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METRIZABILITY ON TOPOLOGICAL SIMPLE ROUGH GROUPS

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Abstract

In this paper, we focus on the concepts of rough pseudonorm and metrizability in topological simple rough groups. Further, we discuss some results related to these concepts. In particular, we explore the interplay between rough pseudonorms and metrizability, and also highlighting the necessary and sufficient condition to ensure a topological simple rough group is metrizable. These findings contribute to a deeper understanding of the structural dynamics of topological simple rough groups.

Keywords: Simple rough group, Topological simple rough group, Rough pseudonorm, Rough pseudometric, Metrizable, Symmetric neighbourhoods.

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1. Introduction:

In recent years, rough sets have been researched combining with some mathematical theories such as topology and algebra. Rough set theory initiated by Pawlak (1982) [13], is a symmetric approach for the classification of objects. In the past 30 years, many extensions of rough set theory in different directions have been reported. Considering the algebraic structures of rough sets is an interesting topic, which has been studied by many authors, including Bonikowaski. Z [16], Kuroki. N, Wang. PP and Li. Z et al. In 1994, Biswas and Nanda [3] introduced the concept of rough groups and rough subgroups, which depends on the upper approximation and does not depend on the lower approximation.

Bagirmaz et al. [12] the authors introduced the concept of topological rough group and extended the notion of a topological group to include algebraic structures of rough groups. In Lin et al. (2020) [7], the authors studied topological rough groups based on the concept of rough groups.

In this paper, we discuss about the rough pseudonorm and metrizability in topological simple rough groups. A rough pseudonorm serves as a generalized function to measure the size of elements in a rough group, providing a foundation for understanding the rough group's structure. Metrizability, on the other hand, establishes conditions under which the topology of a simple rough group can be defined by a rough pseudometric. Also, we explore the interplay between rough pseudonorms and metrizability in topological simple rough groups. In particular, the necessary and sufficient condition for a topological simple rough group to be metrizable is studied and discussed.

2. Preliminaries:

Definition 2.1. [5] Let $K = (U, R)$ be an approximation space and $*$ be a binary operation defined on U. A subset G of universe U is called a rough group if the following properties are satisfied:

(i) $\forall$ x, y $\in$ G, x$*$y $\in \overline{G}$;

(ii) Association property holds in $\overline{G}$;

(iii) $\exists$e $\in \overline{G}$ such that $\forall$x $\in$ G, x$*$e = e$*$x = x; e is called the rough identity element of rough group G;

(iv) $\forall$x $\in$ G, $\exists$y $\in$ G such that x$*$y = y$*$x = e; y is called the rough inverse element of x in G;

Definition 2.2.[14] Let (U, R) be an approximation space and $\theta(x)$ be the set of all elements of U which are related to the element 'x'. Then the function $d_R: U \times U \to \mathbb{R}$ is called a rough pseudo metric on U if the following conditions are true, for all x, y, z $\in$ U:

(i) $d_R(x,y) \geq 0$.

(ii) $d_R(x,y) = 0 \Rightarrow \theta(x) = \theta(y)$,

(iii) $d_R(x,y) = d_R(y,x)$

(iv) $d_R(x,y) + d_R(y,z) \geq d_R(x,z)$

The pair (U, d_R) is called a rough pseudo metric space.

Definition 2.3. [12] A topological rough group is a rough group (G, $*$) together with a topology T on $\overline{G}$ satisfying the following two properties:

(i) the mapping f: G × G → $\overline{G}$ defined by f(x, y) = xy is continuous with respect to product topology on G × G and the topology T_G on G induced by T,

(ii) the inverse mapping g: G → G defined by g(x) = x^{-1} is continuous with respect to the topology T_G on G induced by T.

Definition 2.4. [15] A rough group $G_{\mathfrak{R}}$ is called a simple rough group if it contains no proper non-trivial rough normal subgroups.

That is, $G_{\mathfrak{R}}$ has only the rough normal subgroups {e} and $G_{\mathfrak{R}}$.

Definition 2.5. [15] A topological simple rough group is a simple rough group ($G_{\mathfrak{R}}$, $*$) together with a topology $\overline{\tau}$ on $\overline{G_{\mathfrak{R}}}$ satisfying the following two properties:

(i) The mapping f: $G_{\mathfrak{R}} \times G_{\mathfrak{R}} \to \overline{G_{\mathfrak{R}}}$ defined by f(x, y) = xy, x, y ∈ $G_{\mathfrak{R}}$ is continuous with respect to the product topology on $G_{\mathfrak{R}} \times G_{\mathfrak{R}}$ and the topology τ on $G_{\mathfrak{R}}$ induced by $\overline{\tau}$

(ii) The inverse mapping g: $G_{\mathfrak{R}} \to G_{\mathfrak{R}}$ defined by g(x) = x^{-1}, x ∈ $G_{\mathfrak{R}}$ is continuous with respect to the topology τ on $G_{\mathfrak{R}}$ induced by $\overline{\tau}$.

Definition 2.6. [15] Let $G_{\mathfrak{R}}$ be a topological simple rough group and $\overline{\mathcal{B}} \subseteq \overline{\tau}$ be a base for $\overline{\tau}$. For x ∈ $G_{\mathfrak{R}}$, the family $\mathcal{B}_x$ = {U ∩ $G_{\mathfrak{R}}$: U ∈ $\overline{\mathcal{B}}$, x ∈ U} ⊆ $\overline{\mathcal{B}}$ is called a base at x in τ.

Proposition 2.7. [6] A T_1- space X is a Tychonoff space or completely regular space if and only if for every $x \in X$ and every neighbourhood V of x there exists a continuous function $f: X \to I$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in X \backslash V$.

Theorem 2.8. [16] Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $\{e\}$ is closed in $\overline{G_{\mathfrak{R}}}$. Then $G_{\mathfrak{R}}$ is a T_2- space.

Proposition 2.9. [15] Let $G_{\mathfrak{R}}$ be a topological simple rough group. If U ⊆ $\overline{G_{\mathfrak{R}}}$ is an open set with e ∈ U, then there exists a symmetric open set V of e in $G_{\mathfrak{R}}$ such that VV ⊆ U.

Throughout this paper, we consider X be the universal set, $G_{\mathfrak{R}}$ be a rough group with identity e and $\overline{G_{\mathfrak{R}}}$ be the upper rough approximation of $G_{\mathfrak{R}}$. Also, the corresponding topologies are denoted by $\overline{\tau}$ for $\overline{G_{\mathfrak{R}}}$ and τ for $G_{\mathfrak{R}}$ induced from $\overline{\tau}$.

3. Rough Pseudonorms:

Definition 3.1. Let $G_{\mathfrak{R}}$ be a rough group with identity e and let $\mathcal{P}$ be a non-negative real valued function on $G_{\mathfrak{R}}$. Then $\mathcal{P}$ is a *rough pseudonorm* if it satisfies the following conditions:

(i) $\mathcal{P}(e) = 0$

(ii) $\mathcal{P}(ab) \leq \mathcal{P}(a) + \mathcal{P}(b)$, for all $a, b \in G_{\mathfrak{R}}$

(iii) $\mathcal{P}(a) = \mathcal{P}(a^{-1})$, for all $a, b \in G_{\mathfrak{R}}$.

Proposition 3.2. Let $\mathcal{P}$ be a rough pseudonorm on a group $G_{\mathfrak{R}}$. Then for all $a, b \in G_{\mathfrak{R}}$, $|\mathcal{P}(a) - \mathcal{P}(b)| \leq \mathcal{P}(ab^{-1})$.

Proof: From the definition of rough pseudonorm, for all $a, b \in G_{\mathfrak{R}}$, $\mathcal{P}(b) \leq \mathcal{P}(a) + \mathcal{P}(a^{-1}b)$ and $\mathcal{P}(a) = \mathcal{P}(a^{-1}) \leq \mathcal{P}(b^{-1}) + \mathcal{P}(a^{-1}b) = \mathcal{P}(b) + \mathcal{P}(a^{-1}b)$. Now combining these, we get the result $|\mathcal{P}(a) - \mathcal{P}(b)| \leq \mathcal{P}(ab^{-1})$.

Proposition 3.3. Let $G_{\mathfrak{R}}$ be a simple rough group and $\mathcal{P}$ be a rough pseudonorm on $G_{\mathfrak{R}}$. Then the following are hold:

(i) $x\mathcal{P}$ is a rough pseudonorm on $G_{\mathfrak{R}}$, for all non-negative real number x.

(ii) For every element $x \in G_{\mathfrak{R}}$, the function $\mathcal{P}_x$ by $\mathcal{P}_x(a) = \mathcal{P}(xax^{-1})$ is a rough pseudonorm on $G_{\mathfrak{R}}$

(iii) Let $\mathcal{P}_1$, $\mathcal{P}_2$ be two rough pseudonorms on $G_{\mathfrak{R}}$. Then $\mathcal{P}_1 + \mathcal{P}_2$ is also a rough pseudonorm on $G_{\mathfrak{R}}$.

Proof:

(i) Since x is a non-negative real number, the result is obvious.

(ii) Let $a, x \in G_{\mathfrak{R}}$. Then $\mathcal{P}_x(e) = \mathcal{P}(xex^{-1}) = 0$.

$$\mathcal{P}_x(ab) = \mathcal{P}(x(ab)x^{-1}) = \mathcal{P}\big((xax^{-1})(xbx^{-1})\big) \leq \mathcal{P}(xax^{-1}) + \mathcal{P}(xbx^{-1})$$

$$\leq \mathcal{P}_x(a) + \mathcal{P}_x(b).$$

Also, $\mathcal{P}_x(a^{-1}) = \mathcal{P}(xa^{-1}x^{-1}) = \mathcal{P}(xax^{-1})^{-1} = \mathcal{P}_x(a)$.

(iii) Consider $\mathcal{P}' = \mathcal{P}_1 + \mathcal{P}_2$. Then the proof follows from the definition of rough pseudonorm.

Proposition 3.4. Let $G_{\mathfrak{R}}$ be a simple rough group and f be any bounded real valued function on $G_{\mathfrak{R}}$. Then the function $\mathcal{P}_f$ by $\mathcal{P}_f(a) = Sup_{b\in G_{\mathfrak{R}}}|f(ba) - f(b)|$, $a \in G_{\mathfrak{R}}$, is a rough pseudonorm on $G_{\mathfrak{R}}$.

Proof: Let $a, b, c \in G_{\mathfrak{R}}$. Now to prove the rough pseudonorm conditions,

$\mathcal{P}_f(e) = Sup_{b\in G_{\mathfrak{R}}}|f(be) - f(b)| = 0.$

$\mathcal{P}_f(ab) = Sup_{c\in G_{\mathfrak{R}}}\left|f\big(c(ab)\big) - f(c)\right| \leq Sup_{c\in G_{\mathfrak{R}}}\left\{\left|f\big((ca)b\big) - f(ca)\right| + |f(ca) - f(c)|\right\}$

$\leq \mathcal{P}_f(a) + \mathcal{P}_f(b).$

Then $\mathcal{P}_f(a^{-1}) = Sup_{b\in G_{\mathfrak{R}}}|f(ba^{-1}) - f(b)| = Sup_{c\in G_{\mathfrak{R}}}|f(c) - f(ca)|$, where $c = ba^{-1}$

$= \mathcal{P}_f(a).$

Proposition 3.5. Let $G_{\mathfrak{R}}$ be a topological simple rough group. A rough pseudonorm $\mathcal{P}$ on $G_{\mathfrak{R}}$ is continuous if and only if for every $\varepsilon > 0$, there is a neighbourhood W of e in $G_{\mathfrak{R}}$ such that $\mathcal{P}(a) < \varepsilon$, for all $a \in W$.

Proof: Consider the rough pseudonorm $\mathcal{P}$ on $G_{\mathfrak{R}}$ is continuous. So, it is also continuous at the identity element e. Then for any $\varepsilon > 0$, there exists a neighbourhood W of e such that $|\mathcal{P}(a) - \mathcal{P}(e)| < \varepsilon$, for all $a \in W$. Substituting $\mathcal{P}(e) = 0$, we get $|\mathcal{P}(a)| < \varepsilon$, for all $a \in W$. Now let us prove the sufficient condition. Let $b \in G_{\mathfrak{R}}$ and $\varepsilon > 0$. Then consider a neighbourhood W of e in $G_{\mathfrak{R}}$ such that $\mathcal{P}(a) < \varepsilon$, for all $a \in W$. Also, bW is a neighbourhood of b. Let $c \in bW$ which implies $b^{-1}c \in W$. By the sufficient condition, $\mathcal{P}(b^{-1}c) < \varepsilon$. Since by proposition 3.2, we get $|\mathcal{P}(b) - \mathcal{P}(c)| < \varepsilon$. Hence $\mathcal{P}$ is continuous.

Proposition 3.6. Let $\{V_n\}_{n\in\mathbb{N}}$ be a sequence of open neighbourhoods of the identity element e in a topological simple rough group $G_{\mathfrak{R}}$, where the sequence satisfying the following conditions:

(i) Each neighbourhood V_n is symmetric, that is $V_n^{-1} = V_n$

(ii) The sequence is decreasing and $V_{n+1}^2 \subseteq V_n$, for all $n \in \mathbb{N}$

Then there exists a continuous rough pseudonorm $\mathcal{P}$ on $G_{\mathfrak{R}}$ such that

$$\left\{a \in G_{\mathfrak{R}} :\ \mathcal{P}(a) < \tfrac{1}{2^n}\right\} \subseteq V_n \subseteq \left\{a \in G_{\mathfrak{R}} :\ \mathcal{P}(a) \leq \tfrac{2}{2^n}\right\} \qquad (1)$$

Proof: Let $V(1) = V_0$, $V\left(\frac{m}{2^n}\right)$ be open neighbourhoods of e, for $n \in \mathbb{N}$ and $m = 1, 2, \dots 2^n$ and define

$$V\left(\frac{1}{2^n}\right) = V_{n+1},$$

$$V\left(\frac{2m+1}{2^{n+1}}\right) = V\left(\frac{m}{2^n}\right) V_{n+1}\,,$$

for $m = 1, 2, \dots 2^n - 1$ and $V\left(\frac{m}{2^n}\right) = G_{\mathfrak{R}}$ for $m > 2^n$. These are open neighbourhoods of $V(\gamma)$ of e, for the dyadic rational number γ and $0 \le \gamma \le 1$. Now let us prove the following condition

$$V\left(\frac{m}{2^n}\right) V\left(\frac{1}{2^n}\right) \subseteq V\left(\frac{m+1}{2^n}\right), \text{ for all } m > 0, n \ge 0 \qquad (2)$$

This condition is true for $m + 1 > 2^n$. So, we prove this result for $m < 2^n$. Let us use the induction method. If $n = 1$, then clearly $m = 1$. Therefore, $V\left(\frac{1}{2}\right) V\left(\frac{1}{2}\right) \subseteq V(1)$. Now assume this result is true for n. Let us verify for $n + 1$. If $m = 2k$ is even, then the result is obvious. If $m = 2k + 1$ is odd, $0 \le 2k + 1 \le 2^{n+1}$, then

$$V\left(\frac{m}{2^{n+1}}\right) V\left(\frac{1}{2^{n+1}}\right) = V\left(\frac{2k+1}{2^{n+1}}\right) V_{n+1} = V\left(\frac{k}{2^n}\right) V_{n+1}^2 \subseteq V\left(\frac{k}{2^n}\right) V\left(\frac{1}{2^n}\right)$$

Here, the result is true for $n + 1$. Hence equation (2) is satisfied for all $n \ge 0$, $m > 0$. Let f be a non-negative real valued function on $G_{\mathfrak{R}}$ such that

$$f(a) = Inf\{s \ge 0 : a \in V(s)\}.$$

Also f is well defined because $a \in V(2) = G_{\mathfrak{R}}$, for all $a \in G_{\mathfrak{R}}$. If $0 \le s \le t$, for the dyadic rational numbers s and t, then $V(s) \subseteq V(t)$. So, we get if $f(a) < s$, then $a \in V(s)$. Therefore, $|f(a)| \le 2$, $a \in G_{\mathfrak{R}}$ and f is bounded. Now by proposition 3.4, define the rough pseudonorm $\mathcal{P}$ on $G_{\mathfrak{R}}$ by

$$\mathcal{P}_f(a) = Sup_{b \in G_{\mathfrak{R}}} |f(ba) - f(b)|, a \in G_{\mathfrak{R}}.$$

Consider $a \in G_{\mathfrak{R}}$ and $\mathcal{P}(a) < \frac{1}{2^n}$. Using the definition of rough pseudonorm and $f(e) = 0$, we get

$$f(a) = |f(ea) - f(e)| \le \mathcal{P}(a) < \frac{1}{2^n}.$$

Then $a \in V\left(\frac{1}{2^n}\right) = V_n$. Hence $\left\{a \in G_{\mathfrak{R}} : \mathcal{P}(a) < \frac{1}{2^n}\right\} \subseteq V_n$.

Let $a \in V\left(\frac{1}{2^n}\right)$, $b \in G_{\mathfrak{R}}$ be arbitrary and $x \in \mathbb{N}$ such that $\frac{x-1}{2^n} \leq f(b) < \frac{x}{2^n}$. Then $b \in V\left(\frac{x}{2^n}\right)$, which implies

$$ba \in V\left(\frac{x}{2^n}\right)V\left(\frac{1}{2^n}\right) \text{ and } ba^{-1} \in V\left(\frac{x}{2^n}\right)V\left(\frac{1}{2^n}\right)^{-1}.$$

By hypothesis (ii) and the equation (2),

$$ba \in V\left(\frac{x+1}{2^n}\right) \text{ and } ba^{-1} \in V\left(\frac{x+1}{2^n}\right).$$

Thus,

$$f(ba) \leq \left(\frac{x+1}{2^n}\right) \text{ and } f(ba^{-1}) \leq \left(\frac{x+1}{2^n}\right).$$

Then we get

$$f(ba) - f(b) \leq \frac{2}{2^n} \text{ and } f(ba^{-1}) - f(b) \leq \frac{2}{2^n}.$$

Replace b by ba in the second inequality, we get $f(b) - f(ba) \leq \frac{2}{2^n}$. Combining these inequalities we get,

$$|f(ba) - f(b)| \leq \frac{2}{2^n} \text{ which implies } \mathcal{P}(a) \leq \frac{2}{2^n}.$$

Hence $V_n \subseteq \left\{a \in G_{\mathfrak{R}} : \mathcal{P}(a) \leq \frac{2}{2^n}\right\}$.

Proposition 3.7. Let $G_{\mathfrak{R}}$ be a topological simple rough group. Then there exists a continuous rough pseudonorm $\mathcal{P}$ on $G_{\mathfrak{R}}$ such that $\{a \in G_{\mathfrak{R}} : \mathcal{P}(a) < 1\} \subseteq V$, where $V \subseteq G_{\mathfrak{R}}$ is a symmetric open neighbourhood of e.

Proof: Let us construct a decreasing sequence of symmetric open neighbourhoods $\{V_n\}_{n\in\mathbb{N}}$ of e in $G_{\mathfrak{R}}$ with the conditions $V_{n+1}^2 \subseteq V_n$ and $V = V_0$. Then by theorem 3.6, there exists a continuous rough pseudonorm $\mathcal{P}$ on $G_{\mathfrak{R}}$ such that

$$\left\{a \in G_{\mathfrak{R}} : \mathcal{P}(a) < \frac{1}{2^n}\right\} \subseteq V_n \subseteq \left\{a \in G_{\mathfrak{R}} : \mathcal{P}(a) \leq \frac{2}{2^n}\right\}, \text{ for all } n \in \mathbb{N}.$$

Put $n = 0$, we get $\{a \in G_{\mathfrak{R}} : \mathcal{P}(a) < 1\} \subseteq V_0 \subseteq V$. Hence $\{a \in G_{\mathfrak{R}} : \mathcal{P}(a) < 1\} \subseteq V$.

Theorem 3.8. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $\{e\}$ is closed in $\overline{G_{\mathfrak{R}}}$. Then $G_{\mathfrak{R}}$ is completely regular.

Proof: Since $G_{\mathfrak{R}}$ is a topological simple rough group such that $\{e\}$ is closed in $\overline{G_{\mathfrak{R}}}$, $G_{\mathfrak{R}}$ is a Hausdorff space. Let $x \in G_{\mathfrak{R}}$ and $V \subseteq G_{\mathfrak{R}}$ be a symmetric open neighbourhood of x. Then $W = x^{-1}V$ is an open neighbourhood of e in $G_{\mathfrak{R}}$. Using theorem 3.7, there exists a rough pseudonorm $\mathcal{P}$ on $G_{\mathfrak{R}}$ such that $\{a \in G_{\mathfrak{R}} : \mathcal{P}(a) < 1\} \subseteq W$. Consider a function f on $G_{\mathfrak{R}}$ by $f(a) = \mathcal{P}(x^{-1}a)$. Then $f(x) = 0$ obviously. Now for some $a \in G_{\mathfrak{R}}$, if $f(a) < 1$, $x^{-1}a \in W = x^{-1}V$ which implies $a \in V$. Otherwise $f(b) \geq 1$, for every $b \notin V$. Hence $G_{\mathfrak{R}}$ is completely regular.

4. Metrizability:

Lemma 4.1. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$ and let W be a neighbourhood of e in $G_{\mathfrak{R}}$. Then there exists an open neighbourhood V of e in $G_{\mathfrak{R}}$ such that $U = U^{-1}$ and $cl(UU) \subseteq W$.

Proof: Since $G_{\mathfrak{R}}$ is a topological simple rough group, there is a symmetric open set V of e in $G_{\mathfrak{R}}$ such that $VV \subseteq W$. By the first closure lemma, we get $cl(VV) \subseteq W$. Let $U = V \cap V^{-1}$. Then V is an open neighbourhood of e in $G_{\mathfrak{R}}$ which implies $U \subseteq V$ is a symmetric open neighbourhood and $cl(UU) \subseteq W$.

Proposition 4.2. Let $G_{\mathfrak{R}}$ be a topological simple rough group satisfying the first axiom of countability. Then there exists a countable basis $\mathcal{B}_e$ of symmetric neighbourhoods $\{V_i\}_{i\in\mathbb{N}}$ such that $cl(V_iV_i) \subseteq V_{i-1}$, for $i \geq 2$.

Proof: Let $\{U_i\}_{i\in\mathbb{N}}$ be a basis of identity neighbourhoods of $G_{\mathfrak{R}}$. Consider the identity neighbourhoods $V_1, V_2, V_3, \dots V_j$, for some $j \in \mathbb{N}$ such that $U_1 = V_1$ and $V_n \subseteq U_n$. Then by lemma 4.1, $cl(V_{i+1}V_{i+1}) \subseteq V_i \cap U_{i+1}$. Since $V_{i+1} \subseteq U_{i+1}$, $cl(V_{i+1}V_{i+1}) \subseteq V_i$. Hence the neighbourhoods $\{V_i\}_{i\in\mathbb{N}}$ of e is a basis.

Definition 4.3. A rough metric or rough pseudometric $d_{\mathcal{P}}$ on a rough group $G_{\mathfrak{R}}$ is said to be

(i) *rough left-invariant* if $d_{\mathcal{P}}^{L}(ax, ay) = d_{\mathcal{P}}(x, y)$, for all $a, x, y \in G_{\mathfrak{R}}$

(ii) *rough right-invariant* if $d_{\mathcal{P}}^{R}(xa, ya) = d_{\mathcal{P}}(x, y)$, for all $a, x, y \in G_{\mathfrak{R}}$.

If $H_{\mathfrak{R}}$ is any closed rough subgroup of $\overline{G_{\mathfrak{R}}}$, then a rough pseudometric $d_{\mathcal{P}}$ on $\overline{G_{\mathfrak{R}}}/H_{\mathfrak{R}}$ is said to be *rough left-invariant* if $d_{\mathcal{P}}^{L}(axH_{\mathfrak{R}}, ayH_{\mathfrak{R}}) = d_{\mathcal{P}}(xH_{\mathfrak{R}}, yH_{\mathfrak{R}})$ for all $a, x, y \in \overline{G_{\mathfrak{R}}}$.

Proposition 4.4. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $H_{\mathfrak{R}}$ be a first countable rough subgroup of $G_{\mathfrak{R}}$. Then the closure of $H_{\mathfrak{R}}$, denoted by $cl(H_{\mathfrak{R}})$, is a first countable rough group of $G_{\mathfrak{R}}$.

Proof: Since the rough subgroup $H_{\mathfrak{R}}$ is first countable, there is a countable neighbourhood basis U_n of e in $H_{\mathfrak{R}}$. Since $H_{\mathfrak{R}}$ is dense in $cl(H_{\mathfrak{R}})$ and $G_{\mathfrak{R}}$ is a topological simple rough group, then for any $x \in cl(H_{\mathfrak{R}})$, xU_n form a countable neighbourhood basis of x in $cl(H_{\mathfrak{R}})$. Therefore, $xU_n \cap cl(H_{\mathfrak{R}})$ is a countable neighbourhood basis in $cl(H_{\mathfrak{R}})$, for any $x \in cl(H_{\mathfrak{R}})$. Hence $cl(H_{\mathfrak{R}})$ is first countable.

Theorem 4.5. A topological simple rough group $G_{\mathfrak{R}}$ is metrizable if and only if $G_{\mathfrak{R}}$ is first countable.

Proof: The necessary condition is obvious. So, let us prove the sufficient condition only. Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable base of e in $G_{\mathfrak{R}}$ and $\{V_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of symmetric neighbourhoods of e such that $V_n \subseteq U_n$ and $V_{n+1}^2 \subseteq V_n$, for all $n \in \mathbb{N}$. Also $\{V_n\}_{n\in\mathbb{N}}$ is a base of e. By theorem 3.6, we get a continuous rough pseudonorm $\mathcal{P}$ on $G_{\mathfrak{R}}$ such that

$$\left\{a \in G_{\mathfrak{R}} :\ \mathcal{P}(a) < \frac{1}{2^n}\right\} \subseteq V_n \subseteq \left\{a \in G_{\mathfrak{R}} :\ \mathcal{P}(a) \leq \frac{2}{2^n}\right\}, \text{for all } n \in \mathbb{N} \qquad (3)$$

Let $a, b \in G_{\mathfrak{R}}$ be arbitrary and define a rough pseudometric $d_{\mathcal{P}}(a, b) = \mathcal{P}(ab^{-1})$. Now let us prove $d_{\mathcal{P}}$ is a rough pseudometric on $G_{\mathfrak{R}}$ and it generates the topology τ on $G_{\mathfrak{R}}$. It is clear that $d_{\mathcal{P}}(a, b) \geq 0$, for all $a, b \in G_{\mathfrak{R}}$. Also $d_{\mathcal{P}}(a, a) = 0$, for all $a \in G_{\mathfrak{R}}$. Suppose $d_{\mathcal{P}}(a, b) = 0$. Then by the equation (3), $ab^{-1} \in V_n$, for all $n \in \mathbb{N}$. Since $\{e\} = \bigcap_{n\in\mathbb{N}} V_n$, $ab^{-1} = e$ which implies $a = b$. Let $a, b, c \in G_{\mathfrak{R}}$. Then

$$d_{\mathcal{P}}(a, c) = \mathcal{P}(ac^{-1}) \leq \mathcal{P}(ab^{-1}) + \mathcal{P}(bc^{-1}) \leq d_{\mathcal{P}}(a, b) + d_{\mathcal{P}}(b, c)$$

Hence $d_{\mathcal{P}}$ is a rough pseudometric on $G_{\mathfrak{R}}$. Let W be a neighbourhood of a in $G_{\mathfrak{R}}$. Since $\{V_n\}_{n\in\mathbb{N}}$ is a base of e in $G_{\mathfrak{R}}$ such that $aV_n \subseteq W$, for all $n \in \mathbb{N}$. If $b \in G_{\mathfrak{R}}$ and $d_{\mathcal{P}}(a, b) < \frac{1}{2^n}$, then $ab^{-1} \in V_n$. From this, it follows that $b = a(a^{-1}b) \in aV_n \subseteq W$ which implies

$$\left\{b \in G_{\mathfrak{R}} :\ d_{\mathcal{P}}(a, b) < \frac{1}{2^n}\right\} \subseteq W$$

Hence a rough pseudometric $d_{\mathcal{P}}$ generates the topology τ on $G_{\mathfrak{R}}$.

Corollary 4.6. If $H_{\mathfrak{R}}$ is a metrizable simple rough subgroup of a topological simple rough group $G_{\mathfrak{R}}$, then $cl(H_{\mathfrak{R}})$ in $G_{\mathfrak{R}}$ is a metrizable simple rough subgroup.

Proof: Since $H_{\mathfrak{R}}$ is metrizable, it is first countable in $G_{\mathfrak{R}}$. Then by proposition 4.2, $cl(H_{\mathfrak{R}})$ is first countable which implies $cl(H_{\mathfrak{R}})$ in $G_{\mathfrak{R}}$ is metrizable in $G_{\mathfrak{R}}$.

Corollary 4.7. Let $G_{\mathfrak{R}}$ be a metrizable topological simple rough group and $H_{\mathfrak{R}}$ be a topological simple rough group. If the open mapping $f: G_{\mathfrak{R}} \to H_{\mathfrak{R}}$ is continuous homomorphism, then $H_{\mathfrak{R}}$ is metrizable.

Proof: By theorem 4.3, $G_{\mathfrak{R}}$ is first countable. Since f is a continuous homomorphism, $H_{\mathfrak{R}}$ is first countable. Again, by using theorem 4.3, $H_{\mathfrak{R}}$ is also metrizable.

Corollary 4.8. For every first countable topological simple rough group $G_{\mathfrak{R}}$, there exist right-invariant and left-invariant rough pseudometrics, that is $d_{\mathcal{P}}^{R}$ and $d_{\mathcal{P}}^{L}$ that are generate the topology τ on $G_{\mathfrak{R}}$.

Proof: Consider a continuous rough pseudonorm $d_{\mathcal{P}}$ and define rough pseudometrics $d_{\mathcal{P}}^{R}\,(a,b) = d_{\mathcal{P}}\,(ab^{-1})$ and $d_{\mathcal{P}}^{L}\,(a,b) = d_{\mathcal{P}}\,(a^{-1}b)$. As it was already proved in theorem 4.3, the right-invariant rough pseudometric $d_{\mathcal{P}}^{R}$ generates the topology τ on $G_{\mathfrak{R}}$. Since the inverse mapping of $G_{\mathfrak{R}}$ onto itself is homeomorphism, the left-invariant rough pseudometric $d_{\mathcal{P}}^{L}$ is generates the topology τ on $G_{\mathfrak{R}}$.

Theorem 4.9. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $\overline{G_{\mathfrak{R}}}$ is a group. Let $H_{\mathfrak{R}}$ be a closed rough subgroup of a metrizable topological group $\overline{G_{\mathfrak{R}}}$. Then the rough quotient $\overline{G_{\mathfrak{R}}}/H_{\mathfrak{R}}$ is metrizable.

Proof: Since $\overline{G_{\mathfrak{R}}}$ is metrizable, a rough pseudometric $d_{\mathcal{P}}$ generates the topology of $\overline{G_{\mathfrak{R}}}$. Now we consider a rough pseudometric $\mu_{\mathcal{P}}$ on the rough quotient $\overline{G_{\mathfrak{R}}}/H_{\mathfrak{R}}$ such that

$$\mu_{\mathcal{P}}(aH_{\mathfrak{R}}, bH_{\mathfrak{R}}) = inf\{d_{\mathcal{P}}(ah_1, bh_2): h_1, h_2 \in H_{\mathfrak{R}}\}, \text{for all } a, b \in \overline{G_{\mathfrak{R}}}$$

$$= inf\,\{\, d_{\mathcal{P}}(a, bh_2h_1^{-1}): h_1, h_2 \in H_{\mathfrak{R}}\}$$

$$= inf\{d_{\mathcal{P}}(a, bh): h = h_2h_1^{-1} \in H_{\mathfrak{R}}\}$$

which implies $\mu_{\mathcal{P}}(aH_{\Re}, bH_{\Re}) = d_{\mathcal{P}}(a, bH_{\Re}) > 0$. Also given that $H_{\Re}$ is a closed rough subgroup and if $\mu_{\mathcal{P}}(aH_{\Re}, bH_{\Re}) = d_{\mathcal{P}}(a, bH_{\Re}) = 0$, then $a \in bH_{\Re}$, that is, $aH_{\Re} = bH_{\Re}$. Since $d_{\mathcal{P}}$ is symmetric,

$$\mu_{\mathcal{P}}(bH_{\Re}, aH_{\Re}) = inf\{d_{\mathcal{P}}(b, ah) : h \in H_{\Re}\}$$

$$= inf\{d_{\mathcal{P}}(bh^{-1}, a) : h \in H_{\Re}\}$$

$$= inf\{d_{\mathcal{P}}(a, bh^{-1}) : h \in H_{\Re}\}$$

$$= d_{\mathcal{P}}(a, bH_{\Re}) = \mu_{\mathcal{P}}(aH_{\Re}, bH_{\Re}).$$

Therefore, $\mu_{\mathcal{P}}$ is symmetric. Let $a, b, c \in \overline{G_{\Re}}$ and let δ be an arbitrary positive real number. Let $h_1, h_2 \in H_{\Re}$ such that

$$\mu_{\mathcal{P}}(aH_{\Re}, bH_{\Re}) < d_{\mathcal{P}}(a, bh_1) + \frac{\delta}{2} \text{ and } \mu_{\mathcal{P}}(bH_{\Re}, cH_{\Re}) < d_{\mathcal{P}}(b, ch_2) + \frac{\delta}{2}$$

So, we get $\mu_{\mathcal{P}}(aH_{\Re}, cH_{\Re}) \leq d_{\mathcal{P}}(a, ch_2h_1) \leq d_{\mathcal{P}}(a, bh_1) + d_{\mathcal{P}}(bh_1, ch_2h_1)$

$$= d_{\mathcal{P}}(a, bh_1) + d_{\mathcal{P}}(b, ch_2)$$

$$\mu_{\mathcal{P}}(aH_{\Re}, cH_{\Re}) < \mu_{\mathcal{P}}(aH_{\Re}, bH_{\Re}) + \mu_{\mathcal{P}}(bH_{\Re}, cH_{\Re}) + \delta$$

Since δ is arbitrary, $\mu_{\mathcal{P}}(aH_{\Re}, cH_{\Re}) \leq \mu_{\mathcal{P}}(aH_{\Re}, bH_{\Re}) + \mu_{\mathcal{P}}(bH_{\Re}, cH_{\Re})$. Therefore, $\mu_{\mathcal{P}}$ is a rough pseudometric. Now let us prove that the rough pseudometric $\mu_{\mathcal{P}}$ generates the topology of the rough quotient $\overline{G_{\Re}}/H_{\Re}$. Let $a \in \overline{G_{\Re}}$ and $\delta > 0$ and consider

$$U(a, \delta) = \{b \in \overline{G_{\Re}} : d_{\mathcal{P}}(a, b) < \delta\}$$

$$V(aH_{\Re}, \delta) = \{bH_{\Re} : b \in \overline{G_{\Re}} : \mu_{\mathcal{P}}(aH_{\Re}, bH_{\Re}) < \delta\}$$

Define the quotient map $\varphi : \overline{G_{\Re}} \to \overline{G_{\Re}}/H_{\Re}$ by $\varphi(a) = aH_{\Re}$, for all $a \in \overline{G_{\Re}}$. Then by the definition of rough pseudometric $\mu_{\mathcal{P}}$, $\varphi(U(a, \delta)) = V(aH_{\Re}, \delta)$, for $a \in \overline{G_{\Re}}$ and $\delta > 0$. Since $U(a, \delta)$ is a base for $\overline{G_{\Re}}$ and the quotient map φ is continuous and open, $V(aH_{\Re}, \delta)$ is a base for the rough quotient $\overline{G_{\Re}}/H_{\Re}$ and also it generates the topology of $\overline{G_{\Re}}/H_{\Re}$. Hence the rough quotient $\overline{G_{\Re}}/H_{\Re}$ is metrizable.

5. Conclusion:

In this paper, we discussed about the rough pseudonorm and metrizability in topological simple rough groups. Also, we explored the interplay between rough pseudonorms and metrizability. Further, we proved the necessary and sufficient condition for a topological simple rough group to be metrizable.

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