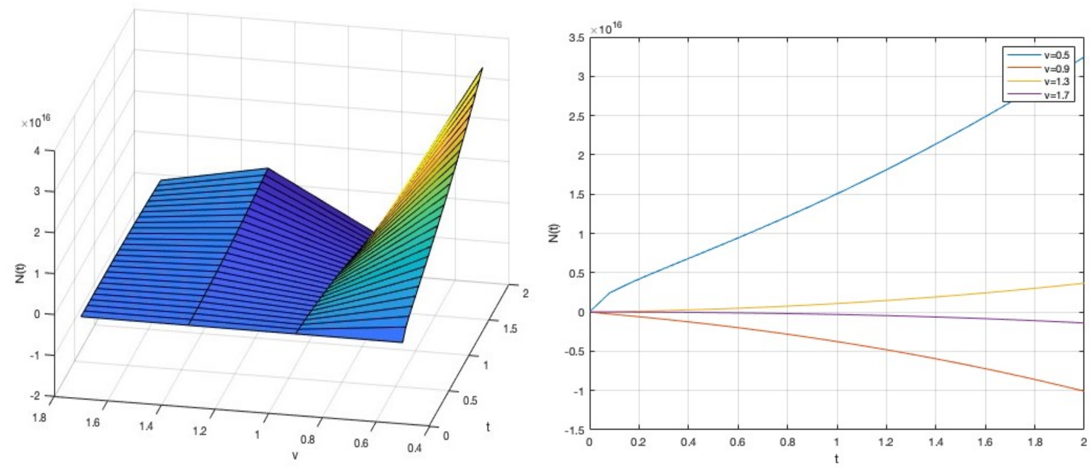

Figure 1 **(a) 3D graph for (10)** **(b) 2D graph for (10)**

Theorem-2: If $c > 0, v > 0, |t| < \infty, \Re(\tau_i + 1) > 0, \Re(\tau) > 0, \Re(\xi) > 0$ $t, \tau_i, \zeta, \eta, \xi \in \mathbb{C}, d \neq c, d > 0, r \in \mathbb{R}, \eta \in (0,1) \cup \mathbb{N}, \Re(\zeta) > 0$, then the FKE

$$N(t) - N_0 \left\{ \mathfrak{J}^{(\vartheta)}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}(d^v t^v) \right\} = -c^v \, {}_0D_t^{-v} \, N(t) \tag{12}$$

and its solution is

$$N(t) = N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma\left(v\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)\right) + 1\right)}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!} \times \left(\frac{dt}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)} E_{v,\left(v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)+1\right)}(-c^v t^v) \tag{13}$$

Proof: Using (5) on (12),

$$N(u) = N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma\left(v\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)\right) + 1\right)}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!} \left(\frac{ud}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)} (1 + c^v u^v)^{-1}$$

$$= N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma\left(v\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)\right)+1\right)}{\Gamma(\ell+\tau_1+1) \ldots \Gamma(\ell+\tau_\vartheta+1)\ell!} \left(\frac{d}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(n+1)\right)} \sum_{\alpha=0}^{\infty} (-c^v u^v)^\alpha u^{\left(v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)\right)}$$

$$= N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma\left(v\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)\right)+1\right)}{\Gamma(\ell+\tau_1+1) \ldots \Gamma(\ell+\tau_\vartheta+1)\ell!} \left(\frac{d}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)} \sum_{\alpha=0}^{\infty} (-c^v)^\alpha u^{\left(v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)+v\alpha\right)}$$

Now using the definition of inverse Sumudu Transform, (8) and then with simple evaluation we achieve the result (13).

Mathematical and Graphical Explanation

We obtain several values of N(t) for (13), by varying t while keeping v constant. These values are interpreted in table 2 and the 3D and 2D graphs 2(a), 2(b), which illustrate the behavior of the result for the kinetic equation (12).

Table 2: The values of $N(t)$ with fix v for (13)

t	N(t) at $v = 0.1$	N(t) at $v = 0.5$	N(t) at $v = 0.9$	N(t) at $v = 1.3$
2	0.001114394	-0.001479266	0.001694018	-0.002810911
2.04	0.001411325	-0.001889534	0.002204592	-0.003683596
2.08	0.001779217	-0.002402634	0.002854421	-0.004803091
2.12	0.002233185	-0.003041721	0.003677696	-0.006232666
2.16	0.002791193	-0.003834604	0.004716125	-0.008050212
2.2	0.003474527	-0.004814576	0.006020399	-0.010351277
2.24	0.004308341	-0.006021378	0.007651925	-0.013252652
2.28	0.005322272	-0.007502306	0.009684839	-0.016896616
2.32	0.006551151	-0.009313494	0.012208367	-0.021455935
2.36	0.008035809	-0.011521387	0.015329575	-0.027139747
2.4	0.009823997	-0.014204434	0.01917656	-0.03420046
2.44	0.011971426	-0.017455028	0.023902162	-0.042941822
2.48	0.014542956	-0.021381732	0.029688251	-0.053728359
2.52	0.017613937	-0.026111817	0.036750692	-0.066996362
2.56	0.021271729	-0.031794168	0.045345067	-0.083266678
2.6	0.025617424	-0.038602589	0.055773274	-0.103159561
2.64	0.030767788	-0.046739569	0.068391107	-0.127411884
2.68	0.036857457	-0.056440575	0.083616974	-0.156897063
2.72	0.044041411	-0.067978914	0.101941881	-0.192648072
2.76	0.052497766	-0.081671262	0.123940883	-0.235883997
2.8	0.062430915	-0.097883936	0.150286173	-0.28804063
2.84	0.074075067	-0.117039992	0.181762044	-0.350805664
2.88	0.087698228	-0.139627267	0.219281966	-0.426159128
2.92	0.103606678	-0.166207477	0.263908058	-0.516419801
2.96	0.122150002	-0.1974265	0.316873269	-0.624298407
3	0.143726743	-0.234025996	0.379606636	-0.752958527

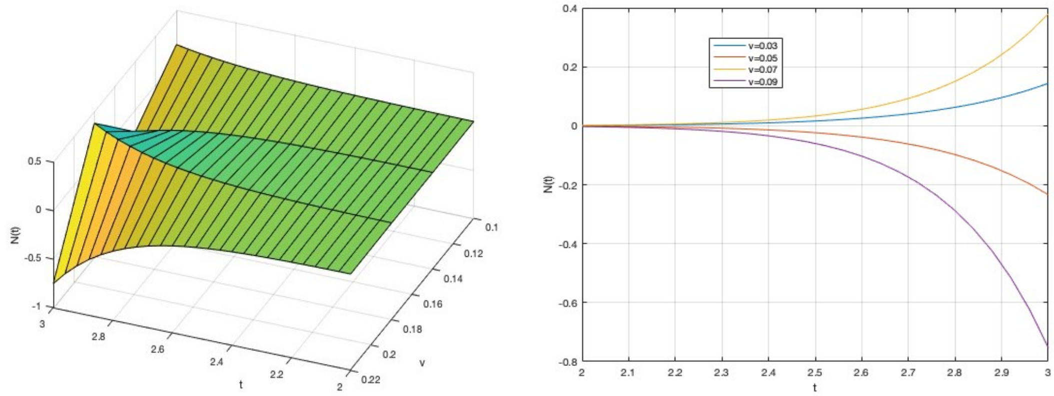

Figure 2 (a) 3D graph for (13) (b) 2D graph for (13)

Theorem-3: If $c > 0, v > 0, |t| < \infty, \Re(\tau_i + 1) > 0, \Re(\tau) > 0, \Re(\zeta) > 0, t, \tau_i, \zeta, \eta, \xi \in \mathbb{C}, r \in \mathbb{R}, \eta \in (0,1) \cup \mathbb{N},\ \Re(\xi) > 0, \lambda \neq v$, then the FKE

$$N(t) - N_0\left({}_0D_t^{\lambda}\left(\mathfrak{J}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}^{(\vartheta)}(t)\right)\right) = -c^v\,{}_0D_t^{-v}\,N(t) \tag{14}$$

and its solution is

$$N(t) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1) + 1\right)t^{\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1) - \lambda}}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1)} E_{v,\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1) - \lambda + 1}(-c^v t^v) \tag{15}$$

Proof: Using (5) on (14),

$$N(s) = N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1) + 1\right)}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1)} (u)^{\left(\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1) - \lambda\right)} \times (1 + c^v u^v)^{-1}$$

$$= N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1) + 1\right)}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1)} (u)^{\left(\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1) - \lambda\right)} \sum_{\alpha=0}^{\infty}(-c^v u^v)^{\alpha}$$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1) + 1\right)}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1)} \sum_{k=0}^{\infty}(-c^v)^{\alpha}(u)^{\left(\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1) - \lambda + v\alpha\right)}$$

Now taking inverse Sumudu Transform, we have

$$N(t) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1) + 1\right)t^{\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1) - \lambda}}{\Gamma(\ell+\tau_1+1)\ldots\Gamma(\ell+\tau_\vartheta+1)\ell!}\left(\frac{1}{\vartheta+1}\right)^{\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1)} \times \sum_{\alpha=0}^{\infty} \frac{(-c^v t^v)^{\alpha}}{\Gamma\left(\sum_{i=1}^{\vartheta}\tau_i + \ell(\vartheta+1) - \lambda + 1 + v\alpha\right)}$$

By using (8), we get the result (15).

Mathematical and Graphical Explanation

We obtain several values of N(t) for (15), by varying t while keeping v constant. These values are interpreted in table 3 and the 3D and 2D graphs 3(a), 3(b), which illustrate the behavior of the result for the kinetic equation (14).

Table 3: The values of $N(t)$ with fix v for (15)

t	$\mathbf{N(t)}$ **at** $\boldsymbol{v = 0.1}$	$\mathbf{N(t)}$ **at** $\boldsymbol{v = 0.5}$	$\mathbf{N(t)}$ **at** $\boldsymbol{v = 0.9}$	$\mathbf{N(t)}$ **at** $\boldsymbol{v = 1.3}$
0	0	0	0	0
0.2	5.88804E+91	-7.86187E+90	1.12174E+90	-1.61069E+89
0.4	9.38709E+91	-1.60481E+91	2.99548E+90	-5.66357E+89
0.6	1.27784E+92	-2.513E+91	5.47438E+90	-1.21451E+90
0.8	1.62491E+92	-3.52106E+91	8.54498E+90	-2.1218E+90
1	1.98662E+92	-4.63432E+91	1.22152E+91	-3.30807E+90
1.2	2.36626E+92	-5.85667E+91	1.65008E+91	-4.79446E+90
1.4	2.76576E+92	-7.19136E+91	2.14209E+91	-6.60273E+90
1.6	3.18643E+92	-8.64145E+91	2.69966E+91	-8.75493E+90
1.8	3.62926E+92	-1.02099E+92	3.32506E+91	-1.12734E+91
2	4.09504E+92	-1.18996E+92	4.02061E+91	-1.41806E+91
2.2	4.58449E+92	-1.37136E+92	4.78874E+91	-1.74992E+91
2.4	5.09825E+92	-1.5655E+92	5.63191E+91	-2.12523E+91
2.6	5.63695E+92	-1.77269E+92	6.55269E+91	-2.5463E+91
2.8	6.2012E+92	-1.99325E+92	7.55367E+91	-3.01548E+91
3	6.79162E+92	-2.22752E+92	8.63755E+91	-3.53517E+91
3.2	7.40882E+92	-2.47584E+92	9.80706E+91	-4.10778E+91
3.4	8.05343E+92	-2.73857E+92	1.1065E+92	-4.73578E+91
3.6	8.7261E+92	-3.01609E+92	1.24143E+92	-5.42169E+91
3.8	9.4275E+92	-3.30878E+92	1.3858E+92	-6.16806E+91
4	1.01583E+93	-3.61704E+92	1.53991E+92	-6.97751E+91
4.2	1.09193E+93	-3.94129E+92	1.70407E+92	-7.85273E+91
4.4	1.17111E+93	-4.28194E+92	1.87861E+92	-8.79644E+91
4.6	1.25345E+93	-4.63946E+92	2.06386E+92	-9.81147E+91
4.8	1.33904E+93	-5.0143E+92	2.26018E+92	-1.09007E+92
5	1.42795E+93	-5.40694E+92	2.46792E+92	-1.20671E+92

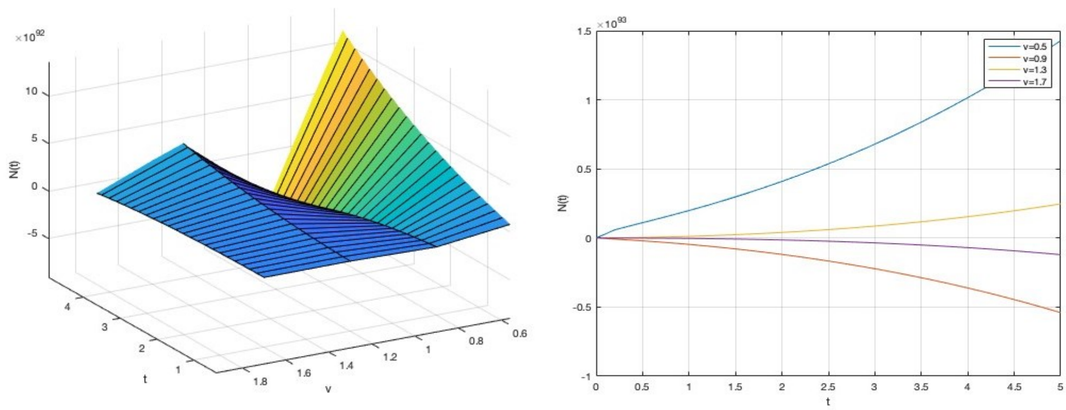

Figure 3 (a) 3D graph for (15) (b) 2D graph for (15)

Theorem-4: If $c > 0, d > 0, v > 0, |t| < \infty, \Re(\tau_i + 1) > 0, \Re(\tau) > 0, \Re(\zeta) > 0, t, \tau_i, \zeta, \eta, \xi \in \mathbb{C}, \eta \in (0,1) \cup \mathbb{N}, \Re(\xi) > 0, \lambda \neq v, d \neq c$, then the FKE

$$N(t) - N_0 \left({}_0D_t^{\lambda} \left(\mathfrak{J}_{\tau_1,\tau_2,\ldots,\tau_\vartheta}^{(\vartheta)} (d^v t^v) \right) \right) = -c^v \, {}_0D_t^{-v} \, N(t) \tag{16}$$

is given as

$$N(t) = N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \Gamma\left(v \left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) \right) + 1 \right)}{\Gamma(\ell + \tau_1 + 1) \ldots \Gamma(\ell + \tau_\vartheta + 1) \ell!} \left(\frac{d}{\vartheta + 1} \right)^{v\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1) \right)}$$

$$\times (t)^{v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)-\lambda} E_{v,v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)-\lambda+1}(-c^v t^v) \quad (17)$$

Proof: Doing the same process as we have done for Theorem 1 and using Lemma 2, we can achieve the result (17).

Mathematical and Graphical Explanation

We obtain several values of N(t) for (17), by varying t while keeping v constant. These values are interpreted in table 4 and the 3D and 2D graphs 4(a), 4(b), which illustrate the behavior of the result for the kinetic equation (16).

Table 4: The values of $N(t)$ with fix v for (17)

t	N(t) at $v = 0.1$	N(t) at $v = 0.5$	N(t) at $v = 0.9$	N(t) at $v = 1.3$
1.5	989.2757986	-1373.487548	1953.719148	-4275.797537
1.52	1361.174549	-1900.522248	2732.823474	-6004.230874
1.54	1865.720527	-2619.686998	3807.102295	-8397.381892
1.56	2547.886046	-3597.621875	5282.937249	-11698.71367
1.58	3467.196015	-4923.032803	7303.288022	-16236.82745
1.6	4702.257497	-6713.712035	10059.75851	-22454.02663
1.62	6356.663494	-9125.729139	13808.51895	-30944.1534
1.64	8566.685676	-12365.45611	18891.29636	-42502.65047
1.66	11511.29846	-16705.29561	25763.02816	-58192.73967
1.68	15425.24578	-22504.25281	35028.28162	-79432.85474
1.7	20616.08617	-30234.85096	47489.21556	-108112.1204
1.72	27486.45034	-40518.36982	64208.75967	-146742.875
1.74	36563.14558	-54171.02501	86593.8889	-198662.1817
1.76	48535.27807	-72264.56394	116505.4846	-268298.2248
1.78	64304.28938	-96205.90739	156403.448	-361522.809
1.8	85049.78541	-127842.0256	209538.6739	-486118.3581
1.82	112316.3733	-169598.3546	280207.486	-652397.5468
1.84	148128.5501	-224661.9443	374089.5801	-874026.9443
1.86	195143.199	-297224.4811	498697.9799	-1169124.134
1.88	256852.7113	-392805.7617	663979.7551	-1561722.584
1.9	337856.561	-518685.7008	883120.4064	-2083732.689
1.92	444225.8423	-684483.3711	1173624.437	-2777574.627
1.94	583994.647	-902936.0961	1558771.96	-3699724.181
1.96	767825.3192	-1190951.947	2069589.389	-4925504.086
1.98	1009913.218	-1571037.608	2747525.97	-6555581.284
2	1329223.026	-2073244.023	3648103.645	-8724810.316

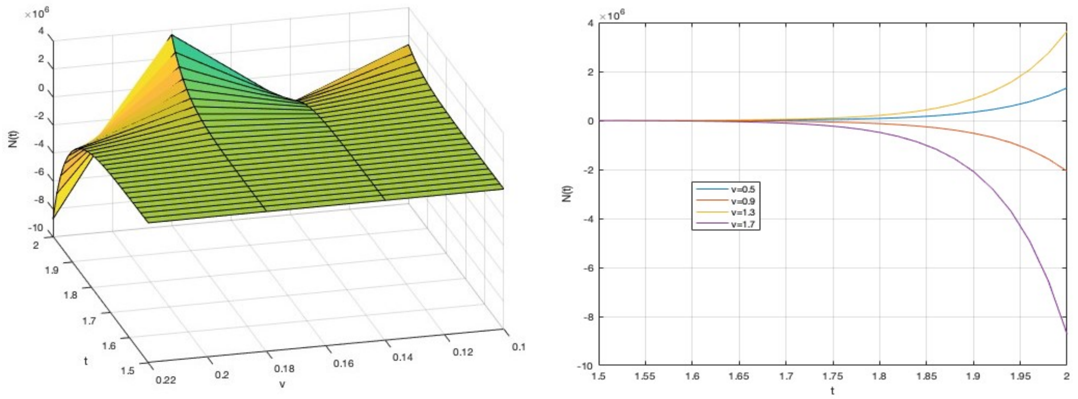

Figure 4 (a) 3D graph for (17) (b) 2D graph for (17)

3. Particular Cases

(i) Substituting $d = c$ in (12), then the FKE reduces as

$$N(t) - N_0 \left\{ \mathfrak{J}^{(\vartheta)}_{\tau_1,\tau_2,\dots,\tau_\vartheta}(c^v t^v) \right\} = -c^v \, {}_0D_t^{-v} \, N(t) \tag{18}$$

and its solution is

$$N(t) = N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma\left(v\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)\right)+1\right)}{\Gamma(\ell+\tau_1+1)\dots\Gamma(\ell+\tau_\vartheta+1)\ell!} \times \left(\frac{ct}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)} \times E_{v,\left(v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)+1\right)}(-c^v t^v) \tag{19}$$

(ii) Substituting $d = c$ in (16), then the FKE reduces in

$$N(t) - N_0 \left({}_0D_t^{\lambda} \left(\mathfrak{J}^{(\vartheta)}_{\tau_1,\tau_2,\dots,\tau_\vartheta}(c^v t^v) \right) \right) = -c^v \, {}_0D_t^{-v} \, N(t) \tag{20}$$

and its solution is

$$N(t) = N_0 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma\left(v\left(\sum_{i=1}^{\vartheta} \tau_i + \ell(\vartheta+1)\right)+1\right)}{\Gamma(\ell+\tau_1+1)\dots\Gamma(\ell+\tau_\vartheta+1)\ell!} \left(\frac{c}{\vartheta+1}\right)^{v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)} \times (t)^{v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)-\lambda} \times E_{v,v\left(\sum_{i=1}^{\vartheta}\tau_i+\ell(\vartheta+1)\right)-\lambda+1}(-c^v t^v) \tag{21}$$

More special cases of our findings can be produced by entering appropriate parametric values in the special function involved here, but we do not record them explicitly here.

Conclusion

In this paper, we suggest applying the Sumudu transform technique to obtain the solution of some novel generalized fractional kinetic equations including Hyper-Bessel function and their fractional derivatives. Additionally, the outcomes are shown in form of the Mittag-Leffler function. Due to the widespread application of fractional kinetic equations, which are applicable in a variety of scientific and engineering fields. Additionally, we have not only published these solutions but also provided a convincing illustration of how their behavior by graphical and numerically representations under the various parametric settings by employing MATLAB. The solutions of four equations (9), (12), (14), and (16) remain non-negative. Furthermore, we may state that the solution is stable and convergent based on the figures N(t) > 0 for t > 0 and N(t) → ∞ and t → ∞. The current work is to suggest a very successful analytical strategy based on the Sumudu Transform which is touched the several different areas of research . The conclusions are significant in both pure mathematics and the applied framework.

Data Availability

The dataset used to support the endings of this study is available from the corresponding author upon request.

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Conflicts of Interest

All authors declare there is no conflicts of interests.

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