UtilitasMathematica

ISSN 0315-3681 Volume 122, 2025

THE MINIMUM RADIUS DOMINATING ENERGY OF A GRAPH

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Abstract

Let G be a graph of order n. A dominating set S is said to be a radius dominating set if for each vertex u not in S there exists a vertex v in S such that $d(u,v) = rad(G)$. The cardinality of the minimum radius dominating set is called the radius domination number; $\gamma_{rd}(G)$. The energy of a graph is closely related to the energy of the chemical structure whose graphical representation is isomorphic to G. In this article, we shall consider a new energy parameter is defined and studied for standard families of graphs, also established some bounds.

Keywords: Minimum radius dominating set, Minimum radius dominating matrix, Minimum radius dominating Eigen values.

2010 Mathematics Subject Classification: Primary 05C50, 05C69.

1. Introduction

Consider a connected graph G of order n. The concept of graph-energy was introduced by I. Gutman in 1978 [4]. This concept did not gain the attention of many scholars in the beginning but later many researchers in graph theory did realize its importance and world-wide mathematical research of graph-energy started. The idea of set energy in fact boosted the research in graph-energy parameter. In this article, we define a matrix called the minimum radius dominating matrix, de- noted by $A_{rd}(G)$ and we study its Eigen values and the energy. Further, we study the mathematical aspects of the minimum radius dominating energy of a graph. It may be possible that the minimum radius dominating energy which we are considering inthis paper have applications in other areas of science such as chemistry and so on. The graphs we are considering are assumed to be finite, simple, un-directed having no isolated vertices.

Consider a simple connected graph $G = (V, E)$. Let v be any arbitrary vertex,the eccentricity of v denoted by $e(v)$ denotes distance of farthest vertex from v.The set of all vertices at a distances $e(v)$ from v is called radius set of v, denoted by $E(v)$. A subset of vertices is said to be radius point set of G if for each v in $V - S$,there is at least one vertex u in $E(v)$. A dominating set S of a graph $G$is said to be radius dominating set of G if each vertex not in S contains an radius vertex. The least cardinality of a radius dominating set in G is called the minimum eccentricity domination number of G, denoted by $\gamma_{rd}(G)$. Any radius dominating set of cardinality $\gamma_{rd}(G)$ is called the $\gamma_{rd} -$ set. For more details on the terms used in this paper refer [5].

1.1. Definition. The crown graph S_n^0 , $(n \geq 2)$ on $2n$ vertices is a graph with vertex set $V = \{u_i, v_j/1 \leq i,\ j \leq n\}$ with an edge from u_i to v_j whenever $i \neq j$.

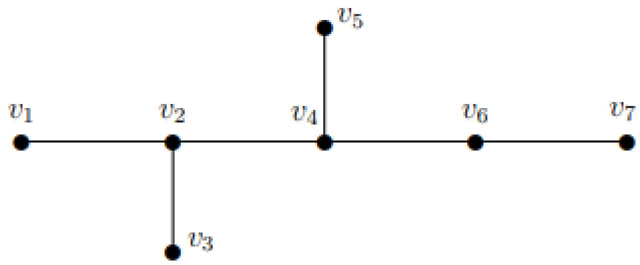

Figure 1: A tree on seven vertices

Definition 1.2. The double star denoted as $S(n, m)$ with $n \geq m \geq 0$, is the graph consisting of the union of two stars$K_n$and K_m together with a line joining their centers.

2. The Minimum Radius dominating energy of a Graph

In this section, we define the minimum radius dominating energy of a graph andstudy some basic properties of the minimum radius dominating energy.

The minimum radius dominating matrix of G is an n−square matrix defined by $A_{rd}(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & if\ \ v_i\ and\ v_j\ \ are\ adjacent, \\ 1 & if\ \ i = j\ \ and\ v_i \epsilon D, \\ 0 & \ \ otherwise \end{cases}$$

The characteristic polynomial of $A_{rd}(G)$ is denoted by $f_n(G, \zeta)$ and is defined by $f_n(G, \zeta) = det(\zeta I - A_{rd}(G))$. The minimum radius dominating Eigen values of the graph G are the Eigen values of the matrix $A_{rd}(G)$. We note that these Eigen values are real numbers since $A_{rd}(G)$ is real and symmetric. So, we can label them in the non-increasing order $\zeta_1 \geqslant \zeta_2 \geqslant \cdots \geqslant \zeta_n$. The minimum radius dominating energy of G is then defined to be the sum of the absolute values of the Eigen values of $A_{rd}(G)$. In symbols, we write

$$E_{rd}(G) = \sum_{i=1}^{n} |\zeta_i|$$

Example 2.1. Consider a tree G of order seven as shown in figure 1, with the vertex set $V = \{v_1, v_2, \dots, v_7\}$ and let $S = \{v_2, v_4, v_6\}$ be a minimum radius dominating set. Then

The minimum radius dominating energy of a graph

$$A_{ed,S}(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

So, $f_n(G,\zeta) = \zeta^7 - 3\zeta^6 - 3\zeta^5 + 9\zeta^4 + 4\zeta^3 - 5\zeta^2 - 2\zeta = 0.$

Using Maple, we are able to find that $E_{rd}(G) = 8.412$.

Now, suppose if we choose another radius dominating set $S' = \{v_2, v_4, v_7\}$ Then

$$A_{ed,S'}(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Clearly, $f_n(G,\zeta) = \zeta^7 - 3\zeta^6 - 3\zeta^5 + 10\zeta^4 + 3\zeta^3 - 7\zeta^2 - 2\zeta = 0.$

Using Maple, we are able to find that $E_{ed,D'}(G) = 8.2732.$

Therefore, it is clear from the above example that the minimum radius dominating energy of a graph G depends on the minimum radius dominating set we choose. Hence, this energy is not the graph invariant.

Theorem 2.1. Consider a connected graph G of order n with an edge set E(G). Let $\gamma_{rd}(G)$ be the radius domination number of G and $f_n(G,\zeta) = c_0\zeta^n + c_1\zeta^{n-1} + c_2\zeta^{n-2} + \cdots c_n$. Then

1. $c_0 = 1.$
2. $c_1 = -\gamma_{rd}(G).$
3. $c_2 = \frac{\gamma_{rd}(\gamma_{rd}-1)}{2} - |E|$

From the properties of Eigen values of a graph, the sum of the squares of the Eigen values *of* $A_{rd}(G)$ will be the trace of$(A_{rd}(G))^2$. Thus, the following result is obvious.

Theorem 2.2. *Let G be any graph and* $\zeta_1, \zeta_2, \cdots, \zeta_n$ *be the Eigen values of* $A_{rd}(G)$*. Then*

$$\sum_{j=1}^{n} {\zeta_j}^2 = 2m + \gamma_{rd}(G)$$

3 The Minimum radius dominating energy of Graphs

In this section, we shall compute the minimum radius dominating energy of somestandard families of graphs. We first define the spectrum of the graph G as follows.

Definition 3.1. Let G be any graph and suppose $\zeta_1, \zeta_2, \cdots, \zeta_k$ are the be distinct Eigen values of G with the multiplicities $m_1, m_2, \dots m_k$ respectively. Then the minimum radius dominating spectrum of the graph G is given by

$$MRDS_{pec}(G) = \begin{pmatrix} \zeta_1 & \zeta_2 & \dots & \zeta_k \\ m_1 & m_2 & \dots & m_k \end{pmatrix}$$

Theorem 3.1. Let $G \cong K_n$ be a complete graph of order $n > 2$. Then $E_{rd}(G) = (n-2) + \sqrt{n^2 - 2n + 5}$.

Proof. Let $G \cong K_n$ be a complete graph of order n. Let the vertex set be$V(G) = \{v_1, v_2, \dots, v_n\}$. By the definition of a complete graph, each vertex is adjacent to every vertex in G. Therefore each vertex of K_n will have eccentricity one and hence any singleton subset leads to minimum radius dominating set in K_n. Let us take $S = \{v_n\}$. Then the minimum radius domination matrix be

$$A_{rd}(K_n) = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 1 & 1 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 1 & 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

The characteristic polynomial be
$f_n(G, \zeta) = (\zeta + 1)^{n-2}(\zeta^2 - (n-1)\zeta - 1)$

The minimum radius dominating spectrum of G be then given by

$$MRDS_{pec}(G) = \begin{pmatrix} -1 & \dfrac{(n-1) + \sqrt{n^2 - 2n + 5}}{2} & \dfrac{(n-1) - \sqrt{n^2 - 2n + 5}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Therefore $E_{rd}(K_n) = (n-2) + \sqrt{n^2 - 2n + 5}$.

Theorem 3.2. Let $G \cong S_n^0$ be a crown graph of order $n > 3$. Then $E_{rd}(G) = 2(n-2) + \sqrt{n^2 - 2n + 5} + \sqrt{n^2 + 2n - 3}$.

Proof. Let $G \cong S_n^0$ be a crown graph of order $n \geq 3$. Let the vertex set be $V(S_n^0) = \{u_1, u_2, u_3, \dots u_n, v_1, v_2, v_3, \dots, v_n\}$ By construction of crown graph, it follows that each vertex will be at a maximum distance of two from any other vertex. Thus $e(v) = 2$, for each vertex v. Thus, any dominating set will be a radius dominating set. Choose $S = \{u_n, v_n\}$. Then the minimum radius dominating matrix will be

$$A_{rd}(S_n^0)=\begin{bmatrix}0&0&0&\dots&0&0&1&1&\dots&1\\0&0&0&\dots&0&1&0&1&\dots&1\\1&1&0&\dots&0&1&1&0&\dots&1\\.&.&.&\dots&.&.&.&.&\dots&1\\.&.&.&\dots&.&.&.&.&\dots&1\\.&.&.&\dots&.&.&.&.&\dots&1\\0&0&0&\dots&1&1&1&1&\dots&0\\0&1&1&\dots&1&0&0&0&\dots&0\\1&0&1&\dots&1&0&0&0&\dots&0\\.&.&.&\dots&.&.&.&.&\dots&0\\.&.&.&\dots&.&.&.&.&\dots&0\\.&.&.&\dots&.&.&.&.&\dots&0\\1&1&1&\dots&0&0&0&0&\dots&1\end{bmatrix}$$

The characteristic polynomial of the matrix will be

$$f_n(G,\zeta) \;=\; (\zeta-1)^{n-2}(\zeta+1)^{(n-2)}(\zeta^2+(n-1)\zeta-1)\,.$$

Hence, the minimum radius dominating spectrum of the crown graph is

$$\begin{pmatrix}1 & -1 & \frac{(n-1)+\sqrt{(n^2-2n+5)}}{2} & \frac{(n-1)+\sqrt{(n^2-2n+5)}}{2} & \frac{(3-n)+\sqrt{(n^2+2n-3)}}{2} & \frac{(3-n)-\sqrt{(n^2+2n-3)}}{2}\\ n-2 & n-2 & 1 & 1 & 1 & 1\end{pmatrix}$$

Therefore$\mathrm{E_{rd}(G)=(n-2)+(n-2)+\sqrt{n^2-2n+5}+\sqrt{n^2+2n-3}}$.

$$\mathrm{E_{rd}(S_n^0\ )=2(n-2)+\sqrt{n^2-2n+5}+\sqrt{n^2+2n-3}}\,.$$

Theorem 3.3. Let $\mathrm{G\cong K_{1,n-1}}$ be a star graph of order $\mathrm{n\geq 2}$. Then $\mathrm{E_{rd}(G)\leq 2+2\sqrt{(n-2)}}$. Equality holds if and only if $\mathrm{n}\ =\ 2$.

Proof. Let $\mathrm{G\cong K_{1,n-1}}$ be a star graph of order $\mathrm{n\geq 2}$. Let the vertex set be $V\,(G)\;=\;\{v_1,v_2,\dots,v_n\}$ with $deg(v_n)\;=\;n-1$. Obviously, the set $S=\{v_n\}$ itself form a radius dominating set of G. Then, the minimum radius dominating matrix will be

$$A_{rd}(K_{1,n-1})=\begin{bmatrix}0&0&0&\dots&0&1\\0&0&0&\dots&0&1\\0&0&0&\dots&0&1\\.&.&.&\dots&.&.\\.&.&.&\dots&.&.\\.&.&.&\dots&.&.\\0&0&0&\dots&0&1\\1&1&1&\dots&1&1\end{bmatrix}$$

$$f_n(K_{1,n-1},\zeta)=\;\zeta^{n-3}(\zeta^3-2\zeta^2-(n-2)\zeta)+(n-2)\,.$$

By using simple analysis, we obtain that

$$\begin{aligned}f_n(K_{1,n-1},\zeta)&=\;\zeta^{n-3}[\zeta^3-2\zeta^2-(n-2)\zeta+(n-2)].\\&=\;\zeta^{n-3}[\zeta^3-2\zeta^2-(n-2)\zeta+2(n-2)-(n-2)].\\&\leq\;\zeta^{n-3}[\zeta^3-2\zeta^2-(n-2)\zeta+2(n-2)].\\&=\;\zeta^{n-3}[(\zeta-2)(\zeta^2-(n-2))].\end{aligned}$$

Hence, $MRDS_{pec}(K_{1,n-1})=\begin{pmatrix}0 & 2 & \sqrt{(n-2)} & -\sqrt{(n-2)}\\ n-3 & 1 & 1 & 1\end{pmatrix}$.

Therefore, $E_{rd}(K_{1,n-1})\leq 2+2\sqrt{(n-2)}$.

Corollary 3.1. The minimum radius dominating energy of $\mathrm{L(K_{1,n-1})}$ is equal to $\mathrm{(n-3)+\sqrt{n^2-4n+8}}$

Proof. We first note that the line graph of $K_{1,n-1}$ is the complete graph K_{n-1}. Now, it follows easily that $E_{rd}(L(K_{1,n-1})) = E_{rd}(L(K_{n-1}))$. Hence from theorem 3.1 (on replacing *n by n-1*), we obtain that, $E_{rd}\left(K_{1,n-1}\right) = (n-3) + \sqrt{n^2 - 4n + 8}$.

Theorem 3.4. For n ≥ 3, the minimum radius dominating energy of $K_{n,n}$ is $(2n - 4) + \sqrt{n^2 - 2n + 5} + \sqrt{n^2 + 2n - 3}$.

Proof. Consider a complete bipartite graph $K_{n,n}$ having the vertex set $V = \{u_j, v_j \,/\, 1 \le i \le m, 1 \le j \le n\}$. Since each vertex will have common eccentricity 2, any dominating set will be a radius dominating set. Let the minimum radius dominating set be $S := \{u_1, v_1\}$. Hence

$$A_{rd}(K_{m,n}) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \\ . & . & . & \dots & . & . & . & . & \dots & 1 \\ . & . & . & \dots & . & . & . & . & \dots & 1 \\ . & . & . & \dots & . & . & . & . & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ . & . & . & \dots & . & . & . & . & \dots & 0 \\ . & . & . & \dots & . & . & . & . & \dots & 0 \\ . & . & . & \dots & . & . & . & . & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

So, the characteristic polynomial is $K_{m,n}$ is given by

$$f_{m,n}\left(K_{m,n}, \zeta\right) = \zeta^{m+n-4}[\zeta^4 - 2\zeta^3 - (mn-1)\zeta^2 + (2mn - m - n)\zeta - (m-1)(n-1)].$$

In particular, for $m = n$ we have,

$$f_{n,n}\left(K_{n,n}, \zeta\right) = (\zeta - 1)^{n-2}(\zeta + 1)^{n-2}(\zeta^2 + (n-3)\zeta - (2n-3))(\zeta^2 - (n-1)\zeta - 1)).$$

Hence, The minimum radius dominating spectrum of $K_{n,n}$is given by

$$\begin{pmatrix} -1 & 1 & \frac{(3-n)+\sqrt{(n^2+2n-3)}}{2} & \frac{(3-n)-\sqrt{(n^2+2n-3)}}{2} & \frac{(n-1)+\sqrt{(n^2-2n+5)}}{2} & \frac{(n-1)-\sqrt{(n^2-2n+5)}}{2} \\ n-2 & n-2 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence, the radius dominating energy of $K_{n,n}$will be

$$E_{rd}(K_{n,n}) = (2n - 4) + \sqrt{n^2 - 2n + 5} + \sqrt{n^2 + 2n - 3} .$$

4 Bounds for $\gamma_{rd}(G)$

Theorem 4.1. Let G be a connected graph of order n and size m. Then $E_{rd}(G)$ is at most $\sqrt{n(2m + \gamma_{rd}(G))}$.

Proof. Consider a connected graph of order n and size m. Let $\zeta_1 \geqslant \zeta_2 \geqslant \cdots \geqslant \zeta_n$. be Eigen values of the minimum radius dominating matrix $A_{rd}(G)$ arranged in the non-increasing order.

We have by Cauchy-Schwarz's inequality that,

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right)$$

If we take $a_j = 1, b_j = |\ \zeta_j\ |$ then

$$\left(\sum_{j=1}^{n} |\ \zeta_j\ |\right)^2 \leq \left(\sum_{j=1}^{n} 1\right)\left(\sum_{j=1}^{n} {\zeta_j}^2\right)$$

Now, by theorem 2.2, we have,

$$(E_{rd}(G))^2 \leq n\left(\sum\nolimits_{j=1}^{n} {\zeta_j}^2\right) = \mathrm{n}(2m + \gamma_{rd}(G))$$

$$E_{rd}(G) \leq \sqrt{n(2m + \gamma_{rd}(G))}.$$

Theorem 4.2. Let G be a connected graph of size m. Then

$$\sqrt{m + \frac{\gamma_{rd}(G)(\gamma_{rd}(G)+1)}{2}} \leq E_{rd}(G) \leq 2\sqrt{\frac{m}{2}}(2m + \gamma_{rd}(G)).$$

Proof. Let G be a connected graph of size m. Then from the definition of minimum radius dominating energy we have,

$$(E_{rd}(G))^2 = \sum_{j=1}^{n} |\ \zeta_j\ |^2 + 2\sum_{j<i} |\ \zeta_i\zeta_j\ |.$$

Now, from Theorem 2.2, we have

$$(E_{rd}(G))^2 = (2m + \gamma_{rd}(G) + 2\sum_{j<i} |\ \zeta_i\zeta_j\ |.$$

Also, we have from Theorem 2.1 that

$$\sum_{j<i} |\ \zeta_i||\ \zeta_j\ | \geq \left|\sum_{j<i} \zeta_i\ \zeta_j\ \right| = \frac{\gamma_{rd}(G)(\gamma_{rd}(G)-1)}{2} - m$$

Therefore, we have

$$(E_{rd}(G))^2 \geq 2m + \gamma_{rd}(G) + \frac{\gamma_{rd}(G)(\gamma_{rd}(G)-1)}{2} - m$$

Hence, we have

$$E_{rd}(G) \geq \sqrt{m + \frac{\gamma_{rd}(G)(\gamma_{rd}(G)+1)}{2}} \qquad (4.1)$$

On the other hand, the maximum number of vertices of in a connected graph of size m is 2m. Hence, from Theorem 4.1, we have

$$E_{rd}(G) \leq 2\sqrt{m\left(m + \frac{\gamma_{rd}(G)}{2}\right)} \qquad (4.2)$$

Now, from equation 4.1 and 4.2, we have

$$\sqrt{m+\frac{\gamma_{rd}(G)(\gamma_{rd}(G)+1)}{2}} \leq E_{rd}(G) \leq 2\sqrt{m\left(m+\frac{\gamma_{rd}(G)}{2}\right)}$$

Theorem 4.3. Let G be a connected graph of size m. Then $E_{rd}(G) \geq \sqrt{2m+\gamma_{rd}(G)}$.

Proof. Let G be a connected graph of size m. For any positive real numbers $a_1, a_{2,} a_3 \dots, a_n$ we have by the consequence of triangle inequality that

$$\left(\sum_{j=1}^{n} a_j\right)^2 \leq \left(\sum_{j=1}^{n} |a_j|^2\right)$$

Let $a_j \; = |\; \zeta_j \;|\; for\ each\ j$. Then

$$\left(\sum_{j=1}^{n} {\zeta_j}^2\right) \leq \left(\sum_{j=1}^{n} |\zeta_j|^2\right) \tag{4.3}$$

Hence,

$$(E_{rd}(G))^2 \geq \left(\sum_{j=1}^{n} {\zeta_j}^2\right) = 2m+\gamma_{rd}(G) \tag{4.4}$$

Therefore,

$$E_{rd}(G) \geq \sqrt{2m+\gamma_{rd}(G)}$$

Theorem 4.4. Let G be a connected graph having n vertices and m edges. Then

$$E_{rd}(G) \geq \sqrt{2m+\gamma_{rd}(G)} + n(n-1)|A_{rd}(G)|^{\frac{2}{n}}.$$

Proof. Let G be a connected graph having n vertices and m edges. By the definition of radius dominating energy of a graph, we have

$$(E_{rd}(G))^2 = \left(\sum_{j=1}^{n} |\zeta_j|\right)^2.$$

$$= \sum_{j=1}^{n} |\; \zeta_j \;|^2 + \sum_{j \neq i} |\; \zeta_i||\zeta_j\;|.$$

Now, by using arithmetic and geometric mean inequality, we get

$$\frac{1}{n(n-1)}\sum_{i \neq j} |\; \zeta_i||\zeta_j\;| \geq \left[\prod_{i \neq j} |\; \zeta_i||\zeta_j\;|\right]^{\frac{1}{n(n-1)}}.$$

$$= \left|\prod_{j=1}^{n} |\zeta_j\;|^{2(n-1)}\right|^{\frac{1}{n(n-1)}}$$

$$= \left|\prod_{j=1}^{n} \zeta_j\right|^{\frac{2}{n}}$$

$$= (|A_{rd}(G)|)^{\frac{2}{n}}$$

Hence,

$$\sum_{i\neq j} |\zeta_i||\zeta_j| \geq n(n-1)(|A_{rd}(G)|)^{\frac{2}{n}}$$

Now consider,

$$[E_{rd}(G)]^2 = \sum_{j=1}^{n} |\zeta_j|^2 + \sum_{j\neq i} |\zeta_i||\zeta_j|$$

$$\geq \sum_{j=1}^{n} |\zeta_j|^2 + n(n-1)(|A_{rd}(G)|)^{\frac{2}{n}}$$

$$i.e., \qquad E_{rd}(G \geq \sqrt{2m + \gamma_{rd}(G) + n(n-1)(|A_{rd}(G)|)^{\frac{2}{n}}}$$

References

1. C. Adiga, A.Bayad, I. Gutman, S. A. Srinivas, The minimum covering energy ofa graph, *Kragujevac J. Sci*. 34 (2012) 39-56.
2. R.B.Bapat, S.Pati, Energy of a graph is never an odd integer.*Bull. Kerala Math.Assoc.* 1, 129 - 132 (2011).
3. Bondy J.A and Murthy.U.S.R, *Graph theory with applications*, American Elsevier,New York.
4. I.Gutman,The energy of a graph.*Ber. Math-Statist. Sekt. Forschungsz.Graz* 103, 1-22,(1978)
5. F. Harary, *Graph Theory*, Narosa Publications House, Reading press.
6. V.R.Kulli, *Theory of domination in Graphs*, Vishwa International Publications
7. M.R.Rajesh Kanna, B.N. Dharmendra, G.Sridhara, *The minimum covering en-ergy of a graph*, IJPAM Vol. 85 No.4 2013, 707-718.