Weibull Generalized Rayleigh Distribution: Properties and Applications

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Abstract

In many practical scenarios, standard probability distributions fall short in modeling complex data structures, particularly in real life data, financial markets, and biological data, etc. To address this gap, we introduce a novel custom probability distribution tailored to real life datasets. This study introduces a novel univariate continuous probability distribution with four parameters, named the Weibull Generalized Rayleigh (WGRL) distribution. This new distribution is formulated by combining the Weibull-H class and the Generalized Rayleigh (GR) distribution. Various statistical properties, including moments, and order statistics. Utilizing the least squares method, Cramer-Von Misses approach, and maximum likelihood estimation, the parameters are computed. A real data set called "the number of deaths per day due to Covid-19 in Nepal during the first wave" is gathered in order to demonstrate the applicability of the model. P-P and Q-Q plot analysis is used to evaluate the model's validity. Some information criteria are used for model comparisons. This work aims to create a novel probability model to get more flexible, innovative and applicable to modern real datasets. We demonstrate that this new distribution outperforms traditional models in terms of fit accuracy, predictive power, and applicable in real life through both theoretical analysis and empirical validation using two real data sets. Model defined here may be useful in studying the characteristics; applications and method of knowing the custom probability model. Statistics from Anderson Darling, Cramer-Von Mises, and Kolmogorov-Smirnov are utilized to assess the suggested model's goodness of fit. R-programming language performs all computations and analysis tasks.

Keywords: *Generalized Rayleigh distribution, Estimation, probability density function, Weibull-H class of distribution*

1. Introduction

Analyzing data is a crucial aspect of research. Various methods exist for studying and interpreting data, with one of the most significant tools being the use of probability distributions. Probability distributions are fundamental tools in statistical modeling, data analysis, and various applications across fields such as engineering, finance, and environmental science. Probability distributions also help researchers to make valid inferences about the potentiality of data. There are many probability models are available in literatures. Researchers have been facing a number of problems in real life concerning the analysis of real data because the classical probability models can analyze all the characteristics of the potentiality and characteristics of the data (Merovci, 2014). To study the data more precisely, different family of distributions as well new probability distributions has been proposed. Development of new family of distributions and new probability models requires the generalization of the existing family of distributions and probability distributions. Other methods of formulation of new models are by compounding the existing models. The main aim of generalization and formulation of new models is to get more flexibility in modeling and data analysis (Usman et al., 2017).

Modification of the existing models can be done by taking exponent in form of parameters, adding extra parameters or by taking the inverse of the variables under study etc. Two parameters Weibull distribution is modified to generate several distributions. The Weibull distribution has been modified into the Exponentiated Weibull distribution

(Mudholkar & Srivastav, 1993). The Weibull distribution, modified by Lai et al. (2003), features a hazard-rate function with a bathtub shape. Surles and Padgett (2005) presented the Rayleigh Generalized distribution which is a member of the generalized Weibull family. The compounding of existing distributions has been found to generalize distributions effectively.

Several researchers have proposed new distribution families by combining or modifying existing ones. For instance, George and Thobias (2017) introduced the Marshall-Olkin Kumaraswamy distribution, while Cordeiro et al. (2016) presented the beta odd log-logistic generalized family, which accommodates various hazard rates including J-shaped, upside-down bathtub, constant, increasing, and decreasing rates. Chaudhary et al. (2023) introduced Modified Upside-Down Bathtub-Shaped Hazard Function Distribution. Bourguignon et al. (2014) introduced the Weibull-G family, and Cordeiro et al. (2017) defined the exponentiated Weibull-H family through an exponentiation of the Weibull-H class.

Further advancements include the Logistic-Rayleigh distribution developed by Chaudhary and Kumar (2020) and the exponentiated Weibull Rayleigh (EWR) distribution by Elgarhy (2019). The extended odd Weibull Rayleigh (EOWR) distribution was introduced by (Almongy et al.,2021) for the analysis of COVID-19 death rates. Ogunsanya et al. (2021) recommended the Weibull Inverse Rayleigh (WIR) distribution. Shen et al. (2022) introduced a generalized Rayleigh distribution for modeling datasets such as Reddit advertising and breast cancer data. The Modified Generalized Rayleigh distribution, which has an inverted bathtub-shaped hazard rate, was created by (Telee & Kumar, 2022). Kumara and Nair (2022) recommended the Additive Log-Inverse Weibull Distribution. Bhat and Ahmad (2023) extended the Exponentiated Rayleigh distribution to include various hazard rate shapes such as decreasing, bathtub, increasing, constant, and J-shaped patterns. Additionally, Bhat et al. (2023) introduced the odd Lindley power Rayleigh distribution, which displays both decreasing and increasing failure rates. The Rayleigh distribution, originally presented by Rayleigh (1882), remains a prominent choice for modeling lifetime data.

2. Material and Methods

In this article, "Weibull generalized Rayleigh (WGRL) distribution" is developed by compounding Weibull-H class of distribution and Generalized Rayleigh distribution (Kundu & Raqab, 2005). The Weibull distribution is a flexible probability distribution that is used to describe failure times, extreme values, and life data in a variety of domains, including survival research and reliability analysis. The Weibull distribution can accommodate different types of failure rates and is particularly useful in analyzing life data and modeling reliability.

The Rayleigh distribution is a continuous probability distribution often used in fields such as signal processing, reliability engineering, and meteorology. It is particularly useful in modeling the magnitude of a vector with two independent, normally distributed components and is commonly used in scenarios involving the distribution of noise or random signals. Rayleigh distribution, often used for modeling the magnitude of random vectors, each serve distinct purposes. However, there are scenarios where neither distribution adequately captures the underlying data characteristics, necessitating the development of a more flexible model. In other words, Traditional distributions like the Weibull and Rayleigh, while useful, have limitations when applied to complex datasets. The Weibull distribution, though adaptable, may not always capture the specific tail behaviors or skewness of the data. Conversely, the Rayleigh distribution is effective for modeling magnitude data but lacks the flexibility to accommodate varying hazard rates. The WGRL aims to address these limitations by combining the adaptability of the Weibull distribution with the simplicity of the Rayleigh distribution, providing a more robust tool for modeling a wider range of data characteristics. In response to this need, we introduce the Weibull Generalized Rayleigh Distribution (WGRL), a novel probability distribution that integrates the strengths of both the Weibull and Rayleigh distributions. The WGRL extends the Rayleigh distribution by incorporating a Weibull component, allowing for enhanced flexibility in modeling data with varying tail behaviors and hazard rates. This article aims to improve the flexibility of the Weibull-H family distribution. In addition, the model includes other significant attributes including moment, moment generating function, order statistics, etc. The least squares approach, Cramer-Von Misses technique, and technique of maximum likelihood estimation are used parameter estimation. Data analysis, comparisons and inferential studies are explained are also included in the study. Analytical methods as well as the R programming language are used for the computation and graphics.

Objectives

Customization of the traditional probability model is advancement of the modern probability theory. As number of parameters increases, in some cases, the model obtained become more flexible and valid compared to the classical model. In many cases, we find data with large number of outliers that make hard to fit with classical model. In such case, customized model fits more adequately than traditional model. In literature, there are various customized probability model available which helps in fitting new data sets better.

The purpose of this study is to define and explore the Weibull Generalized Rayleigh Distribution. We seek to:

a. Develop a comprehensive mathematical formulation of the WGRL, as well as to study key statistical properties.
b. Analyze the distribution's behavior under different parameter settings to show its flexibility and applicability.
c. Demonstrate the utility of the WGRL through practical applications in various fields, highlighting its advantages over traditional distributions.

Methodology

We derive the Weibull Generalized Rayleigh Distribution by integrating the Weibull distribution's shape parameter into the Rayleigh framework. The resulting distribution is characterized by a flexible shape parameter that adjusts the distribution's tail behavior and hazard function. We provide a detailed derivation of the WGRL's mathematical properties, including its moments, entropy, and hazard function. Additionally, we employ applied the model on two real data sets to verify its validity and the applicability. Analytical method and R programming is used to numerical calculations and for graphical studies.

Significance and Applications

The WGRL offers several advantages over existing distributions. Its flexibility makes it particularly useful for modeling data with complex patterns that are not well captured by simpler distributions. We explore its applications in reliability analysis, environmental modeling, signal processing, where traditional models may fall short. By demonstrating the WGRL's superior fit and interpretability, we highlight its potential to enhance analysis and decision-making in these areas.

Model Formulation

The Weibull-H class has CDF and PDF as

$$F(x;\beta,\theta)=1-e^{\left[-\beta u\right]}, \tag{2.1}$$

$$f(x;\beta,\theta)=\beta\theta H(x;\xi)^{\theta-1}\left(\overline{H}\left(x,\xi\right)^{-(\theta+1)}\right)h(x:\xi)\exp\left[-\beta u\right] \tag{2.2}$$

where $u=H(x;\xi)^{\theta}\,\overline{H}\left(x,\xi\right)^{-\theta}$

Also, $H(x;\xi)$, $\overline{H}\left(x,\xi\right)^{\theta+1}$ and $h(x:\xi)$ stand for the base line distribution's CDF, PDF, and reliability function, respectively. Generalized Rayleigh distribution (Kundu & Raqab, 2005) has cdf and pdf as given by eqs. (2.3) and (2.4) respectively.

$$H(x;\alpha,\lambda)=\left(1-e^{-(\lambda x)^2}\right)^{\alpha};\left(x,\alpha,\lambda>0\right) \tag{2.3}$$

$$h(x;\alpha,\lambda)=2\alpha\lambda^2 xe^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\alpha} \tag{2.4}$$

Thus, CDF and PDF of WGRL distribution has function

$$F(x:\alpha,\beta,\lambda,\theta)=1-\exp\left[-\beta\left\{\left(1-v\right)^{-\alpha}-1\right\}^{-\theta}\right];(\alpha,\beta,\lambda,\theta,x)>0 \tag{2.5}$$

$$f(x:\alpha,\beta,\lambda,\theta)=2v\theta\alpha\beta\lambda^{2}v\left(1-v\right)^{\alpha\theta-1}\exp\left[-\beta\left\{\left(1-v\right)^{-\alpha}-1\right\}^{-\theta}\right]\left(1-\left(1-v\right)^{\alpha}\right)^{-(\theta+1)};x>0,(\alpha,\beta,\lambda,\theta)>0 \tag{2.6}$$

Where, $v= e^{-(\lambda x)^{2}}$
Here, α, θ be shape and λ, β are two scale parameters.

Survival function:
Expression (2.8) provides the survival function of the WGRL, which is represented by R(x).

$$R(x)=\exp\left[-\beta\left\{\left(1-v\right)^{-\alpha}\right\}^{-\theta}\right];(\alpha,\beta,\lambda,\theta,x)>0 \tag{2.7}$$

Hazard rate:

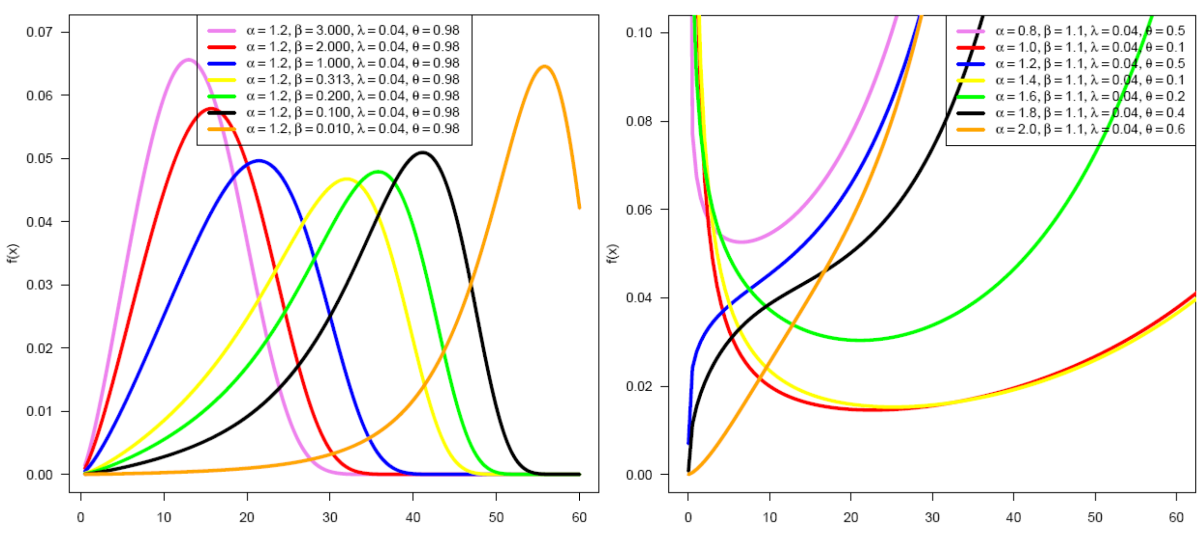
Expression (2.8) provides the hazard rate function (HRF), which is represented by (h(x)).

$$h(x)=2\alpha\beta\lambda^{2}\theta xv\left(1-v\right)^{\alpha\theta-1}\exp\left[-\beta\left\{\left(1-v\right)^{-\alpha}-1\right\}^{-\theta}\right]\left(1-\left(1-v\right)^{\alpha}\right)^{-(\theta+1)}T_{1} \tag{2.8}$$

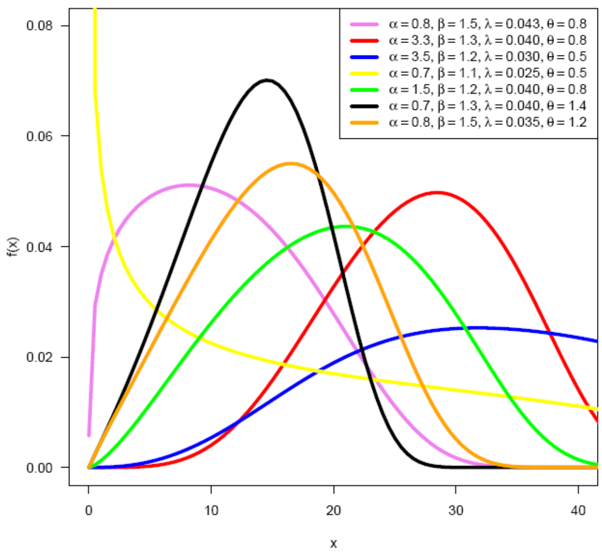
Where, $T_{1}=\left[\exp\left[-\beta\left\{\left(1-v\right)^{-\alpha}\right\}^{-\theta}\right]\right]^{-1};x>0,(\alpha,\beta,\lambda,\theta)>0$

Figure 1 illustrates the density plots and hazard rate plots corresponding to various parameter values. For fixed (α=1.2, λ=0.04, θ = 0.98) density curve is symmetrical at β = 0.31257, positively skewed for β > 0.31257 and negatively skewed for β < 0.31257(figure 1(a)). Changing the values of parameters, different density curves can exist (figure 1(c)).

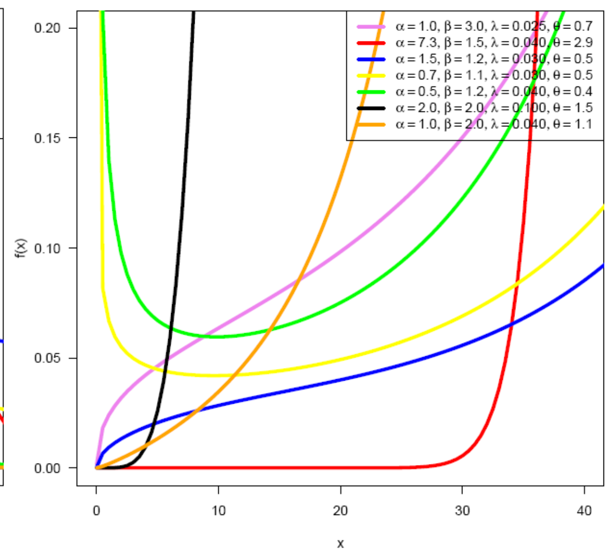
The hazard rate functions vary in shape depending on the parameter values. Shape may be j-shaped, decreasing, increasing, increasing-then-decreasing, or bathtub-shaped, as illustrated in Figures 1(b) and 1(c).



(a) (b)



(c)



(d)

Fig.1: pdf (Right side) and cdf (Right side)

Reversed hazard rate:

The Reversed hazard rate denoted by r(x) is given by expression (2.9)

$$r(x) = 2\alpha\beta\lambda^2\theta x v \exp\left[-\beta\left\{(1-v)^{-\alpha}-1\right\}^{-\theta}\right](1-v)^{\alpha\theta-1}\left(1-(1-v)^{\alpha}\right)^{-(1+\theta)} T_2$$

$$\text{Where, } T_2 = \left[1-\exp\left[-\beta\left\{(1-v)^{-\alpha}\right\}^{-\theta}\right]\right]^{-1};\, x>0, \tag{2.9}$$

It is a useful tool for understanding the behavior of survival times and the changes in risk or hazard rates over time, under the event of interest not occurred by a certain time.

3. Major Properties of Distribution

Probability distributions are fundamental concepts in probability and statistics, describing how probabilities are distributed over possible outcomes. The probability model has many different features.

Some major properties:

Quantile function:

In probability and statistics, the quantile function is an effective tool that may be used to identify certain points or thresholds in a distribution. It is essential for data analysis, risk assessment, and many other applications. It is alternative function to CDF. Quantile function Q (u) = F^{-1}(u), where u follows U (0, 1). It is given by expression (3.1).

$$Q(u) = \left[\left(-1/\lambda^2\right)\ln\left\{1-\left(1+\left(1-(1/\beta)\log(1-u)\right)^{(-1/\theta)}\right)^{(-1/\alpha)}\right\}\right]^{1/2}, 0<u<1 \tag{3.1}$$

Median:

Taking u= ½ to find the median of the model using above quantile function,

$$\text{Median} = \left[\left(-\frac{1}{\lambda^2}\right)\ln\left\{1-\left(1+\left(1-\left(\frac{1}{\beta}\right)\log\left(\frac{1}{2}\right)\right)^{(-1/\theta)}\right)^{(-1/\alpha)}\right\}\right]^{1/2} \tag{3.2}$$

Random Deviate Generation:

Generating random deviates involves using methods that transform uniform random variables or other simple distributions into the desired distribution. The choice of method depends on the complexity of the target distribution, computational efficiency, and ease of implementation. Each method has its own set of advantages and use cases, making it necessary to select the suitable one depending on the application. The model's random deviation generation is,

$$x = \left[\left(-1/\lambda^2\right)\ln\left\{1-\left(1+\left(1-(1/\beta)\log(1-u)\right)^{(-1/\theta)}\right)^{(-1/\alpha)}\right\}\right]^{1/2}, 0 < u < 1 \tag{3.3}$$

Asymptotic behavior:

The asymptotic behavior of a probability model refers to the behavior of the model or the probabilities it describes as some parameter or aspect of the model grows large or approaches a certain limit. Understanding these behaviors is crucial for statistical inference, modeling, and applications across various fields. The distribution's asymptotic behavior is determined by $\lim_{x\to 0} f(x) = 0 = \lim_{x\to\infty} f(x)$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{2\alpha\beta\lambda^2\theta x v\left(1-v\right)^{\alpha\theta-1}}{\left(1-\left(1-v\right)^{\alpha}\right)^{(\theta+1)}} \exp\left[-\beta\left\{\frac{1}{\left(1-v\right)^{\alpha}}-1\right\}^{-\theta}\right] = 0$$

$$\lim_{x\to\infty} f(x) = 2\lim_{x\to\infty} \frac{2\alpha\beta\lambda^2\theta x v\left(1-v\right)^{\alpha\theta-1}}{\left(1-\left(1-v\right)^{\alpha}\right)^{(\theta+1)}} \exp\left[-\beta\left\{\frac{1}{\left(1-v\right)^{\alpha}}-1\right\}^{-\theta}\right] = 0$$

Here, $\lim_{x\to 0} f(x) = 0 = \lim_{x\to\infty} f(x)$. This suggests that the suggested distribution has a modal value since it has asymptotic features.

Mode:

Mode of the model can be calculated by solving for $f'(x) = 0$. For simplicity, taking log of the density function, we get

$$\ln f(x) = \ln(2\alpha\beta\lambda^2\theta) + \ln x - (\lambda x)^2 + (\alpha\theta - 1)\ln\left(1-v\right) - \left(\theta+1\right)\ln\left(1-\left(1-v\right)^{\alpha}\right) + \ln\left[-\beta\left\{\left(1-v\right)^{-\alpha}-1\right\}^{-\theta}\right] = 0$$

Differentiating and applying $f'(x) = 0$, we get

$$e^{(\lambda x)^2} - 2\lambda x v + 2x^2(\alpha\theta-1)\lambda^2(1\text{-}v)^{-1} + 2(\theta+1)\lambda^2 x^2(1\text{-}v)^{\alpha-1}(1-(1\text{-}v)^{\alpha})^{-1} + 2\alpha\beta\theta\lambda^2 x^2((1\text{-}v)^{-\alpha}-1)^{-(\theta+1)}(1\text{-}v)^{-(\alpha+1)} = 0 \tag{3.4}$$

Equation (3.4) cannot be solved analytically, so Newton-Raphson method (Ypma ,1995) can be used to solve it for *x* which will be modal value.

Skewness and kurtosis:

Here, the Bowley's coefficient of skewness technique based on quantiles (Al-saiary et al., 2019) has been used as,

$$S_{KB} = \frac{Q3 + Q1 - 2*Q2}{Q3 - Q1}$$

where, Q denotes for quantile function. Using the quantile function defined in eq (3.1), we have calculated the coefficient of skewness for some particular sets of parameters to know about the nature of the density curve.

Table 1: Coefficient of skewness for various values of parameters

α	β	λ	θ	SK
2	1.20000000	1.3000	0.40	0.120000
1	0.33390015	0.0435	0.82	0.000063
2	0.50000000	0.0700	0.06	-0.074600

Coefficient of skewness presented in table 1 shows that density curve may be positively, symmetrical and negatively skewed depending on the parameters value.
The following relationship may be used to derive the Coefficient of Octiles Kurtosis by (Moors ,1998).

$$K_u = \frac{Q(7/8) - Q(1/8) - Q(5/8) + Q(3/8)}{Q(6/8) - Q(2/8)}$$

Some set of values of variable are generated using random deviate generation (3.3) and some basic characteristics for different set of parameters are calculated.

Table 2: Some basic characteristics of the model on basis of generated data

α	β	λ	θ	Q_1	Median	Mean	Mode	Q_3	Skewness	Kurtosis
1.20	3.00	0.04	0.98	23.2614	24.3129	24.9622	23.0143	26.1580	1.0336	3.1742
1.20	1.00	0.04	0.98	24.2998	26.8433	27.7077	25.1145	30.4657	0.6782	2.3970
1.20	0.10	0.04	0.98	32.1538	38.9324	38.3877	40.0218	44.6251	− 0.0864	1.9266
1.20	0.01	0.04	0.98	46.6746	53.4129	51.9826	56.2735	58.3214	− 0.6995	3.1735
1.20	3.00	2.00	0.10	31.9511	33.3590	34.0503	31.9764	35.625	0.8814	2.8076
1.20	2.50	2.00	0.50	0.5689	0.7783	0.8532	0.6285	1.0854	0.6723	2.3413
1.20	1.50	2.00	1.40	0.4703	0.4995	0.5146	0.4693	0.5474	0.9113	2.8713
1.20	1.00	2.00	1.80	0.4693	0.4953	0.5069	0.4721	0.5359	0.8246	2.6851
1.20	0.01	2.00	2.60	0.6423	0.7056	0.6955	0.7258	0.7552	− 0.4614	2.6405
0.05	3.00	0.28	1.20	0.0048	0.0182	0.0171	0.0204	0.0192	2.4085	8.7699

From table above, it is clear that distribution possess different characteristics of the skewness and kurtosis. For some set of data, density curve is negatively skewed and for some sets it is positively skewed. In same way, curves are platykurtic and leptokurtic. Also, the curve has distinct value of mode.

Useful Expansions:

The distribution that follows is obtained for examining the different aspects of the model using binomial series in generalized form. Assuming $|Z|<1$; $n>0$, we may produce

$$(1-w)^r = \sum_{t=0}^{\infty} (-1)^t \binom{r}{t} w^t ; r>0$$

Here given below is the exponential function in power series expansion :

$$e^{-aw} = \sum_{t=0}^{\infty} \frac{(-1)^t (aw)^t}{t!}$$

Using the previously described exponential and binomial expansion, the proposed model's cumulative distribution function and probability density function in series expansion form become,

$$F(x) = 1 - \phi_{ijk} e^{-k(\lambda x)^2} \tag{3.5}$$

$$\text{Where, } \phi_{ijk} = \sum_{(i,j,k)=0}^{\infty} (-1)^k (-\beta)^i \binom{\theta i + j - 1}{\theta i - 1} \binom{\alpha\theta i + \alpha j}{k}$$

And,
$$f(x) = \phi_{ijk} x e^{-(1+i)\lambda^2 x^2} \tag{3.6}$$

$$\text{Where, } \phi_{ijk} = 2\alpha\beta\theta\lambda^2 \sum_{(i,j,k)=0}^{\infty} (-1)^{i+j} (\beta)^j \binom{\alpha\theta + \alpha k + \theta\alpha j - 1}{i} \binom{k + \theta j + \theta}{\theta j + \theta}$$

Expressions for moments:

In statistics, "moments" are specific quantitative measures that describe various aspects of a probability distribution or dataset. Moments provide a comprehensive way of explanation for a distribution, including its central tendency, variability, asymmetry, and shape. The r^{th} raw moment μ'_r of model is,

$$\mu'_r = E(X^r) = \int_0^\infty x^r f(x) dx$$

Using equation (3.6) in equation (3.7), we get

$$\mu'_r = \int_0^\infty x^r \phi_{ijk} x e^{-(1+i)\lambda^2 x^2} dx$$

The r^{th} arbitrary moment of the MGRL obtained is as,

$$\mu_r' = E(X^r) = \phi_{ijk} \frac{\Gamma\left(1 + \frac{r}{2}\right)}{2b^{\left(\frac{r+1}{2}\right)}}; \quad r = 1, 2, 3, 4, \ldots \tag{3.7}$$

Where, b = (i+1)λ^2 and $\Gamma x = \int_0^\infty x^{s-1} e^{-x} dx = (x-1)!,$ is gamma function of x.

First order raw moment of the MGRL when r = 1 is as follows:

$$\mu_1' = \phi_{ijk} \frac{\sqrt{\pi}}{4b}.$$

The MGRL's second order raw moment will be as follows when r = 2.

$$\mu_2' = \frac{\phi_{ijk}}{2b^{3/2}}.$$

Variance of the model is given as,

$$\text{Var(x)} = \mu_2' - (\mu_1')^2$$

$$= \left(\frac{\phi_{ijk}}{2b^{3/2}}\right) . - \left(\frac{\phi_{ijk}\sqrt{\pi}}{4b}\right)^2 = . \frac{\phi_{ijk}}{2b^{3/2}} \left(1 - \frac{\phi_{ijk}\pi}{8\sqrt{b}}\right)$$

The lower incomplete moments $\varphi_s(t)$ can be calculated using relation,

$$\varphi_S(t) = \int_0^t x^S f(x) dx \quad = \phi_{ijk} \int_0^t x^S x e^{-(1+i)\lambda^2 x^2} dx$$

Hence, $\varphi_s(t) = \dfrac{\gamma\left(\frac{s}{2}+1, bt^2\right)}{2b^{\left(\frac{s}{2}+1\right)}}$

where, $\gamma(s,t) = \int_0^t x^{s-1} e^{-x} dx$ is lower incomplete gamma function

Similarly, the conditional moments of WGRL are given as.

$$\tau_s(t) = \int_t^\infty x^s f(x) dx = \int_t^\infty x^s \phi_{ijk} x e^{-(1+i)\lambda^2 x^2} dx$$

$$\tau_s(t) = \phi_{ijk} \frac{\Gamma\left(\frac{s}{2}+1, bt^2\right)}{2b^{\left(\frac{s}{2}+1\right)}} \tag{3.8}$$

where, $\Gamma(s,t) = \int_0^t x^{s-1} e^{-x} dx$ is upper incomplete gamma function.

Expression for Moment Generating Function (MGF)

The MGF is a versatile and important concept in probability theory that provides a way to encapsulate and work with the moments. It is useful for deriving properties of distributions, computing moments, and analyzing the behavior of sums of random variables. Understanding and using the MGF can simplify many problems in both theoretical and applied statistics. Moment generating function (MGF) of model is given as;

$$M_X(t) = E[e^{tX}] = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \sum_{r=0}^{\infty} t^r \frac{\Gamma\left(1+\frac{r}{2}\right)}{2r! b^{\left(\frac{1+r}{2}\right)}} \tag{3.9}$$

Residual Life function

The residual lifetime (or residual life) is an important concept in survival analysis and reliability theory. It provides a measure of the remaining time until an event occurs, given that the event has not yet occurred. Residual life of the decision variable has nth moments given by expression below

$$m_n(t) = \frac{1}{R(t)} \int_t^\infty (x-t)^n f(x) dx$$

Using expansion for binomial series, $(x-t)^n$ can expanded as by expression,

$$(x-t)^n = \sum_{l=0}^{n} (-1)^l \binom{n}{l} x^{n-l} t^l$$

$$m_n(t) = \frac{\phi_{ijk}}{R(t)} \sum_{l=0}^{\infty} \binom{n}{l} (-t)^l \frac{\Gamma\left(\frac{n-l+2}{2}\right), bt^2}{2b^{\left(\frac{n-l+2}{2}\right)}}$$

Similarly, the WGRL's n^{th} moment of revised residual life function may be computed as,

$$M_n(t)=\frac{1}{F(t)}\int_0^t (t-x)^n f(x)dx$$

$$M_n(t)=\frac{\phi_{ijk}}{F(t)}\sum_{l=0}^{\infty}\binom{n}{l}(-1)^{l+t}(t)^l\frac{\gamma\left(\frac{n-l+2}{2}\right),bt^2}{2b^{\left(\frac{n-l+2}{2}\right)}}$$

Mean past lifetime (MPL):

MPL is denoted as k(x), is defined as the expected remaining time after the failure of a component with a total lifetime of x or less. It is expressed as the conditional expectation of X − x for X ≤ x. The MPL can be computed using the formula:

$$k(x)=E[x-X\,/\,X\le x]=\frac{\int_0^{\infty}F(x)dx}{F(x)}$$

$$=x-\frac{\int_0^{t}xf(x)dx}{F(x)}=x-\left(\frac{\phi_{ijk}}{2b^{3/2}F(x)}\right)\gamma\left(\frac{3}{2},bt^2\right)$$

Deviation from Average:

The mean deviation quantifies the average distance between data points and a central value (mean or median). It is useful for understanding the spread of data and is simpler and more intuitive than variance and standard deviation. By providing insights into variability, the mean deviation helps in descriptive analysis and comparing distributions. Mean deviation is the measure of variability of the observational values from central values of distribution.

(i.) **Mean deviation under mean [MD(μ)]:** Mean deviation under mean is ,

$$MD(\mu)=\int_0^{\infty}|x-\mu|f(x)dx=\int_0^{\mu}\left\{(\mu-x)+(x-\mu)\right\}f(x)dx$$

$$=2\mu F(\mu)-2\mu+\int_{\mu}^{\infty}xf(x)dx\ =2\mu F(\mu)-2\mu+\int_{\mu}^{\infty}x\phi_{ijk}xe^{-bx^2}dx$$

$$=2\mu F(\mu)-2\mu+\phi_{ijk}\int_{\mu}^{\infty}x^2e^{-bx^2}dx=2\mu F(\mu)-2\mu+\frac{\phi_{ijk}}{2b^{3/2}}\Gamma\left(\frac{3}{2},b\mu^2\right)$$

(ii.) **Mean deviation under median :** Under median , it is as

$$MD(m_d)=\int_0^{\infty}|x-m_d|f(x)dx=\int_0^{m_d}\left\{(m_d-x)+(x-m_d)\right\}f(x)dx$$

$$=2\mu F(m_d)-\mu-m_d+\frac{\phi_{ijk}}{b^{3/2}}\Gamma\left(\frac{3}{2},b\,m_d{}^2\right)$$

Order statistics:

In life testing and in field of reliability, there is immense use of the order statistics. Let $x_{(i)}$,1 ≤ i ≤ n is a random sample of size n from order statistics $X_{i:n}$. Let $X_{r:n}$; 1 ≤ r ≤ n is the pdf $f_{r;n}$ (x) of the rth order statistics. Here, rth order statistics can be defined as,

$$f_{r;n}(x)=\frac{n!}{(n-r)!(r-1)!}f(x)[1-F(x)]^{n-r}[F(x)]^{r-1}$$

$$= \frac{n!\phi_{ijk} x e^{-bx^2}}{(n-r)!(r-1)!}[\omega_{ijk} e^{-k\lambda^2 x^2}]^{n-r}[1-\omega_{ijk} e^{-k\lambda^2 x^2}]^{1-r} \tag{3.10}$$

Using expression (3.5), Where, $\omega_{ijk} = \sum_{i,j,k=0}^{\infty} (-1)^k (-\beta)^i \binom{\alpha\theta i + \alpha j}{k}\binom{\theta i + j - 1}{\theta i - 1}$

Biggest order statistics $f_{n;n}(x)$ has pdf as follows for r = n.

$$f_{n:n}(x) = n\phi_{ijk} x e^{-bx^2}[1-\omega_{ijk} e^{-k\lambda^2 x^2}]^{n-1} \quad ; \; x_{(n)} > 0$$

Least order statistics $f_{1;n}(x)$ has pdf as follows when r = 1.s

$$f_{1:n}(x) = n\phi_{ijk} x e^{-bx^2}[\omega_{ijk} e^{-k\lambda^2 x^2}]^{n-1} \quad ; x_{(1)} > 0$$

4. Parameter estimation

Estimating parameters of probability distributions is essential for effective statistical analysis and modeling. Various techniques are available for this purpose, each offering distinct advantages and limitations. The selection of a particular method typically hinges on the specifics of the problem, the characteristics of the data, and the intended properties of the estimators. Among the commonly employed methods for parameter estimation are,

Maximum Likelihood Estimation

Log likelihood function $\ell(x;\alpha,\phi,\lambda,\theta)$ for a random sample $x_1,\ldots,x_n$ from $WGRL(\alpha,\beta,\lambda,\theta)$ is;

$$\ell(x;\alpha,\phi,\lambda,\theta) = n\ln(2\alpha\beta\lambda^2\theta) + \sum_{i=1}^{n}\ln x_i - \lambda^2\sum_{i=1}^{n} x_i^2 + (\lambda\theta - 1)\sum_{i=1}^{n}\ln\left(1-e^{-(\lambda x_i)^2}\right)$$
$$-(\theta+1)\sum_{i=1}^{n}\ln\left[1-\left(1-e^{-(\lambda x_i)^2}\right)^{\alpha}\right] - \beta\left[\left(1-e^{-(\lambda x_i)^2}\right)^{-\alpha} - 1\right]^{-\theta} \tag{4.1}$$

First order partial derivatives of the eq. (4.1) are

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - \theta\sum_{i=1}^{n}\log u_i + (\theta+1)\sum_{i=1}^{n} u_i^{\alpha}\left(1-u^{\alpha}\right)^{-1}\ln u_i - \beta\theta\sum_{i=1}^{n} u_i^{-\alpha}\left(1-u_i^{-\alpha}\right)^{-(\theta+1)}$$

where, $u_i = \left(1\text{-}e^{-(\lambda x_i)^2}\right)$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} - \left[\left(1-u\right)^{-\alpha} - 1\right]^{-\theta}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{2n}{\lambda} - 2\lambda\sum_{i=1}^{n} x^2{}_i + \theta\sum_{i=1}^{n}\ln u_i - 2\lambda(\lambda\theta-1)\sum_{i=1}^{n} u_i^{-1} x^2{}_i e^{-(\lambda x_i)^2} - 2\alpha\lambda(\theta+1)u_i^{\alpha-1}\left(1-u_i^{\alpha}\right)^{-1} x_i^2 e^{-(\lambda x_i)^2}$$
$$-2\alpha\beta\lambda\theta u_i^{-(\alpha+1)}\left(u_i^{-\alpha}-1\right)^{-(\theta+1)} x_i^2 e^{-(\lambda x_i)^2}$$

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} + \lambda\sum_{i=1}^{n}\ln u_i - \sum_{i=1}^{n}\ln\left(1-u^{\alpha}\right) - \beta\left(u^{-\alpha}-1\right)^{-\theta}\ln\left(u^{-\alpha}-1\right)$$

Applying $\frac{\partial\ell}{\partial\alpha} == \frac{\partial\ell}{\partial\beta} = \frac{\partial\ell}{\partial\lambda} = \frac{\partial\ell}{\partial\theta} = 0$ and solving for parameters will give the estimated values of the parameters. Since above partial derivatives are not linear so it is not possible to solve analytically and can be solved using suitable computer programming. The estimated parameter $\hat{\Theta} = (\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})$ under MLE of $WGRL(\alpha,\beta,\lambda,\theta)$ with the parameter vector $\Theta = (\alpha, \beta, \theta, \lambda)$, then the asymptotic normality results in, $\left(\hat{\Theta}-\Theta\right) \to N_3\left[0,\left(I(\Theta)\right)^{-1}\right]$,where $I(\Theta)$ stands for the Fisher's information matrix given by,

$$I(\Theta) = -\begin{pmatrix} E\left(\frac{\partial^2 \ell}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \ell}{\partial \alpha\, \partial \beta}\right) & E\left(\frac{\partial^2 \ell}{\partial \alpha\, \partial \lambda}\right) & E\left(\frac{\partial^2 \ell}{\partial \alpha\, \partial \theta}\right) \\ E\left(\frac{\partial^2 \ell}{\partial \beta\, \partial \alpha}\right) & E\left(\frac{\partial^2 \ell}{\partial \beta^2}\right) & E\left(\frac{\partial^2 \ell}{\partial \beta\, \partial \lambda}\right) & E\left(\frac{\partial^2 \ell}{\partial \beta\, \partial \theta}\right) \\ E\left(\frac{\partial^2 \ell}{\partial \lambda \partial \alpha}\right) & E\left(\frac{\partial^2 \ell}{\partial \lambda\, \partial \beta}\right) & E\left(\frac{\partial^2 \ell}{\partial \lambda^2}\right) & E\left(\frac{\partial^2 \ell}{\partial \lambda\, \partial \theta}\right) \\ E\left(\frac{\partial^2 \ell}{\partial \theta\, \partial \alpha}\right) & E\left(\frac{\partial^2 \ell}{\partial \theta\, \partial \beta}\right) & E\left(\frac{\partial^2 \ell}{\partial \theta\, \partial \lambda}\right) & E\left(\frac{\partial^2 \ell}{\partial \theta^2}\right) \end{pmatrix}$$

In practice, it is often impossible to determine the value of Θ, making the asymptotic variance $\left(I(\Theta)\right)^{-1}$ of the maximum likelihood estimator (MLE) meaningless. However, by using estimated parameter values, we can approximate the asymptotic variance. This is accomplished by obtaining the observed Fisher information matrix, represented as $O(\hat{\Theta})$ which functions as an approximation of the information matrix $I(\Theta)$, based on the Hessian matrix H provided below.

$$O(\hat{\Theta}) = -\begin{pmatrix} \left(\frac{\partial^2 \ell}{\partial \alpha^2}\right) & \left(\frac{\partial^2 \ell}{\partial \alpha\, \partial \beta}\right) & \left(\frac{\partial^2 \ell}{\partial \alpha\, \partial \lambda}\right) & \left(\frac{\partial^2 \ell}{\partial \alpha\, \partial \theta}\right) \\ \left(\frac{\partial^2 \ell}{\partial \beta\, \partial \alpha}\right) & \left(\frac{\partial^2 \ell}{\partial \beta^2}\right) & \left(\frac{\partial^2 \ell}{\partial \beta\, \partial \lambda}\right) & \left(\frac{\partial^2 \ell}{\partial \beta\, \partial \theta}\right) \\ \left(\frac{\partial^2 \ell}{\partial \lambda \partial \alpha}\right) & \left(\frac{\partial^2 \ell}{\partial \lambda\, \partial \beta}\right) & \left(\frac{\partial^2 \ell}{\partial \lambda^2}\right) & \left(\frac{\partial^2 \ell}{\partial \lambda\, \partial \theta}\right) \\ \left(\frac{\partial^2 \ell}{\partial \theta\, \partial \alpha}\right) & \left(\frac{\partial^2 \ell}{\partial \theta\, \partial \beta}\right) & \left(\frac{\partial^2 \ell}{\partial \theta\, \partial \lambda}\right) & \left(\frac{\partial^2 \ell}{\partial \theta^2}\right) \end{pmatrix}_{\Big|\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta}\right)} = -H(\Theta)_{\Big|\left(\Theta=\hat{\Theta}\right)}$$

After using the Newton-Raphson method of maximizing likelihood, equation (4.2) yields the observed information matrix, which defines the variance-covariance. matrix.

$$\left[-H(\underline{\Theta})_{|(\Theta=\hat{\Theta})}\right]^{-1} = \begin{pmatrix} \text{variance}(\hat{\alpha}) & \text{co variance}(\hat{\alpha},\hat{\beta}) & \text{co variance}(\hat{\alpha},\hat{\lambda}) & \text{co variance}(\hat{\alpha},\hat{\theta}) \\ \text{co variance}(\hat{\beta},\hat{\alpha}) & \text{variance}(\hat{\beta}) & \text{co variance}(\hat{\beta},\hat{\lambda}) & \text{co variance}(\hat{\beta},\hat{\theta}) \\ \text{co variance}(\hat{\lambda},\hat{\alpha}) & \text{co variance}(\hat{\lambda},\hat{\beta}) & \text{variance}(\hat{\lambda}) & \text{co variance}(\hat{\lambda},\hat{\theta}) \\ \text{co variance}(\hat{\theta},\hat{\alpha}) & \text{co variance}(\hat{\theta},\hat{\beta}) & \text{co variance}(\hat{\theta},\hat{\lambda}) & \text{variance}(\hat{\theta}) \end{pmatrix} \tag{4.2}$$

Least Square Estimation

Let $X_{(i)},\ 1 \le i \le n$ is a ordered random variable having distribution function F (.) and $x_{(i)}$ is a random sample of size n from a distribution. Let C is a function defined by

$$C(x;\alpha,\beta,\lambda,\theta) = \sum_{i=1}^{n}\left[F(X_{(i)}) - \frac{i}{n+1}\right]^2 \tag{4.3}$$

For estimation of parameters of the $WGRL(\alpha,\beta,\lambda,\theta)$ we should minimize the function defined in (4.3) . That is

$$C\left(x_{(i)}\right) = \sum_{i=1}^{n}\left[1-\exp\left\{-\beta\left\{\left(1-z\right)^{-\alpha}-1\right\}\right\}^{-\theta} - \frac{i}{(n+1)}\right]^2;\ x>0,(\alpha,\beta,\lambda,\theta)>0$$

Differentiating with respect to α, β, λ, and θ a, we get

$$\frac{\partial C}{\partial \alpha} == 2\beta\theta\sum_{i=1}^{n} u_{(i)}^{-\alpha}\left(u_{(i)}^{-\alpha}-1\right)^{-(\theta+1)} e^{-\beta\left(u_{(i)}^{-\alpha}-1\right)^{-\theta}} \ln u_{(i)}\left[F(X_{(i)})-\frac{i}{n+1}\right]$$

$$\frac{\partial C}{\partial \beta} = -2\beta\sum_{i=1}^{n}\left(u_{(i)}^{-\alpha}-1\right)^{-\theta} e^{-\left(u_{(i)}^{-\alpha}-1\right)^{-\theta}}\left[F(x_{(i)})-\frac{i}{n+1}\right]$$

$$\frac{\partial C}{\partial \lambda} = 4\alpha\beta\lambda\theta\sum_{i=1}^{n} e^{-\beta\left(u_{(i)}^{-\alpha}-1\right)^{-\theta}}\left(u_{(i)}^{-\alpha}-1\right)^{-(\theta+1)}\left(u_{(i)}^{-\alpha}-1\right)x^{2}_{(i)}e^{-\left(\lambda x_{(i)}\right)^{2}}\left[F(X_{(i)})-\frac{i}{n+1}\right]$$

$$\frac{\partial C}{\partial \theta} = 2\beta\sum_{i=1}^{n} e^{-\beta\left(u_{(i)}^{-\alpha}-1\right)^{-\theta}}\left(u_{(i)}^{-\alpha}-1\right)^{-\theta}\ln\left(u_{(i)}^{-\alpha}-1\right)\left[F(x_{(i)})-\frac{i}{n+1}\right]$$

Solution of $\frac{\partial C}{\partial \alpha}=0, \frac{\partial C}{\partial \beta}=0, \frac{\partial C}{\partial \lambda}=0$, $\frac{\partial C}{\partial \theta}=0,$ will give LSE.

Parameters can be also be estimated by using weighted least square method. For this we define a function D as,

$$D\left(X;\alpha,\beta,\lambda,\theta\right)=\sum_{i=1}^{n} w_i\left[F(X_{(i)})-\left\{\frac{i}{n+1}\right\}\right] \text{Where, } w_i=\frac{1}{Var[X_{(i)}]}=\frac{(n+2)\left(n^2+2n+1\right)}{i\left(n-i+1\right)}$$

Function *D* becomes,

$$D\left(X;\alpha,\beta,\lambda,\theta\right)=\sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i\left(n-i+1\right)}\left[1-\exp\left\{-\beta\left\{\left(1-e^{-(\lambda x)^2}\right)^{-\alpha}-1\right\}\right\}^{-\theta}-\left\{\frac{i}{n+1}\right\}\right]^2$$

Cramer-Von-Mises estimation

Let $X_{(i)};1\le i\le n$ is a ordered random variable having distribution function F (.) and $x_{(i)}$ is a sample with size n . The CVME of α, β, λ, and θ can be found by minimizing R defined as,

$$R\left(X;\alpha,\beta,\lambda,\theta\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{i:n}\mid\alpha,\beta,\lambda,\theta\right)-\left\{\frac{2i-1}{2n}\right\}\right]^2$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\exp\left\{-\beta\left\{\left(1-e^{-(\lambda x)^2}\right)^{-\alpha}-1\right\}\right\}^{-\theta}-\frac{2i-1}{2n}\right]^2;\ x>0,(\alpha,\beta,\lambda,\theta)>0 \tag{4.4}$$

Differentiating equation (4.4) w.r. to α, β, λ, and θ yields:

$$\frac{\partial R}{\partial \alpha} == 2\beta\theta\sum_{i=1}^{n} u_{(i)}^{-\alpha}\left(u_{(i)}^{-\alpha}-1\right)^{-(\theta+1)} e^{-\beta\left(u_{(i)}^{-\alpha}-1\right)^{-\theta}} \ln u_{(i)}\left[F(X_{(i)})-\frac{2i-1}{2n}\right]$$

$$\frac{\partial R}{\partial \beta} = -2\beta\sum_{i=1}^{n}\left(u_{(i)}^{-\alpha}-1\right)^{-\theta} e^{-\left(u_{(i)}^{-\alpha}-1\right)^{-\theta}}\left[F(x_{(i)})-\frac{2i-1}{2n}\right]$$

$$\frac{\partial R}{\partial \lambda} = 4\alpha\beta\lambda\theta\sum_{i=1}^{n} e^{-\beta\left(u_{(i)}^{-\alpha}-1\right)^{-\theta}}\left(u_{(i)}^{-\alpha}-1\right)^{-(\theta+1)}\left(u_{(i)}^{-\alpha}-1\right)x^{2}_{(i)}e^{-\left(\lambda x_{(i)}\right)^{2}}\left[F(X_{(i)})-\frac{2i-1}{2n}\right]$$

$$\frac{\partial R}{\partial \theta} = 2\beta\sum_{i=1}^{n} e^{-\beta\left(u_{(i)}^{-\alpha}-1\right)^{-\theta}}\left(u_{(i)}^{-\alpha}-1\right)^{-\theta}\ln\left(u_{(i)}^{-\alpha}-1\right)\left[F(x_{(i)})-\frac{2i-1}{2n}\right]$$

Solution of $\frac{\partial R}{\partial \alpha}=0, \frac{\partial R}{\partial \beta}=0, \frac{\partial R}{\partial \lambda}=0$, and $\frac{\partial R}{\partial \theta}=0$ will give CVME

5. Application to real data set

Testing custom probability models on real data sets is essential for validating assumptions, evaluating predictive performance, detecting over fitting, and refining the model. It ensures that the model is practical, reliable, and effective in real-world applications, and it helps to identify and address any limitations or issues that may arise when applying the model outside of theoretical or controlled environments. Two real data sets are collected in order to evaluate the model's applicability.

Data Set I

The dataset is daily COVID-19 deaths in Nepal's first wave, spanning from 23rd January to 24th December 2020 as recorded by (Ministry of Health and Population, Government of Nepal ,2020).
3, 4, 2, 5, 5, 3, 2, 4, 4, 2, 2, 2, 2, 2, 2, 3, 2, 3, 8, 4, 4, 3, 2, 3, 7, 6, 6, 11, 9, 3, 8, 7, 11, 8, 12, 12, 14, 7, 11, 12, 6, 14, 9, 9, 11,6, 6, 5, 5, 14, 9, 15, 11, 8, 4, 7 , 11, 10, 16, 2, 7, 17, 6, 8, 10, 4, 10, 7, 11, 11, 8, 7, 19, 9, 15, 12, 10, 14, 22, 9, 18, 12, 19, 21, 17, 23, 24, 20, 30, 18, 18, 17, 21, 18, 22, 26, 12, 12, 18, 8, 26, 21, 17, 13, 5, 15, 14, 11, 17, 16, 15, 13, 13, 6, 9, 17, 12, 17, 22, 7, 13, 16, 10, 17, 20, 23, 14, 12, 11, 15, 9, 18, 14, 13, 6, 16, 12, 11, 7, 3, 5, 5,16, 16,2 4, 28, 23, 23, 19, 25, 29, 21, 9

Exploratory data analysis (EDA):

EDA is used to summarize, visualize, and understand the underlying structure of a dataset. The goal of EDA is to uncover patterns, spot anomalies, test assumptions, and check the validity of statistical models before performing analyses that are more formal. That is, through exploratory data analysis, important variables and hidden patterns are found in the data. For the provided data, figure 2 displays the Total Time Test (TTT) and boxplot plot. Dataset's eligibility for a certain probability model is evaluated using the TTT plot. Expression for TTT plot is shown below.

$$T\left(\frac{r}{n}\right)=\sum_{i=1}^{n} y(i:n)+(n-r)y_{i:n}\left(\sum_{i=1}^{n} y(i:n)\right)^{-1}$$

When sample order statistics are represented by $y(i:n)\,(i=1,2,...,r)$ and r = 1, 2... n. The recommended distribution's hazard rate shape will increase since the data's TTT plot is concave.

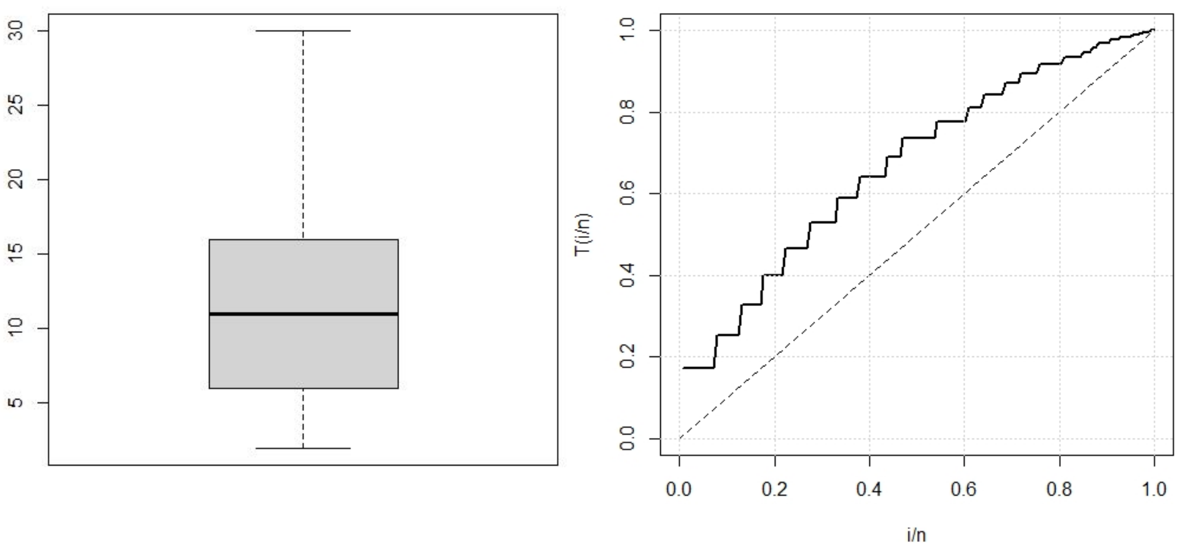


Fig 2. TTT plot (right Plot) and Boxplot (left panel)

Table 3: Descriptive measures

Smallest	1st quartile	Mean	2nd Quartile	3rd Quartile	SD	Skewness	K	Largest
2	6	11.61	11	16	6.76	0.51	2.55	29

The dataset is non-normally shaped and positively skewed

Parameter estimation

The nonlinearity of the partial derivatives makes it hard to estimate parameters using an analytical technique. The R platform's optim() function (R Core Team, 2023) is used to estimate parameters. Table 4 displays the parameters' MLE and accompanying standard error of estimation (SE).

Table 4: MLE and standard errors for the parameters

Parameters	MLE	Standard Error
Alpha	2.2565	1.1769
Beta	1.4937	0.5717
Lambda	0.0678	0.0225
Theta	0.4059	0.1636

Figure 3 illustrates the P-P and Q-Q plots for the model.

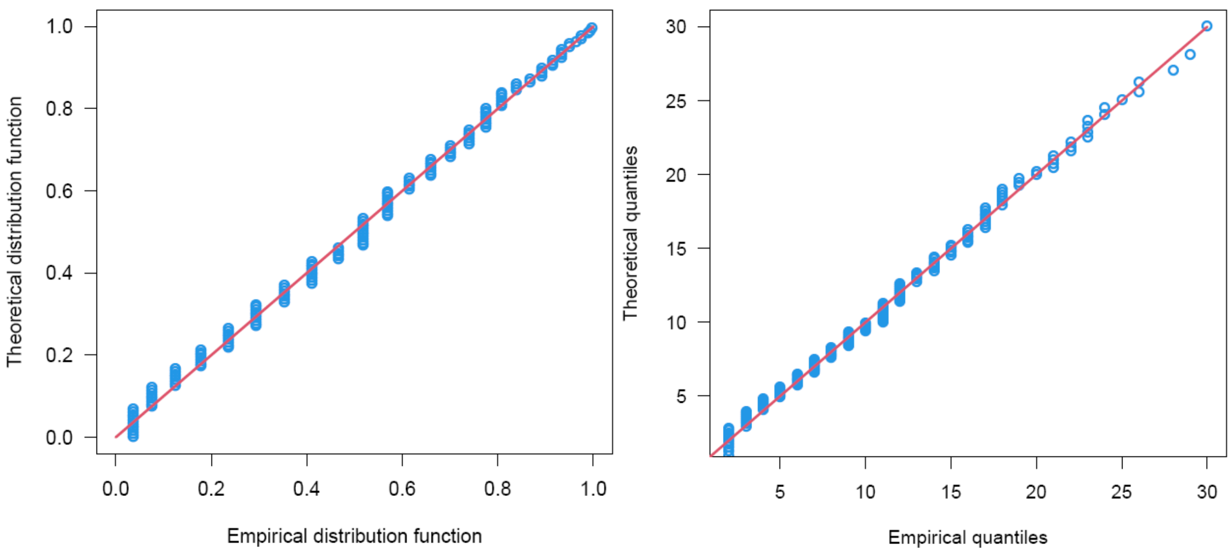


Fig 3: P-P plot on the left and the Q-Q plot on the right

The parameters have been estimated and summarized in Table 5**.** This table includes AIC, BIC, CAIC, and HQIC for model comparison. Additionally, the test statistics are evaluated using AD, KS, and W^2 statistics, with detailed results provided in Table 5.

Methods	MLE	LSE	CVM
$\hat{\alpha}$	2.2565	0.9726	1.4662
$\hat{\beta}$	1.4937	2.3611	1.5746
$\hat{\lambda}$	0.0678	0.0381	0.0565
$\hat{\theta}$	0.4059	0.7532	0.5412
LL	-495.445	-496.3115	-496.4857
AIC	998.8890	1000.6230	1000.9710
BIC	1011.0110	1012.7450	1013.0930
CAIC	999.1592	1000.8930	1001.2420
HQIC	1003.8130	1005.5470	1005.8960
D(p value)	0.0544(0.7565)	0.0518(0.8065)	0.0530(0.7828)
An(p value)	0.4521(0.7957)	0.4089(0.8396)	0.4149(0.8336)
W^2(p value)	0.0503(0.8754)	0.0402(0.9326)	0.0406(0.9304)

The MLE fits the data more accurately than the LSE and CVM, according to the findings. Figure 4 displays the data's histogram as well as the fitted pdf from recommended model.

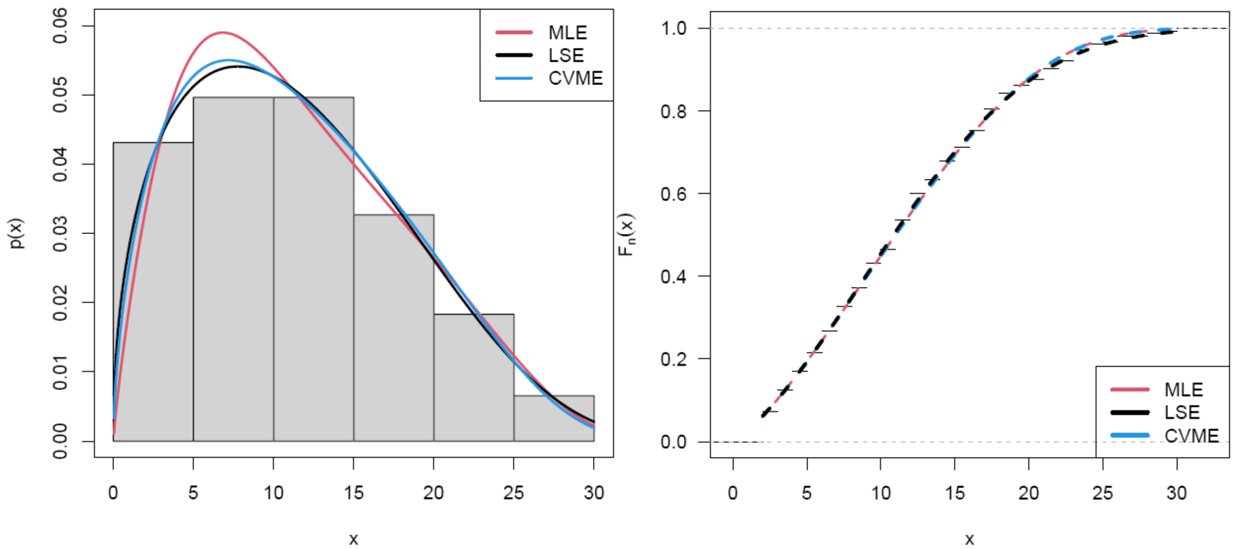


Fig.4: Histogram versus PDF (left panel) and ECDF versus CDF (right panel)

To compare our model with five other probability distributions, we consider several information criteria and inferential statistics based on goodness-of-fit measures. The distributions under comparison include:

(i.) Marshall-Olkin Logistic Exponential (MOLE) Distribution (Monsoor et al., 2019)

(ii.) Exponentiated Half Logistic Exponential (EHLE) Distribution (Almarashi et al., 2018)

(iii.) Exponentiated Weibull (EW) Distribution (Mudholkar & Srivastava, 1993)

(iv.) Odd Lomax Exponential (OLE) Distribution (Ogunsanya et al., 2019)

(v.) Lomax Exponentiated Weibull (LEW) Distribution (Ansari & Nofal, 2021)

The evaluation is based on various criteria and statistical tests that assess the fit of these models to the data. Table 6 displays the estimated parameters of the models and their standard error of estimates

Table 6: Estimates and standard error of estimates.

Models	**alpha**	**beta**	**lambda**	**theta**
	SE	**SE**	**SE**	**SE**
WGRL	.2.2565	1.4937	0.0678	0.4059
	1.1768	0.5717	0.0226	0.1636
EHLE	1.6739	1.6872	0.0903	
	0.2170	1.6870	0.6966	
MOLE	1.7640	0.0551	0.6462	
	1.2355	0.1214	3.5299	
OLE	1.3787	0.1317	6.0933	
	8.9637	0.3793	16.9276	
EW	0.0586	1.2263	1.6152	
	0.1468	0.7111	1.9064	
LEW	21.9102	0.1905		123.0411
	1.4641	0.0175		71.9578

Log-likelihood (LL), AIC, CAIC, BIC, and HQIC are computed for WGRL and rival models in order to compare them. The information criterion values are shown in Table 7.

Table 7: BIC, CAIC, AIC, Log-likelihood (LL), and HQIC of WGRL

Models	AIC	BIC	HQIC	CAIC	LL
MWGRL	998.889	1011.010	1003.813	999.1592	-495.445
EHLE	1007.178	1016.270	1010.871	1007.339	-500.589
MOLE	1012.658	1021.750	1016.351	1012.819	-503.329
OLE	1006.831	1015.920	1010.524	1006.992	-500.416
EW	1006.403	1015.490	1010.096	1006.564	-500.201
LEW	1004.011	1013.100	1007.704	1004.172	-499.006

The suggested model fits the data more accurately than rival models, as shown by smaller criteria values. Figure 5 is the histogram versus density fit of model as well the empirical distribution curves versus theoretical distribution curves.

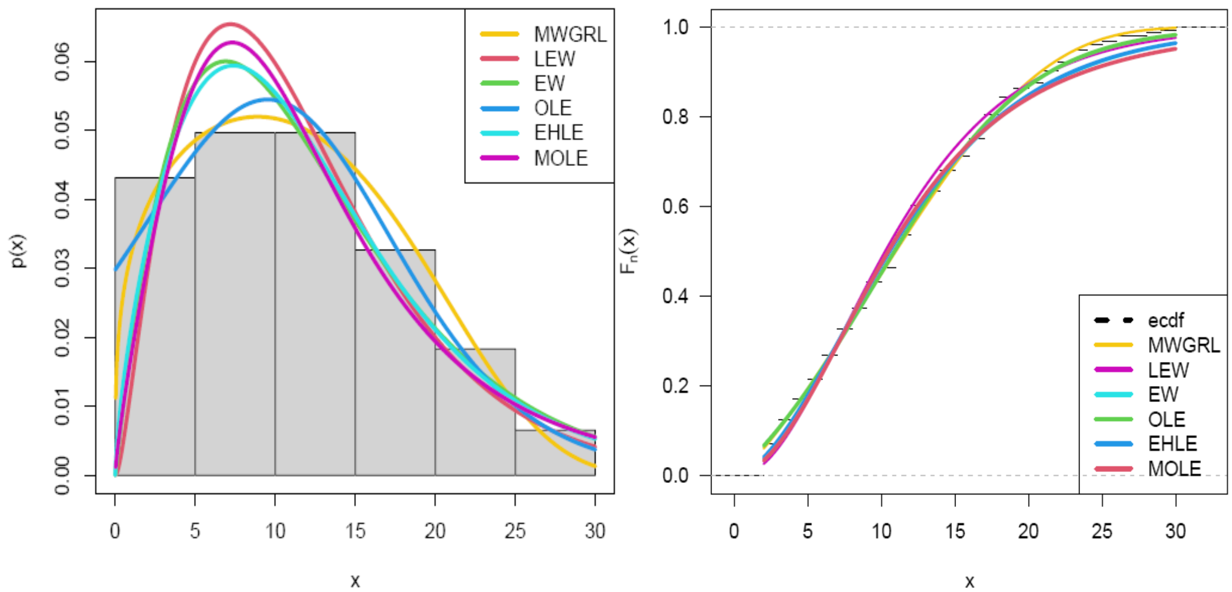


Fig 5. Hist. versus density fit (Left) and ECDF versus theoretical distribution curve (Left)

Table 8 exhibits Anderson-Darling (A^2), Cramer-Von Mises (W^2), and Kolmogorov - Smirnov (KS), statistics and as well as their p-values for different distributions, including the suggested model.

Table 8.: Test statistics and the associated p-values

Models	K-S(p-Values)	CVM (p-Values)	AD (p- Values)
MWGRL	0.0544 (0.7565)	0.0503 (0.8754)	0.4521 (0.7957)
EHLE	0.0592 (0.6572)	0.0761 (0.7156)	0.8240 (0.4639)
MOLE	0.0651 (0.5352)	0.1124 (0.5278)	1.1875 (0.2723)
OLE	0.0665 (0.5078)	0.0490(0.8828)	0.5932 (0.6542)
EW	0.0599 (0.6414)	0.0845 (0.6672)	0.8997 (0.414)
LEW	0.0790 (0.2952)	0.1318 (0.4504)	1.0401 (0.3368)

It is found that the test statistics of the model has lesser value with larger p values compared to the competing models concluding that model fits data better than competing models.

Data Set II:

We have also shown how the $WGRL(\alpha, \beta, \lambda, \theta)$ distribution can be applied using another real dataset, which consists of the survival periods (in days) of seventy-two virulent tuberculin-infected guinea pigs (Bjerkedal, 1960). Data has been used in many researches like in logistic-X family of distribution by (Tahir et al.,2016), Lindely exponential distribution (Umar et al., 2019) it is also applied in Weibull exponential distribution (Raya,2019).

10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231, 240, 245, 251, 253, 254, 255, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555.

Exploratory data analysis:

Boxplots and TTT plots, which are shown in Figure 6, are used to examine the characteristics of the data. Boxplot demonstrates the positive skewness while the TTT plot's concave form suggests that the data's hazard rate curve is increasing.

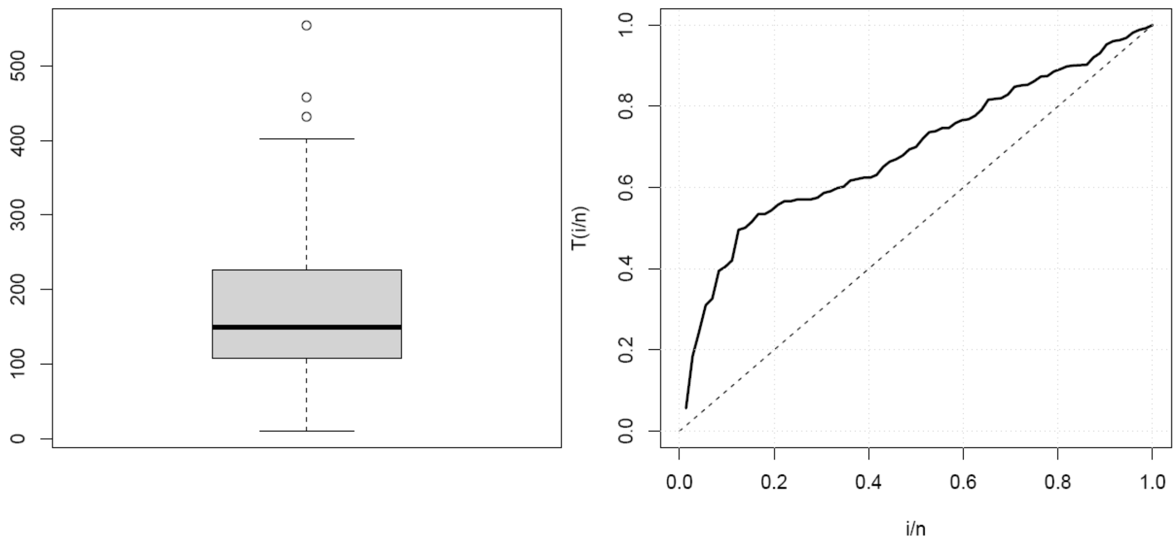


Fig.6: The Boxplot (Right) and the TTT plot in the (Right)

Some vital characteristics of the data is analyzed by calculating the summary statistics and are tabulate in table 9. The curve is non-normal and positively skewed, as the summary demonstrates.

Table 9: Summary statistics

Min.	Q_1	Median	Mean	Q_3	SD	Sk	k	Max.
10.00	108.00	149.50	176.80	224.00	103.47	1.13	4.99	550.00

Parameters of the fitted model are obtained to fit the model. To estimate parameters optim () function of R programming language is used here also. Estimated parameters using Maximum likelihood estimation methods and SE are in table 10.

Table 10: Calculated parameter values and their standard error estimates

Parameters	MLE	SE
Alpha	9.3745	5.2956
Beta	2.5064	0.5729
Lambda	0.0048	0.0008
Theta	0.1528	0.0674

For the second set of data, the model's P-P and Q-Q plots are displayed in Figure 7.

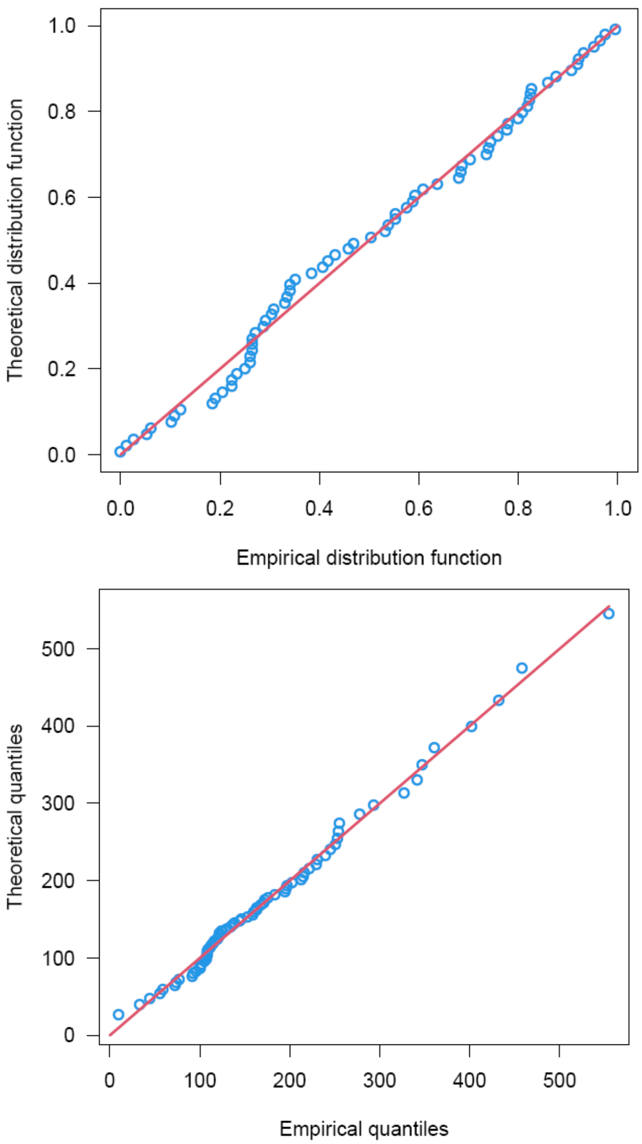


Fig 7: P-P plot (Left) and Q-Q plot (Right) of the proposed model

Using MLE, CVME, and LSE approaches, LL, AIC, BIC, CAIC, and HQIC are computed for model validation. To compare the method of estimation, test statistics using K-S, AD and CVME are also obtained. Since MLE method has least value of test statistics with greater p values indicating MLE method gives better fit to the data. Calculated values are tabulated in table11.

Table 11: Constants, LL, AIC, CAIC, BIC, HQIC with p- values

Methods	MLE	LSE	CVM
$\hat{\alpha}$	9.3745	9.3745	9.3745
$\hat{\beta}$	2.5064	2.5064	2.5064
$\hat{\lambda}$	0.0048	0.0047	0.0047
$\hat{\theta}$	0.1528	0.1528	0.1528
LL	-424.4647	-424.4651	-424.4652
AIC	856.9295	856.9302	856.9304

BIC	866.0361	866.0369	866.0370
CAIC	857.5265	857.5273	857.5274
HQIC	860.5549	860.5556	860.5558
D(p-value)	0.0744 (0.8207)	0.0736 (0.8297)	0.0736 (0.8305)
An(p-value)	0.3512 (0.8949)	0.3495 (0.8965)	0.3494 (0.8966)
W^2(p-value)	0.0543 (0.8512)	0.0543 (0.8515)	0.0543 (0.8516)

Figure 8 presents a comparison between the empirical and theoretical distributions, featuring both the histogram and the fitted pdfs. The density function was estimated using MLE, CVME and LSE.

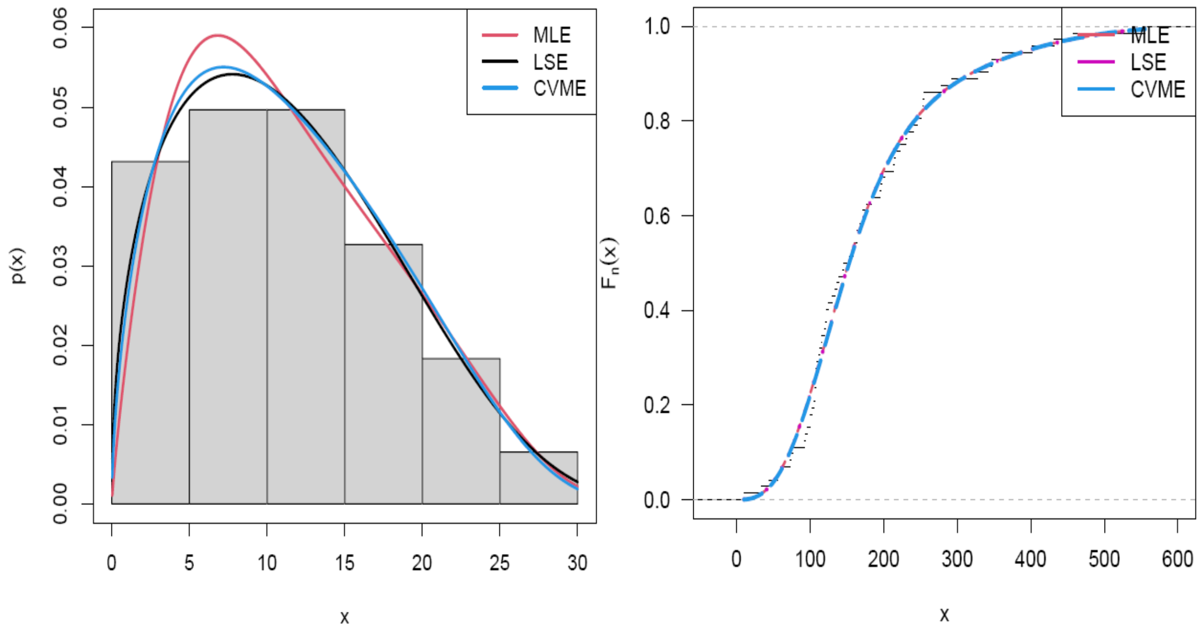


Fig 8. Hist. versus density fit (left) and empirical versus theoretical distribution (right)

(i.) To assess the applicability of our model, we compare it with five other distributions:
(ii.) Exponentiated Generalized Inverted Exponential (EGIE) Distribution (Oguntunde et al., 2014)
(iii.) Generalized Inverted Generalized Exponential (GIGE) Distribution (Oguntunde et al., 2015)
(iv.) Marshall-Olkin Logistic Exponential (MOLE) Distribution (Monsoor et al., 2019)
(v.) Generalized Weibull Extension (GWE) Distribution (Sarhan & Apaloo, 2013).
(vi.) Exponentiated Half Logistic Exponential (EHLE) Distribution (Almarashi et al., 2018)

The parameter estimates for these models were calculated using R programming and are detailed in Table 12.

Table 12: Parameters and standard error of estimates

Models	alpha	beta	lambda	theta	gamma
WGRL	9.3745	2.5064	0.0048	0.1528	
EHLE	0.1283	2.7348	0.1043		
MOLE	1.4771		0.0103	7.8302	
GWE	18.6453	0.1708	9.2162		
EGIE	3.0379	0.7799	245.5836		
GIGE	2.8880		20.4530		10.2910

The fit of the recommended model was assessed using HQIC, AIC, CAIC, BIC, and negative log likelihood values; the results are shown in Table 13.

Table 13: AIC, BIC, CAIC, HQIC, and LL of WGRL

Models	AIC	BIC	CAIC	HQIC	LL
MWGRL	856.9295	866.0361	857.5265	860.5549	-424.4647
EHLE	856.9537	863.7837	857.8067	859.6728	-425.4769
MOLE	858.9337	865.7637	859.2866	861.6527	-426.4668
GWE	859.9908	866.8208	860.3438	862.7099	-426.9954
EGIE	883.2235	890.0535	883.5765	885.9426	-438.6118
GIGE	885.189	892.0190	885.5419	887.5419	-439.5945

A smaller criteria value means that, in comparison to rival models, the suggested model fits the data more accurately. Figure 9 displays fitted pdf & Histogram as well the empirical distribution curves versus theoretical distribution curves.

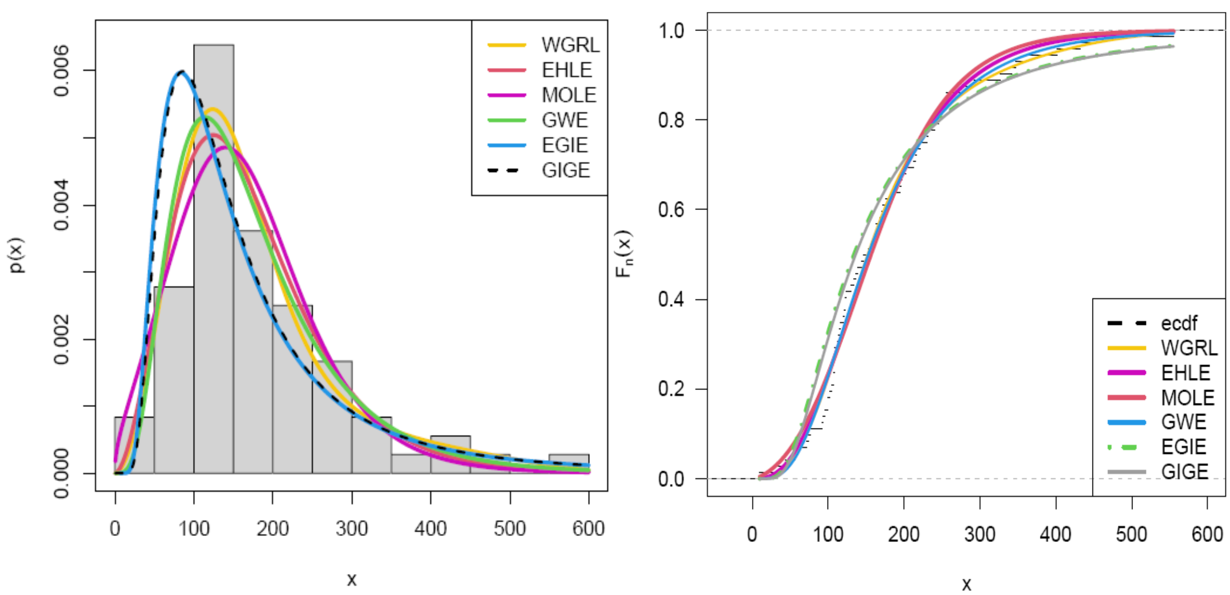


Fig 9. Hist. versus density fit (Left) and ECDF versus theoretical distribution curve (Left)

Table 13 displays KS, A^2, and W^2 statistics, along with p-values, for various distributions, including the proposed model.

Table 14: The associated p values for the test statistics

Models	**K-S(p-Values)**	**CVM (p-Values)**	**AD (p- Values)**
MWGRL	0.0744 (0.8208)	0.0543 (0.851)	0.3512(0.8950)
EHLE	0.0823 (0.7141)	0.0699(0.7535)	0.5064(0.7397)
MOLE	0.0900(0.6038)	0.0985(0.5935)	0.7824(0.4934)
GWE	0.0777(0.7773)	0.0550(0.8472)	0.4322(0.8158)
EGIE	0.1742(0.0253)	0.4087(0.0683)	2.3442(0.0600)
LEW	0.1602(0.0497)	0.3243(0.1155)	1.9897(0.0932)

It is found that the test statistics of the model has lesser value with larger p values compared to the competing models concluding that model fits data better than competing models.

6. Results Discussion and Conclusion

In this article, we have formulated an innovative flexible distribution called Weibull generalized Rayleigh (WGRL) distribution. This is developed by compounding Weibull-H class of distribution and Generalized Rayleigh distribution. We have derived and analyzed some of the suggested distribution's statistical properties. The PDF curve of the WGRL model displays an increasing-decreasing shape with a right-skewed distribution. To assess its applicability, two data sets are considered, and parameter estimation are performed using three methods: maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME). The validity of the model has tested using Q-Q plots, P-P plots, and other information criteria. The suggested model demonstrated a good fit to real-life data under models considered. Furthermore, the hazard function plot varies in shape based on the model parameters, displaying a decreasing, monotonically increasing, and bathtub shape. The proposed distribution's applicability and suitability outperforms other considered distributions, demonstrating its flexibility. This paper bridges the gap between the Weibull and Rayleigh distributions, offering a more versatile tool for statistical modeling. We provide a thorough analysis of the WGRL's properties and applications, underscoring its potential to address challenges encountered with traditional distributions. Future research will focus on further applications, parameter estimation techniques, and comparisons with other advanced distributions.

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