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Analytical Investigation of Fingero-Imbibition Phenomenon with Gravitational and Heterogeneity Effect: A Mathematical Approach

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Abstract

The current study presents a comprehensive mathematical analysis of fingero-imbibition phenomenon within a heterogeneous porous medium. Fingero-imbibition, characterized by the preferential flow of a wetting liquid through porous media, plays a pivotal role in various natural and engineered systems. In this study, we focus on vertical heterogeneity, acknowledging the significance of gravity-driven effects on fluid movement in porous structures. The variational iteration method is employed to conduct a rigorous mathematical analysis for this study, considering suitable initial and boundary conditions. The findings are presented through numerical analyses and graphical representations, utilizing MATLAB for the visualization of results.

Keywords: Fingero-imbibition, Heterogeneous porous medium, Variational iteration method, Secondary oil retrieval process, Non-linear partial differential equation (PDE)

Introduction

Understanding fingero-imbibition is crucial in various applications, including secondary oil recovery. In the oil industry, for instance, optimizing recovery processes requires a detailed comprehension of how fluids move through porous rock formations. By studying the dynamics of fingero-imbibition, researchers and engineers can develop strategies to enhance oil recovery efficiency and minimize environmental impact. This paper discusses the fingero-imbibition phenomenon within vertically oriented porous medium of heterogeneous.

The phenomenon of fingero-imbibition occurs when a wetting fluid infiltrates a porous medium, such as soil or rock, and establishes preferential flow paths due to variations in the medium's properties. These variations can include differences in pore size, permeability, or capillary forces. As the wetting fluid advances, it tends to follow these preferential paths, creating finger-like structures within the porous medium.

If a porous medium saturated with one phase (such as oil) comes into interaction with another phase (water), which exhibits a preference for wetting, a natural movement of the wetting phase infiltrates the medium, accompanied by the simultaneous counter movement of the native phase without the need for exterior forces. This phenomenon is recognized as imbibition. Additionally, when a phase (oil) confined within a porous medium is supplanted by another phase with lower viscosity, rather than a uniform supplanting of the entire front, protuberances may form and rapidly traverse the porous medium, resulting in the fingering phenomenon. The co-occurrence of both imbibition and fingering phenomenon was coined as "fingero-imbibition" by Verma [2].

Numerous researchers have explored fingero-imbibition by seeking both analytical and numerical solutions. Meher et. al. (2012) employed the approach of Adomian decomposition to study phenomena of fingero-imbibition in two-phase movement within porous media [13]. Parikh et. al. (2013) investigated the fingero-imbibition phenomena in a vertically oriented homogeneous porous matrix using Functional separable method [2]. Lyiola (2013) introduced the method of q-Homotopy analysis to study phenomena of fingero-imbibition in two-phase movement within porous media [11]. Lyiola and Folarin (2014) explored the time-fractional aspect of fingero-imbibition in two-phase movement within porous media, presenting an approximate analytical study exploiting Homotopy analysis approach [12]. Choksi and Singh (2017) examined the fingero-imbibition in the context of two-phase movement within homogeneous porous media with the inclusion of a magnetic field impact [4]. Prajapati and Desai (2017) determined an analytical approximate solution for the phenomenon of fingero-imbibition by employing the method of Optimal homotopy analysis [5]. Patel and Desai (2017) extended the application of the method Homotopy analysis to investigate fingero-imbibition in a Heterogeneous porous medium [9]. Pathak and Singh (2018) focused on the phenomenon of fingero-imbibition in slanting porous media, utilizing the method of Optimal homotopy analysis [14].

This study investigates the phenomenon of fingero-imbibition, where in compactible fluids (oil and water) traverse a heterogeneous porous material. The water is introduced vertically downloads, influenced by both gravity and mean capillary forces. The governing equation for this one-dimensional, non-linear, partial differential equation (PDE) is addressed using the method of Variational iteration [7, 8] with appropriate initial and boundary conditions. Notably, prior research on this topic has lacked consideration of heterogeneity and gravitational effects vertically. The primary aim of current study is to elucidate the saturation of the wetting fluid (water) during the occurrence of fingero-imbibition in the oil retrieval procedure.

Statement of the problem

Imagine a scenario where a consistent water injection takes place in an oil-saturated heterogeneous porous medium. In the course of the secondary oil retrieval process, imbibition and fingering phenomena happen concurrently. Fingering occurs for the less viscous phase of the fluid and preferentially wetting phase, characterizing the fingero-imbibition phenomenon. This process begins with water injection through imbibition, leading to the formation of fingers during the displacement. In a heterogeneous porous medium, properties like permeability and porosity can differ from one location to another. The permeability and porosity of this medium are treated as functions solely dependent on the variable z.

In this context, we presume the applicability of Darcy's rule to the examined flow system for mathematically formulating the fingero-imbibition phenomenon. The fingers, characterized by distinct and irregular sizes and shapes, are approximated as rectangles. Only the average cross-sectional region employed by the fingers is taken into consideration, overlooking the individual finger's specific size and shape. The saturation (Sw) of the injected water in the fingero-imbibition phenomenon is consequently expressed as the average cross-sectional region employed by the injected water at a given time *t* and depth *z*.

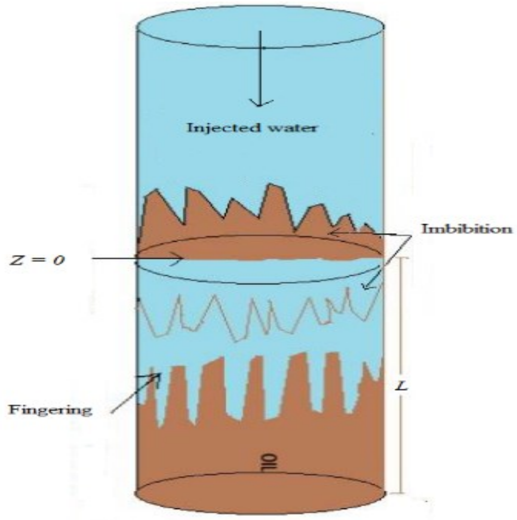

Figure – 1: Phenomenon of Fingero-imbibition

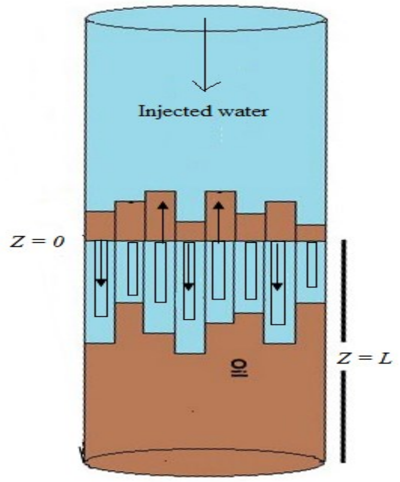

Figure – 2: Schematic diagram of Phenomenon of Fingero-imbibition

Mathematical Framework

In the secondary oil retrieval process, under the assumption that Darcy rule is applicable to the analyzed flow system, the velocities of injected water (V_w) and oil (V_o) can be signified as follows [1, 6]:

$$V_w = -\frac{K_w}{\mu_w} K \left(\frac{\partial P_w}{\partial z} + \rho_w g \right) \tag{1}$$

$$V_o = -\frac{K_o}{\mu_o} K \left(\frac{\partial P_o}{\partial z} + \rho_o g \right) \tag{2}$$

The continuity equation for the injected water can be expressed as

$$\varphi \left(\frac{\partial S_w}{\partial t} \right) + \frac{\partial V_w}{\partial z} = 0$$

(3)

The disparity in pressure between the non-wetting phase (oil) and the wetting phase (water) is denoted by capillary pressure.

$$P_c(S_w) = P_o - P_w \tag{4}$$

Capillary pressure is a continuous linear function of the form [10],

$$P_c = -\beta S_w$$

(5)

In accordance with Scheidegger and Johnson [1], the conventional correlation between phase saturation and relative permeability is taken into account as:

$$K_w = S_w,\ K_o = 1 - \alpha S_w \quad (\alpha = 1.11) \tag{6}$$

In the case of a uniformly heterogeneous porous medium, we consider the permeability and porosity to be functions solely dependent on the variable z [3].

Permeability: $K(z) = K_0(1 + bz)$

Porosity: $\varphi(z) = \dfrac{1}{a_1 - a_2 z}$

where $K_0, b,\ a_1$ and a_2 are positive constants.

Here $\varphi(z)$ is incapable to surpass unity, we suppose that $a_1 - a_2 z \geq 1$.

In simplified manner, $K \propto \varphi$ [15]

Therefore, $K = K_c \varphi$ (7)

At the common interface, condition for Counter-current imbibition phenomenon is presented by,

$$V_w + V_o = 0$$

From (1) and (2),

$$\left(\frac{K_w}{\mu_w}\right) K \left(\frac{\partial P_w}{\partial z} + \rho_w g\right) + \left(\frac{K_o}{\mu_o}\right) K \left(\frac{\partial P_o}{\partial z} + \rho_o g\right) = 0 \tag{8}$$

From (8) and (4),

$$\left(\frac{K_w}{\mu_w} + \frac{K_o}{\mu_o}\right)\frac{\partial P_w}{\partial z} + \left(\frac{K_o}{\mu_o}\right)\frac{\partial P_c}{\partial z} = -\left(\frac{K_o}{\mu_o}\rho_o + \frac{K_w}{\mu_w}\rho_w\right) g$$

Therefore,

$$\frac{\partial P_w}{\partial z}=-\left[\frac{\left(\frac{K_w}{\mu_w}\rho_w+\frac{K_o}{\mu_o}\rho_o\right)g+\left(\frac{K_o}{\mu_o}\right)\frac{\partial P_c}{\partial z}}{\frac{K_w}{\mu_w}+\frac{K_o}{\mu_o}}\right] \tag{9}$$

By using the equation (9) into equation (1), we acquire

$$V_w=-\left(\frac{K_w}{\mu_w}\right)K\left[\frac{\left(\frac{K_o}{\mu_o}\right)(\rho_w-\rho_o)g-\left(\frac{K_o}{\mu_o}\right)\frac{\partial P_c}{\partial z}}{\frac{K_w}{\mu_w}+\frac{K_o}{\mu_o}}\right] \tag{10}$$

Using the equation (10) into equation (3), we get

$$\varphi\left(\frac{\partial S_w}{\partial t}\right)+\frac{\partial}{\partial z}\left[K\left(\frac{\frac{K_wK_o}{\mu_w\mu_o}}{\frac{K_w}{\mu_w}+\frac{K_o}{\mu_o}}\right)\frac{\partial P_c}{\partial S_w}\frac{\partial S_w}{\partial z}\right]-\frac{\partial}{\partial z}\left[K(\rho_w-\rho_o)g\left(\frac{\frac{K_wK_o}{\mu_w\mu_o}}{\frac{K_w}{\mu_w}+\frac{K_o}{\mu_o}}\right)\right]=0 \tag{11}$$

Viscous oil and water are contained in the current analysis, so we have [1],

$$\frac{\frac{K_wK_o}{\mu_w\mu_o}}{\frac{K_w}{\mu_w}+\frac{K_o}{\mu_o}}\approx\frac{K_o}{\mu_o}=\frac{1-\alpha S_w}{\mu_o} \tag{12}$$

Using values from (5), (6), (7) and (12) into equation (11), we obtain

$$\varphi\left(\frac{\partial S_w}{\partial t}\right)=\frac{K_c\beta}{\mu_o}\frac{\partial}{\partial z}\left[\varphi(1-\alpha S_w)\frac{\partial S_w}{\partial z}\right]+\frac{K_c(\rho_w-\rho_o)g}{\mu_o}\frac{\partial}{\partial z}\left[\varphi(1-\alpha S_w)\right] \tag{13}$$

For simplification, taking $S=1-\alpha S_w$ in equation (13), we find

$$\frac{\partial S}{\partial t}=\frac{K_c\beta}{\mu_o}\left[\frac{\partial}{\partial z}\left(S\frac{\partial S}{\partial z}\right)+S\frac{\partial S}{\partial z}\frac{a_2}{a_1}\right]-\frac{\alpha K_c(\rho_w-\rho_o)g}{\mu_o}\left(\frac{\partial S}{\partial z}+S\frac{a_2}{a_1}\right) \tag{14}$$

$\left(\because \frac{1}{\varphi}\frac{\partial\varphi}{\partial z}=\frac{\partial(\log\varphi)}{\partial z}=\frac{\partial}{\partial z}\left(-\log a_1+\frac{a_2}{a_1}z\right)=\frac{a_2}{a_1}\right)$ (Omitting higher order terms of z)

Utilizing dimensionless variables,

$Z = \frac{z}{L}$ and $T = \frac{K_c \beta t}{\mu_w L^2}$

Equation (14) reduces to,

$$\frac{\partial S}{\partial T} = \frac{\partial}{\partial Z}\left(S \frac{\partial S}{\partial Z} \right) - A \frac{\partial S}{\partial Z} + BS \frac{\partial S}{\partial Z} - ABS \tag{15}$$

where $S(Z,\ T) = 1 - \alpha S_w(Z,\ T)$, $A = \frac{\alpha L(\rho_w - \rho_o) g}{\beta}$, and $B = L \frac{a_2}{a_1}$

Equation (15) delineates a governing non-linear partial differential equation for the phenomenon of fingero – imbibition within a vertically oriented porous medium of heterogeneous.

The appropriate initial and boundary conditions are outlined below:

If $Z > 0,\ S(Z,\ 0) = S_0(Z)$

If $T > 0,\ S(0,\ T) = S_1(T)$

If $T > 0,\ S(L,\ T) = S_2(T)$

Problem Solution

Following the approach of variational iteration [7, 8], the correction functional of equation (15) is presented as follows:

$$S_{n+1}(Z,T) = S_n + \int_0^T \lambda(T) \left[\frac{\partial S_n}{\partial \tau} - \frac{\partial}{\partial z}\left(S_n \frac{\partial S_n}{\partial z} \right) + A \frac{\partial S_n}{\partial z} - BS_n \frac{\partial S_n}{\partial z} + ABS_n \right] d\tau$$

Here 'λ' represents a Lagrange's multiplier, determined as follows:

As, $S_n(Z,\tau)$ is showed restricted variation, $\delta S_n(Z,\tau) = 0$.

Computing variation with respect to S_n, remarking that $\delta S_n(0) = 0$, gives

$$\delta S_{n+1}(Z,T) = \delta S_n + \delta \int_0^T \lambda(T) \left[\frac{\partial S_n}{\partial \tau} - \frac{\partial}{\partial z}\left(S_n \frac{\partial S_n}{\partial z} \right) + A \frac{\partial S_n}{\partial z} - BS_n \frac{\partial S_n}{\partial z} + ABS_n \right] d\tau$$

Utilizing integration by parts, we get

$$\delta S_{n+1}(Z,T) = \delta S_n + \left[\lambda(z) \delta S_n \right]_{\tau = T} - \int_0^T \lambda'(T) \delta S_n d\tau + \int_0^T \lambda AB \delta S_n d\tau$$

Stationary conditions are determined as follows:

$$\left[1+\lambda(\tau)\right]_{\tau=T}=0,$$

$$\left[-\lambda'(\tau)+AB\lambda(\tau)\right]_{\tau=T}=0$$

Thus, Lagrange multiplier $\lambda=-e^{AB(\tau-T)}$

The iteration formula given can be derived as follows,

$$S_{n+1}(Z,T)=S_n-\int_0^T e^{AB(\tau-T)}\left[\frac{\partial S_n}{\partial\tau}-\frac{\partial}{\partial z}\left(S_n\frac{\partial S_n}{\partial z}\right)+A\frac{\partial S_n}{\partial z}-BS_n\frac{\partial S_n}{\partial z}+ABS_n\right]d\tau \tag{16}$$

We select the initial approximation,

$$S_0=e^{-Z}\ \text{[2]} \tag{17}$$

Taking $n=0,\ 1,\ 2,\ldots..$ we attain the following approximations by using above approximation (17) in iterative formula.

$$S_1=e^{-Z}+\frac{1}{AB}\left(2e^{-2Z}-Be^{-2Z}+Ae^{-Z}-ABe^{-Z}\right)\left(1-e^{-ABT}\right)$$

$$\begin{aligned}S_2&=e^{-Z}+\frac{1}{AB}\left(2e^{-2Z}-Be^{-2Z}+Ae^{-Z}-ABe^{-Z}\right)\left(1-e^{-ABT}\right)\\&-\frac{1}{(AB)^2}\Big[\left(18e^{-3Z}-9Be^{-3Z}+4Ae^{-2Z}-4ABe^{-2Z}\right)+A\left(4e^{-2Z}-2Be^{-2Z}+Ae^{-Z}-ABe^{-Z}\right)\\&-B\left(6e^{-3Z}-3Be^{-3Z}+2Ae^{-2Z}-2ABe^{-2Z}\right)\Big]\left(1-e^{-ABT}\right)^2\\&+\frac{1}{(AB)^3}\Big[\left(2e^{-2Z}-Be^{-2Z}+Ae^{-Z}-ABe^{-Z}\right)\left(8e^{-2Z}-4Be^{-2Z}+Ae^{-Z}-ABe^{-Z}\right)\\&+\left(4e^{-2Z}-2Be^{-2Z}+Ae^{-Z}-ABe^{-Z}\right)^2\\&-B\left(2e^{-2Z}-Be^{-2Z}+Ae^{-Z}-ABe^{-Z}\right)\left(4e^{-2Z}-2Be^{-2Z}+Ae^{-Z}-ABe^{-Z}\right)\Big]\left(1-e^{-ABT}\right)^3\end{aligned} \tag{18}$$

Likewise, by employing the iterative formula (16), additional iterations can be attained. Equation (18) represents an analytical approximate solution to equation (15).

Now, $S=1-\alpha S_w \quad \therefore\ S_w=\frac{1-S}{\alpha}$

Hence, the sought-after approximate analytical solution to the current problem is achieved through the following process:

$$S_{w_1}=\frac{1}{\alpha}\left\{1-\left[e^{-Z}+\frac{1}{AB}\left(2e^{-2Z}-Be^{-2Z}+Ae^{-Z}-ABe^{-Z}\right)\left(1-e^{-ABT}\right)\right]\right\}$$

$$
\begin{aligned}
S_{w_2} = \frac{1}{\alpha}\Bigg\{1 - \Bigg[& e^{-Z} + \frac{1}{AB}\left(2e^{-2Z} - Be^{-2Z} + Ae^{-Z} - ABe^{-Z}\right)\left(1 - e^{-ABT}\right) \\
& - \frac{1}{(AB)^2}\Big[\left(18e^{-3Z} - 9Be^{-3Z} + 4Ae^{-2Z} - 4ABe^{-2Z}\right) + A\left(4e^{-2Z} - 2Be^{-2Z} + Ae^{-Z} - ABe^{-Z}\right) \\
& - B\left(6e^{-3Z} - 3Be^{-3Z} + 2Ae^{-2Z} - 2ABe^{-2Z}\right)\Big]\left(1 - e^{-ABT}\right)^2 \\
& + \frac{1}{(AB)^3}\Big[\left(2e^{-2Z} - Be^{-2Z} + Ae^{-Z} - ABe^{-Z}\right)\left(8e^{-2Z} - 4Be^{-2Z} + Ae^{-Z} - ABe^{-Z}\right) \\
& + \left(4e^{-2Z} - 2Be^{-2Z} + Ae^{-Z} - ABe^{-Z}\right)^2 \\
& - B\left(2e^{-2Z} - Be^{-2Z} + Ae^{-Z} - ABe^{-Z}\right)\left(4e^{-2Z} - 2Be^{-2Z} + Ae^{-Z} - ABe^{-Z}\right)\Big]\left(1 - e^{-ABT}\right)^3\Bigg]\Bigg\}
\end{aligned}
\tag{19}
$$

Result and Discussion

MATLAB is used to conduct numerical analysis and generate graphical representations for the solution (19). The specific constants provided below have been chosen from standard literature.

Gravity $(g) = 9.8$, Density of injected water $(\rho_w) = 0.1$, Density of native oil $(\rho_o) = 0.8$, $(\beta) = 10$ Length $(L) = 1$,

Thus, $A = \dfrac{\alpha L(\rho_w - \rho_o)g}{\beta} \approx 1$.

Table – 1 presents the numerical solutions for the saturation of injected water (S_w) at different depth (Z) for a predetermined time (T). Fig. 3 exhibits the plots of the saturation of injected water (S_w) at different depth (Z) for predetermined time (T). Fig. 4 expresses the plots of the saturation of injected water (S_w) versus T for predetermined depth (Z). Fig. 5 displays the 3D plot of the solution. It has been determined from the Fig. 3, Fig. 4 and Fig. 5 that saturation of injected water decreases with time and increases with depth.

Table – 1: Numerical results for Saturation of injected water (S_w)

Z	$T=0$	$T=0.02$	$T=0.04$	$T=0.06$	$T=0.08$	$T=0.1$
0.1	0.857E – 1	0.711E – 1	0.568E – 1	0.428E – 1	0.290E – 1	0.155E – 1
0.2	1.633E – 1	1.513E – 1	1.396E – 1	1.281E – 1	1.169E – 1	1.058E – 1
0.3	2.335E – 1	2.237E – 1	2.141E – 1	2.047E – 1	1.955E – 1	1.864E – 1
0.4	2.970E – 1	2.890E – 1	2.811E – 1	2.734E – 1	2.659E – 1	2.585E – 1
0.5	3.545E – 1	3.479E – 1	3.415E – 1	3.352E – 1	3.290E – 1	3.229E – 1
0.6	4.065E – 1	4.011E – 1	3.958E – 1	3.907E – 1	3.856E – 1	3.807E – 1
0.7	4.535E – 1	4.491E – 1	4.448E – 1	4.406E – 1	4.364E – 1	4.324E – 1
0.8	4.961E – 1	4.925E – 1	4.890E – 1	4.855E – 1	4.821E – 1	4.788E – 1
0.9	5.346E – 1	5.317E – 1	5.288E – 1	5.254E – 1	5.232E – 1	5.205E – 1
1.0	5.695E – 1	5.671E – 1	5.647E – 1	5.624E – 1	5.601E – 1	5.579E – 1

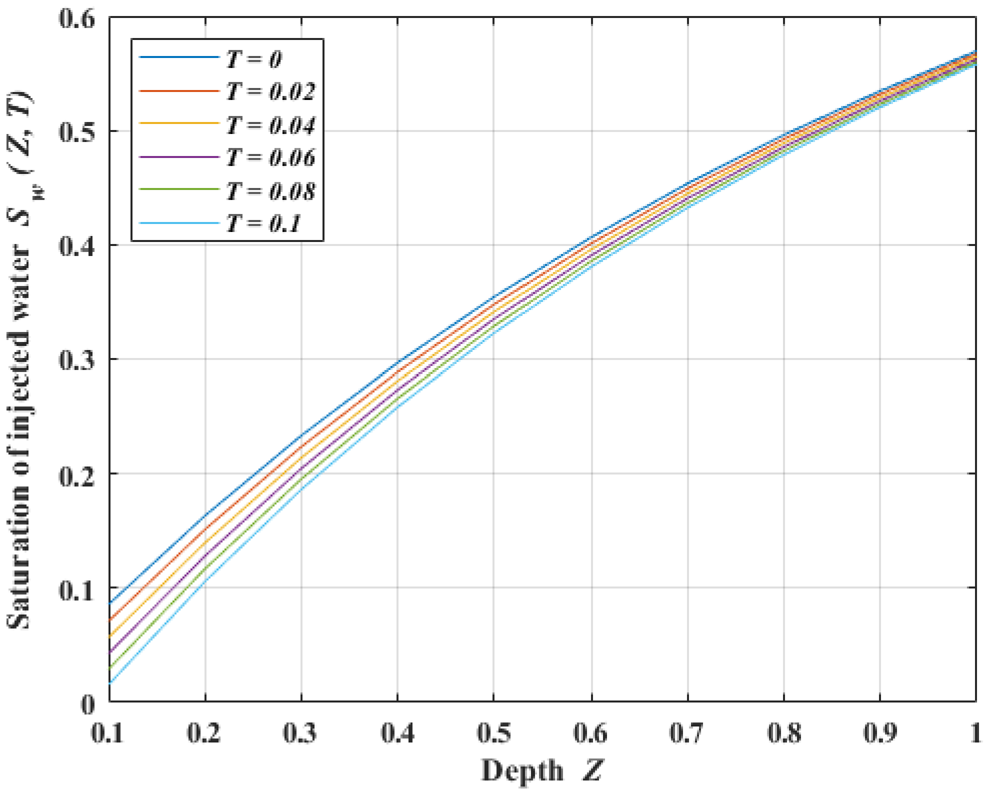

Figure – 3: Saturation of injected water Vs. Depth Z

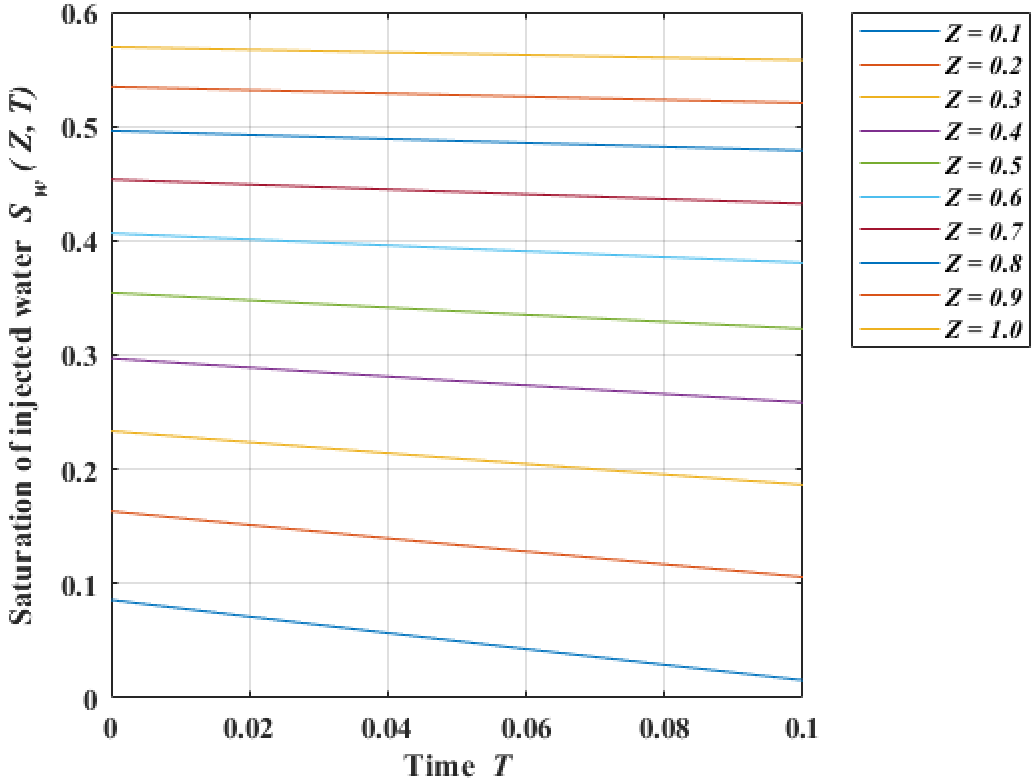

Figure – 4: Saturation of injected water Vs. Time T

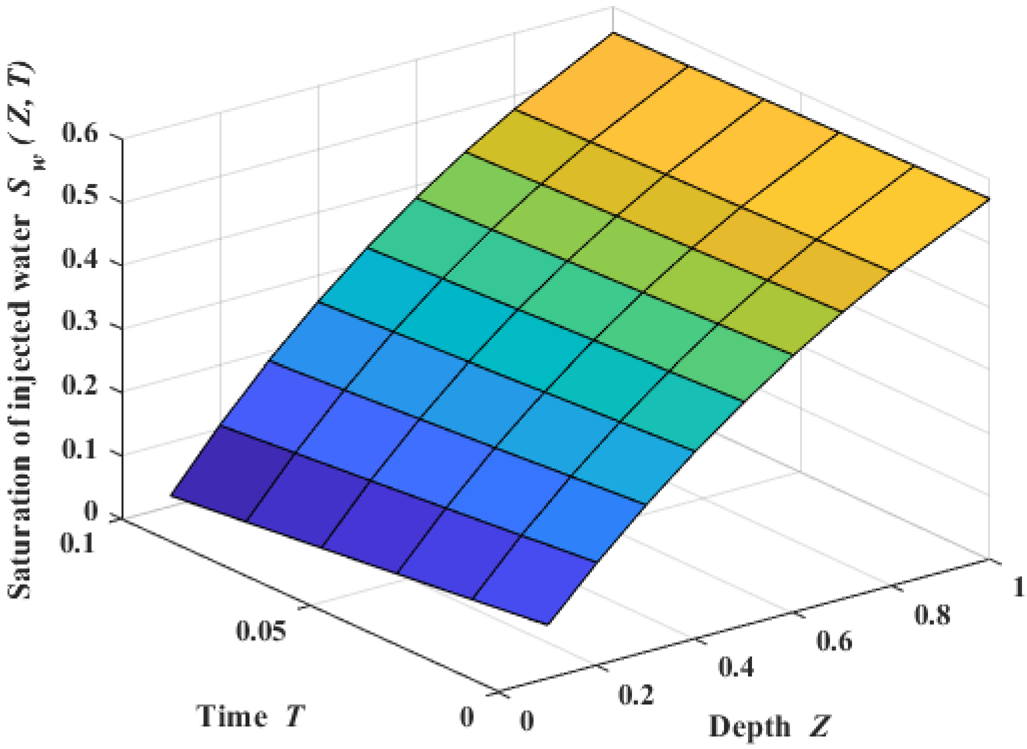

Figure – 5: 3D plot of the Saturation of injected water

Conclusion

In current study, we have elaborated mathematical model to describe the fingero-imbibition phenomenon within vertically oriented heterogeneous porous medium. The approach of Variational iteration is employed to acquire an approximate analytical result for the governing equation of this phenomenon. This solution satisfies both the initial as well as boundary conditions. It can be detected from the graphical and numerical elucidations that saturation of injected water decreases with respect to time as well as increases with respect to depth. This is consistent with the physical characteristics of the problem. The presented analytical framework not only enhances our understanding of fingero-imbibition in vertically heterogeneous porous medium but also offers a valuable tool for predicting and optimizing fluid transport in such systems. The findings have implications in diverse fields, including soil science, petroleum engineering, and environmental remediation, where accurate predictions of fluid flow dynamics are crucial for informed decision-making.

Nomenclature

V_o – Velocity of oil	φ – Porosity
V_w – Velocity of water	S_o – Saturation of oil
K_o – Relative permeability of oil	S_w – Saturation of water
K_w – Relative permeability of water	$\beta,\ K_c$ – Constant of proportionality
K – Permeability of heterogeneous porous medium	$K_0,\ a_1,\ a_2,\ b$ – Positive constants
P_o – Pressure of oil	L – Length of cylindrical porous matrix
P_w – Pressure of water	λ – Lagrange multiplier
ρ_o – Density of oil	$\tilde{S}_n$ – Restricted variation
ρ_w – Density of water	T – Time
μ_o – Kinematic viscosity of oil	Z – Depth
μ_w – Kinematic viscosity of water	

References

1. A. E. Scheidegger and E. F. Johnson, The Statistical Behavior of Instabilities in Displacement Processes in Porous Media, Canadian Journal of Physics, 39(2), 326 – 334, 1961.
2. A. K. Parikh, M. N. Mehta, V. H. Pradhan, Mathematical Modeling and Analysis of Fingero-imbibition Phenomenon in Vertical Downward Cylindrical Homogeneous Porous Matrix, Conference proceedings of Nirma University International Conference on Engineering (NUiCONE), 2013.
3. A. P. Verma, Statistical Behaviour of Fingering in a Displacement in Heterogeneous Porous
Medium with Capillary Pressure, Canadian Journal of Physics, 47(3), 319 – 324, 1969.

4. B. G. Choksi and T. R. Singh, Fingero-Imbibition Phenomenon in Double Phase Flow through Homogeneous Porous Media with Magnetic Field Effect, Elixir Appl. Math. 106, 46597-46601, 2017.

5. D. J. Prajapati and N. B. Desai, Approximate Analytical Solution for the Fingero-Imbibition Phenomenon by Optimal Homotopy Analysis Method, International Journal of Computational and Applied Mathematics, 12(3), 751-761, 2017.

6. J. Bear and A. H. Chang, Modelling Groundwater Flow and Contaminant Transport, Dynamics of Fluids in Porous Media, Springer Science Business, Media B. V., 2010.

7. J. H. He, Variational Iteration Method: A Kind of Nonlinear Analytical Technique: Some Examples, International Journal of Nonlinear Mechanics, 34(4), 699 – 708, 1999.

8. J. H. He, Variational Iteration Method: Some Recent Results and New Interpretations, International Journal of Computational and Applied Mathematics, 207(1), 3 – 17, 2007.

9. M. A. Patel and N. B. Desai, Homotopy Analysis Method for the Fingero-imbibition phenomenon in Heterogeneous Porous Medium, Nonlinear Sci. Lett. A, 8(1), 90-100, 2017.

10. M. N. Mehta, Asymptotic Expansion of fluid flow through porous media, Ph. D. Thesis, South Gujarat University, Surat, India, 1977.

11. O. S. Lyiola, q-Homotopy Analysis Method and Application to Fingero-imbibition Phenomena in Double Phase Flow through Porous Media, Asian Journal of Current Engineering and Maths, 2(34), 2013.

12. O. S. Lyiola and S. B. Folarin, Approximate Analytical Study of Fingero-imbibition Phenomena of Time-Fractional Type in Double Phase Flow through Porous Media, European Journal of Pure and Applied Mathematics, 7(2), 210 – 229, 2014.

13. R. Meher, M. N. Mehta, and S. K. Meher. Adomian Decomposition Approach to Fingero-Imbibition Phenomena in Double Phase Flow through Porous Media, International Journal of Applied Mathematics and Mechanics. 6 (9): 34-46, 2012.

14. S. Pathak and T. Singh, Study on Fingero Imbibition Phenomena in Inclined Porous Media by Optimal Homotopy Analysis Method, Ain Shams Engineering Journal, 9, 1181-1187, 2018.

15. Z. Cheng, Reservoir Simulation: Mathematical Techniques in Oil Recovery, Society for Industrial and Applied Mathematics, Philadelphia, 1 – 25, 2007.