Analysis of Existence and Uniqueness for Riesz-Caputo Fractional Differential Equation with State-Dependent Delay

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Abstract

This study atmost the existence and uniqueness of solutions for Riesz-Caputo fractional differential equations with State dependent Delays. The goal is to set up enough conditions, for the extant and individuality of solutions, investigating the interaction between the State dependent delay and Riesz-Caputo fractional derivative. An illustrative example is provided to demonstrate the result.

Keywords: Delay, Riesz-Caputo derivative, Schaefer's fixed point Theorem, Arzela-Ascoli theorem.

1 Introduction

In recent decades, fractional calculus has gained significant interest from researchers across various disciplines [1, 2, 3, 4, 5, 6]. For instance, in many dynamic processes, the effects of influencing factors can persist even after they have disappeared. The idea of memory plays a significant role in such cases and employing fractional calculus is largely functional in disciplines that includes physics, chemistry, mechanics, and economics . Most available research has focal point on the Riemann-Liouville (R-L) or Caputo derivatives that are fractional one-sided operators that express either future or past memory effects. To recent procedures that rely on both, the latest applications of this derivative have been explained in [7, 8, 9, 10, 11, 12, 13, 14].

More over in these studies, the Riesz-Caputo $(\mathrm{R}-\mathrm{C})$ fractional derivative is formulated in a consistent way within the idealogy of the R-L fractional derivative. The Caputo fractional derivative has been explained to give an apt generalization of the extremum principle [15]. Moreover, there are few results available in the literature pertaining to the existence of solutions for the $\mathrm{R}-\mathrm{C}$ derivative. For example Zhou et al. [16, 17] that explored the existence and uniqueness answers of selective fractional differential equations. In [18, 19, 20] Baleanu et al. studied the availability of solutions for fractional derivative. Fulai Chen [21] explored the boundary value problems of $\mathrm{R}-\mathrm{C}$ fractional derivative with anti-periodic conditions. This paper focus on investigating the $\mathrm{R-C}$ fractional differential equations and state dependent delays.

$$ {}_{0}^{RC}D_{x}^{\alpha}\mathrm{z}(\mathbf{t}) = A\mathrm{z}(\mathbf{t}) + G(\mathbf{t}, z_{\varrho(\mathbf{t},\mathrm{z}_{\mathbf{t}})}) \qquad (1.1) $$

$$ z(0) = \Upsilon(0) \in \mathrm{B}, \mathbf{t} \in \mathrm{I} = [0, a] \qquad (1.2) $$

This research is organised as follows:
We emphasizes certain ideas and review few preliminary facts pertaining to the $\mathrm{R-C}$ fractional derivative in the first Section.
We encompasses the solutions of (1.1)-(1.2) in the second Section.
It is based on the theorems namely Schaefer's fixed point and Banach contraction principle.

2 PRIMARY CONCEPTS

In this particular section, we inform certain representations, descriptions, and preliminary ideas that are all implemented in this work.
The Banach space $C(\theta, \mathcal{R})$ of all continuous functions defined as,

$$ ||\xi||_{\infty} = \sup\{|\xi(v)| : v \in \theta\} $$

Consider the Banach space

$$ \mathcal{C}(\mathcal{J}, \mathcal{R}) = \{\mathrm{z}: \mathcal{J} \to \mathcal{R}; \mathrm{z}_{\Omega j} = \xi_j; j = 1,2, \ldots m, \mathrm{z}_{\Omega j} \in \mathcal{C}(\Omega_j, \mathcal{R}); j = 0,1, \ldots . m $$

and

$$ \mathrm{z}(\mathbf{t}_j^-), \mathrm{z}(\mathbf{t}_j^+), \mathrm{z}(x_j^-), \mathrm{z}(x_j^+) \text{ with } \mathrm{z}(\mathbf{t}_j^-) = \mathrm{z}(\mathbf{t}_j), \} $$

with the norm

$$ \| \mathrm{z} \|_{\mathcal{C}} = \{sup_{\mathbf{t} \in \mathcal{J}} |\mathrm{z}(\mathbf{t})|\} $$

Definition 2.1 *[22] Let $\alpha > 0$. The Riemann Liouville fractional integral of a function $\varphi \in C(\theta, \mathcal{R})$ of order α is given by*

$$ {}_{0}I_{v}^{\alpha}\varphi(v) = \frac{1}{\Gamma(\alpha)} \int_{0}^{v} (v - \varrho)^{\alpha - 1} \varphi(\varrho) d\varrho $$

Definition 2.2 *[22] Let $\alpha > 0$. The Riesz fractional integral of a function $\varphi \in C(\theta, \mathcal{R})$ of order α is given by*

$$ {}_{0}I_{v}^{\alpha}\varphi(v) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} |v - \varrho|^{\alpha - 1} \varphi(\varrho) d\varrho $$

$$ ={}_{0} I_{v}^{\alpha}\varphi(v) + {}_{v}I_{x}^{\alpha}\varphi(v) $$

Definition 2.3 *[22] Let $\alpha \in (n, n+1), n \in \mathcal{N}_0$.* The Caputo fractional derivative of a function $\varphi \in C(\theta, \mathcal{R})$ of order α are given by

$$ {}_{0}{}^{C}D_{v}{}^{\alpha}\varphi(v) = \frac{1}{\Gamma(n + 1 - \alpha)} \int_{0}^{v} (v - \varrho)^{n - \alpha} \varphi^{(n+1)}(\varrho) d\varrho $$

Definition 2.4 *[22] Let* $\alpha \in (n, n+1), n \in \mathcal{N}_0$. The Riesz fractional derivative of a function $\xi \in C(\theta, \mathcal{R})$ of order α are given by

$$ {}_0^{RC}D_x^{\alpha}\varphi(v) = \frac{1}{\Gamma(n+1-\alpha)} \int_0^x |v-\varrho|^{n-\alpha} \varphi^{(n+1)}(\varrho) d\varrho $$

$$ = \tfrac{1}{2}[{}_0^{C}D_v^{\alpha}\varphi(v) + (-1)^{n+1}{}_0^{RC}D_x^{\alpha}\varphi(v)] $$

where ${}_0^{C}D_v^{\alpha}$ is the Caputo derivative .
If we take $0 < \alpha \le 1$ and $\varphi \in \mathcal{C}(\theta, \mathcal{R})$, we obtain

$$ {}_0^{RC}D_x^{\alpha}\varphi(v) = \frac{1}{2}[{}_0^{C}D_v^{\alpha}\varphi(v) - {}_0^{RC}D_x^{\alpha}\varphi(v)] $$

Lemma 2.1 *[22] If* $\xi \in \mathcal{C}^{n+1}(\theta, \mathcal{R})$ *and* $\alpha \in (n, n+1]$ *, then we have*

$$ {}_0I_v^{\alpha}\ {}_0^{C}D_v^{\alpha}\xi(v) = \xi(v) - \textstyle\sum_{j=0}^{n} \frac{\xi^{j}(0)}{j!} v^j \quad \text{and} $$

$$ {}_vI_x^{\alpha}\,{}_v^{C}D_x^{\alpha}\xi(v) = (-1)^{n+1}[\xi(v) - \textstyle\sum_{j=0}^{n} \frac{(-1)^j\xi^j(x)}{j!}(x-v)^j] $$

$$ {}_0I_x^{\alpha}\ {}_0^{RC}D_x^{\alpha}\xi(v) = \tfrac{1}{2}[{}_0I_v^{\alpha}\ {}_0^{C}D_v^{\alpha}\xi(v) + (-1)^{n+1}{}_vI_x^{\alpha}\ {}_v^{C}D_x^{\alpha}\xi(v)] $$

In particular, if $0 \le \alpha \le 1$,then we obtain

$$ {}_0I_x^{\alpha}\ {}_0^{RC}D_x^{\alpha}\xi(v) = \xi(v) - \tfrac{1}{2}(\xi(0) + \xi(x)) $$

Lemma 2.2 *[22] Let* $\omega \in \mathcal{C}(\theta, \mathcal{R})$ *and* $0 \le \alpha \le 1$. *Then* $z \in \mathcal{C}(\theta, \mathcal{R})$ *is a solution of*

$$ {}_0^{RC}D_x^{\alpha}\mathrm{z}(v) = \omega(v), v \in \theta $$

if and only if y verifies the following integral equations :

$$ \mathrm{z}(v) = \mathrm{z}(0) - \tfrac{1}{\Gamma(\alpha)}\int_0^x \varrho^{\alpha-1}\omega(\varrho)d\varrho + \tfrac{1}{\Gamma(\alpha)}\int_0^x |v-\varrho|^{\alpha-1}\omega(\varrho)d\varrho $$

In this paper, we will employ an axiomatic definition for the space Q which is similar to those introduce in [23,24,25,26,27] .

More precisely, B will be a linear space of all functions from $(-\infty, 0]$ *to* $\mathcal{R}$ endowed with a semi norm $||.||_B$ satisfying the following axioms :

(i) If $\mathrm{z}: (-\infty, b] \longrightarrow \mathcal{R}, b > 0$,is continuous on J and $\mathrm{z}_0 \in Q$ (A) , then for every $t \in J$, the following conditions hold :

(ii) $\mathrm{z}_{\mathbf{t}} \in \mathrm{B}$ and $||\mathrm{z}_{\mathbf{t}}||_B \le K(\mathbf{t})\ \sup\{|\mathrm{z}(s)|: 0 \le s \le \mathbf{t}\} + M(t)||\mathrm{z}_0||_B$

(iii) $|\mathrm{z}(\mathbf{t})| \le \mathcal{H}||\mathrm{z}_{\mathbf{t}}||_B$ where $\mathcal{H} > 0$ is a constant , $\mathcal{K}: [0, \infty) \longrightarrow [1, \infty)$ is continuous , $\mathcal{K}: [0, \infty) \longrightarrow [1, \infty)$. locally bounded and $\mathcal{H}, \mathcal{K}, \mathcal{M}$ are independent of $\mathrm{z}(.)$

(iv) For the function z(.) in (A) , $\mathrm{z}_{\mathbf{t}}$ is a B valued continuous function on [0,b].

(v) The space B is complete.

The next lemma is a consequence of the phase space axioms and proved in [28].

Lemma 2.3 *[23] Let $\varphi \in B$and $I = (\gamma, 0]$ such that $\varphi \in B$ for every $t \in I$*
Assume that there exist a locally bounded function $\mathcal{J}^{\varphi}: I \longrightarrow [0, \infty)$
such that $||\varphi_{\mathbf{t}}||_B \leq \mathcal{J}^{\varphi}(\mathbf{t})||\varphi||_B$ for every $t \in B$.
It z: $(\infty, b] \longrightarrow \mathcal{R})$ is continuous on $\mathcal{J}$ and $\mathrm{z}_0 = \varphi$ then,

$$||\mathrm{z}_{\mathbf{t}}||_B \leq (\mathcal{M}_b + \mathcal{J}^{\varphi}(max\{\gamma, |s|\})||\varphi||_{\mathcal{Q}} + \mathcal{K}_b sup\{|\mathrm{z}(\theta)|: \theta \in [0, max\{0, s\}]\}, s \in (\gamma, b]$$

where we denoted $\mathcal{K}_b = sup_{t \in J}\mathcal{K}(\mathbf{t})$ & $\mathcal{M}_b = sup_{\mathbf{t} \in J}\mathcal{M}(t)$

In this section , the non linear alternative of Leray Schauder type is used to investigate the existence solution of problem . Let us start by defining what we mean by a solution of problem (1.1-1.2)

Definition 2.5 *A function* $z: (-\infty, b] \longrightarrow \mathcal{R}$ *is said to be a solution of if* $\mathrm{z}_0 = \Upsilon, \mathrm{z}_{\varrho(s,\mathrm{z}_s)} \in B$
for every $s \in \mathcal{J}$ and

$$\mathrm{z}(\mathbf{t}) = \Upsilon(0) - \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} As^{\alpha-1}\mathrm{z}(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} s^{\alpha-1}\mathrm{G}(s, \mathrm{z}_{\varrho(s,\mathrm{z}(s))})ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} A(\mathbf{t}-s)^{\alpha-1}\mathrm{z}(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} (\mathbf{t}-s)^{\alpha-1}\mathrm{G}(s, \mathrm{z}_{\varrho(s,\mathrm{z}(s))})ds$$

In what follows we assume that $\varphi: \mathcal{J} \times B \longrightarrow (-\infty, b]$ is continuous and $\varphi \in B$ and G satisfies the following hzpotheses :

H1 : G is a continuous function ;

H2 : There exist $p, q \in \mathcal{C}(\mathcal{J}, \mathcal{R}^+)$ such that $|\mathrm{G}(\mathbf{t}, u)| \leqslant p(\mathbf{t}) + q(\mathbf{t})||u||_{\mathfrak{B}}$
for $\mathbf{t} \in \mathcal{J}$ and each $u \in B$, and $||\mathcal{I}^{\beta}p||_{\infty} < +\infty$;

H3 : The function $\mathbf{t} \longrightarrow \varphi_{\mathbf{t}}$ is well defined and continuous from the set
$\mathcal{R}(\varpi) = \{\omega(s, \psi): (s, \psi) \in \mathcal{J} \times B, \omega(s, \psi) \leq 0\}$ into B.
Moreover , there exists a continuous and bounded function
$\mathcal{J}^{\varphi}: \mathcal{R}(\varpi) \longrightarrow (0, \infty)$ such that $||\varphi_{\mathbf{t}}||_{\mathfrak{B}} \leq \mathcal{J}^{\varphi}(\mathbf{t})||\varphi||_{\mathfrak{B}}$ for every $\mathbf{t} \in \mathcal{R}(\varpi)$.

H4 : Lipschitz condition

$$|\mathrm{G}(s, \mathrm{z}_{n(\varrho)(s,\mathrm{z}_{n(s)})}) - \mathrm{G}(s, \mathrm{z}_{\varrho(s,\mathrm{z}(s))})| \leq Ł_f \beta \sup_{(0 \leq s \leq T)}|\mathrm{z}_n - \mathrm{z}|$$

Theorem 2.1 *Assume that the hypothesis (H1) to (H4) hold.*
Let f: $\mathcal{J} \times B \longrightarrow \mathcal{R}$ be an L' Caratheodory function. Assume that

(a). There exist a constant μ such that $||A|| \leq \mu$ for each $t \geq 0$;

(b). There exist a continuous function $\varphi: \mathcal{J} \times B \longrightarrow (-\infty, b]$ such that $\varphi \in B$ and

$$\mathcal{M}_B \leq \mathcal{H}[\mathcal{M}_b + \mathcal{J}^{\varphi}||\varphi||_{\mathfrak{B}} + K_b]$$

$$\mathcal{N}_B \leq P(s) + q(s)\{(\mathcal{M}_b + \mathcal{J}^{\varphi})||\varphi_{|}|\mathfrak{B} + K_b r\}$$

for each bounded $B \subseteq \mathcal{C}(\mathcal{J}, \mathcal{R})$ and $t \in \mathcal{J}$, the set

$$\{z(\mathbf{t}) = \Upsilon(0) - \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} As^{\alpha-1}z(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} s^{\alpha-1}G(s, z_{\varrho(s,z(s))})ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} A(\mathbf{t}-s)^{\alpha-1}z(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} (\mathbf{t}-s)^{\alpha-1}G(s, z_{\varrho(s,z(s))})ds\}$$

is relatively compact in $\mathcal{R}$. Then this problem has at least one solution.

Proof: Let $Y = \{z \in B(\mathcal{J},\mathcal{R}): z(0) = \varphi(0)\}$ endowed with the uniform topology and $N: Y \longrightarrow Y$ be the operator defined by

$$Nz(\mathbf{t}) = \Upsilon(0) - \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} As^{\alpha-1}z(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} s^{\alpha-1}G(s, z_{\varrho(s,z(s))})ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} A(\mathbf{t}-s)^{\alpha-1}z(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} (\mathbf{t}-s)^{\alpha-1}G\left(s, z_{\varrho(s,z(s))}\right)ds$$

Clearly, the fixed points of N are mild solutions to (1.1-1.2) .
The proof will be given in several steps.

***STEP* 1 :** Let z_n be a sequence in Z , such that $z_n \longrightarrow z$,
we will prove that $N(z_n) \longrightarrow N(z)$, for each $t \in \mathcal{J}$, we have

$$|Nz_n(\mathbf{t}) - Nz(\mathbf{t})| \leq \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} |As^{\alpha-1}[z_n(s) - z(s)]|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} |s^{\alpha-1}\left[G\left(s, z_{n\left(\varrho(s,z_n(s))\right)}\right) - G\left(s, z_{\left(\varrho(s,z(s))\right)}\right)\right]|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} |(t-s)^{\alpha-1}\left[G\left(s, z_{n\left(\varrho(s,z_n(s))\right)}\right) - G\left(s, z_{\left(\varrho(s,z(s))\right)}\right)\right]|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} |A(t-s)^{\alpha-1}[z_n(s) - z(s)]|ds$$

B is bounded and f is an L' Caratheodory function we have by the Lebesgue dominated convergence theorem

$$\leq \frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}\mu|z_n - z| + \frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}Ł_f\beta \sup_{0\leq s\leq T}|z_n - z|$$

$$\leq \frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}[\mu + Ł_f\beta]|z_n - z| \quad \text{as } n \longrightarrow \infty.$$

Thus N is continuous.

***STEP* 2**: N maps bounded sets into bounded sets in $\mathcal{C}(\mathcal{J},\mathcal{R})$,.
Indeed it is enough to show that for any $q > 0$, there exits a positive constant ł

such that for each $z \in B_q = \{z \in \mathcal{C}(\mathcal{J},\mathcal{R}): ||z||_{\mathcal{C}} \leq q\}$ one has $||N(z)||_{\mathcal{C}} \leq ł$.

Let $z \in B_q$ the fact that f is an L' Carathodory function ,we have for each $t \in \mathcal{J}$

$$|N\mathrm{z}(\mathbf{t})| \leq |\Upsilon(0)| + \left|\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} As^{\alpha-1}\mathrm{z}(s)ds\right| + \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} |s^{\alpha-1}\mathrm{G}(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))})ds|$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} |A(\mathbf{t}-s)^{\alpha-1}\mathrm{z}(s)ds| + \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} |(\mathbf{t}-s)^{\alpha-1}\mathrm{G}(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))})ds|$$

$$\leq |\Upsilon(0)| + \frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}|\mathrm{z(s)}| + \frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}\left|\mathrm{G}(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))})\right|$$

$$\leq |\Upsilon(0)| + \frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}H||z_s||_{\mathfrak{B}} + \frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}[p(\mathbf{s})+q(\mathbf{s})]||\mathrm{G}(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))})||_{\mathfrak{B}}$$

$$\leq |\Upsilon(0)| + \frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}[H[\mathcal{M}_b + \mathcal{J}^{\varphi}||\varphi||_{\mathfrak{B}} + \mathcal{K}_b] + p(\mathbf{s}) + q(\mathbf{s})\{[\mathcal{M}_b + \mathcal{J}^{\varphi}||\varphi||_{\mathfrak{B}} + \mathcal{K}_b r]\}]$$

$$\leq |\Upsilon(0)| + \frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}[\mathcal{M}_B + \mathcal{N}_B]$$
$$= l$$

***STEP* 3**: N maps bounded sets into equicontinuous sets of $\mathcal{C}(\mathcal{J},\mathcal{R})$.
Let $\tau_1,\tau_2 \in \mathcal{J}', \tau_1 < \tau_2$ and let B_q be a bounded set of $\mathcal{C}(\mathcal{J},\mathcal{R})$. as in STEP 2.
Let $\mathrm{z} \in \mathrm{B}_q$ then for each $\mathbf{t} \in \mathcal{J}$,we have

$$|N\mathrm{z}(\tau_2) - N\mathrm{z}(\tau_1)| = |-\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2} As^{\alpha-1}\mathrm{z}(s)ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2} s^{\alpha-1}\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)ds + \frac{1}{\Gamma(\alpha)}\int_0^{\tau_2} A(\tau_2-s)^{\alpha-1}\mathrm{z}(s)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2} (\tau_2-s)^{\alpha-1}\mathrm{G}(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))})ds + \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} As^{\alpha-1}\mathrm{z}(s)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} s^{\alpha-1}\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} A(\tau_1-s)^{\alpha-1}\mathrm{z}(s)ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} (\tau_1-s)^{\alpha-1}\mathrm{G}(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))})ds|$$

$$|N\mathrm{z}(\tau_2) - N\mathrm{z}(\tau_1)| = |-\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} As^{\alpha-1}\mathrm{z}(s)ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} A(\tau_1-s)^{\alpha-1}\mathrm{z}(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^{\tau_2} (\tau_2-s)^{\alpha-1}\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} (\tau_1-s)^{\alpha-1}\mathrm{G}(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))})ds|$$
$$\leq \frac{1}{\Gamma(\alpha+1)}[(\tau_2)^{\alpha} - (\tau_1)^{\alpha}][|\mathrm{A}||\mathrm{z(s)}| + \left|\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)\right|]$$
$$+\frac{1}{\Gamma(\alpha+1)}[(\tau_2)^{\alpha} - (\tau_1)^{\alpha}][|\mathrm{A}||\mathrm{z(s)}| + \frac{1}{\Gamma(\alpha+1)}[(\tau_2)^{\alpha} - (\tau_1)^{\alpha}]\left|\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)\right|$$
$$\leq \frac{2}{\Gamma(\alpha+1)}[(\tau_2)^{\alpha} - (\tau_1)^{\alpha}][|\mathrm{A}||\mathrm{z(s)}| + \left|\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)\right|]$$

$$\leq \frac{2}{\Gamma(\alpha+1)}\{[(\tau_2)^\alpha-(\tau_1)^\alpha][\mu H[\mathcal{M}_b+\mathcal{J}^\varphi||\varphi||_{\mathfrak{B}}+\mathcal{K}_b]\ +p(\mathbf{s})+ q(\mathbf{s})\{[\mathcal{M}_b+\mathcal{J}^\varphi||\varphi||_{\mathfrak{B}}+\mathcal{K}_b r]\}\}$$

$$\leq \frac{\mathbf{2t}^\alpha}{\Gamma(\alpha+1)}[(\tau_2)^\alpha-(\tau_1)^\alpha][\mu[\mathcal{M}_B]+\mathcal{N}_B]$$

The right hand side tends to zero as $\tau_2-\tau_1\longrightarrow 0$.

This proves the equicontinuitz for the case where $\mathbf{t}\neq\mathbf{t}_i, i=1,2,\dots,m+1$.
It remains to examine the equicontinuitz at $\mathbf{t}=\mathbf{t}_i$. First we prove equicontinuitz at $\mathbf{t}=\mathbf{t}_i^-$.
Fix $\delta_1>0$ such that $\{\mathbf{t}_k: k\neq i\}\cap[\mathbf{t}_i-\delta_1,\mathbf{t}_i+\delta_1]=\Phi$. For $0<h<\delta_1$, we have that

$$|N\mathrm{z}(t_i)-N\mathrm{z}(t_i-\mathrm{h})| = |-\frac{1}{\Gamma(\alpha)}\int_{t_i-\mathrm{h}}^{t_i} As^{\alpha-1}\mathrm{z}(s)ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_{t_i-\mathrm{h}}^{t_i} s^{\alpha-1}\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)ds+\frac{1}{\Gamma(\alpha)}\int_0^{t_i} A(t_i-s)^{\alpha-1}\mathrm{z}(s)ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^{t_i-\mathrm{h}} A(t_i-\mathrm{h}-s)^{\alpha-1}\mathrm{z}(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{t_i} (t_i-s)^{\alpha-1}\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^{t_i-\mathrm{h}} (t_i-\mathrm{h}-s)^{\alpha-1}\mathrm{G}(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))})ds|$$

$$\leq \frac{1}{\Gamma(\alpha+1)}[(t_i)^\alpha-(t_i-\mathrm{h})^\alpha][|\mathrm{A}||\mathrm{z}(\mathrm{s})|+\left|\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)\right|]$$

$$+\frac{1}{\Gamma(\alpha+1)}[(t_i)^\alpha-(t_i-\mathrm{h})^\alpha][|\mathrm{A}||\mathrm{z}(\mathrm{s})|$$

$$+\frac{1}{\Gamma(\alpha+1)}[(t_i)^\alpha-(t_i-\mathrm{h})^\alpha]\left|\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)\right|$$

$$\leq \frac{2}{\Gamma(\alpha+1)}[(t_i)^\alpha-(t_i-\mathrm{h})^\alpha][|\mathrm{A}||\mathrm{z}(\mathrm{s})|+\left|\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)\right|]$$

$$\leq \frac{2}{\Gamma(\alpha+1)}\{[(t_i)^\alpha-(t_i-\mathrm{h})^\alpha][\mu H[\mathcal{M}_b+\mathcal{J}^\varphi||\varphi||_{\mathfrak{B}}+\mathcal{K}_b]\ +p(\mathbf{s})+ q(\mathbf{s})\{[\mathcal{M}_b+\mathcal{J}^\varphi||\varphi||_{\mathfrak{B}}+\mathcal{K}_b r]\}\}$$

$$\leq \frac{\mathbf{2t}^\alpha}{\Gamma(\alpha+1)}[(t_i)^\alpha-(t_i-\mathrm{h})^\alpha][\mu[\mathcal{M}_B]+\mathcal{N}_B]$$

The right hand side tends to zero as $h\longrightarrow 0$.

Next we prove the equicontinuitz at $\mathbf{t}=\mathbf{t}_i^+$.

Fix $\delta_2>0$ such that $\{\mathbf{t}_k: k\neq i\}\cap[\mathbf{t}_i-\delta_2,\mathbf{t}_i+\delta_2]=\Phi$. For $0<h<\delta_2$,
we have that

$$|N\mathrm{z}(t_i+\mathrm{h})-N\mathrm{z}(t_i)| = |-\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t_i+h} As^{\alpha-1}\mathrm{z}(s)ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_{t_i-\mathrm{h}}^{t_i+h} s^{\alpha-1}\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{0}^{t_i+h} A(t_i+h-s)^{\alpha-1}\mathrm{z}(s)ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_{0}^{t_i} A(t_i-s)^{\alpha-1}\mathrm{z}(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{0}^{t_i+h} (t_i+h-s)^{\alpha-1}\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_{0}^{t_i} (t_i-s)^{\alpha-1}\mathrm{G}(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))})ds|$$

$$\leq \frac{1}{\Gamma(\alpha+1)}[(t_i+\mathrm{h})^{\alpha}-(t_i)^{\alpha}][|\mathrm{A}||\mathrm{z(s)}|+\left|\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)\right|]$$

$$+\frac{1}{\Gamma(\alpha+1)}[(t_i+h)^{\alpha}-(t_i)^{\alpha}][|\mathrm{A}||\mathrm{z(s)}|$$

$$+\frac{1}{\Gamma(\alpha+1)}[(t_i+\mathrm{h})^{\alpha}-(t_i)^{\alpha}]\left|\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)\right|$$

$$\leq \frac{2}{\Gamma(\alpha+1)}[(t_i+h)^{\alpha}-(t_i)^{\alpha}][|\mathrm{A}||\mathrm{z(s)}|+\left|\mathrm{G}\left(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))}\right)\right|]$$

$$\leq \frac{2}{\Gamma(\alpha+1)}\{[(t_i+\mathrm{h})^{\alpha}-(t_i)^{\alpha}][\mu H[\mathcal{M}_b+\mathcal{J}^{\varphi}||\varphi||_{\mathfrak{B}}+\mathcal{K}_b] + p(\mathbf{s})+ q(\mathbf{s})\{[\mathcal{M}_b+\mathcal{J}^{\varphi}||\varphi||_{\mathfrak{B}}+\mathcal{K}_b r]\}\}$$

$$\leq \frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}[(t_i+\mathrm{h})^{\alpha}-(t_i)^{\alpha}][\mu[\mathcal{M}_B]+\mathcal{N}_B]$$

The right hand side tends to zero as $h\longrightarrow 0$.

As a consequence of STEP 1 to 3 and (a) together with Arzela Ascoli theorem, we can conclude that $N:\mathcal{C}(\mathcal{J},\mathcal{R})\longrightarrow\mathcal{C}(\mathcal{J},\mathcal{R})$ is a completelz continuous operator.

***STEP* 4**:

Now it remains to show that the set

$\xi(N)=\{\mathrm{z}\in\mathcal{C}(\mathcal{J},\mathcal{R}):\mathrm{z}=\lambda N(\mathrm{z})$, for some $0<\lambda<1\}$ is bounded.

Let $\mathrm{z}\in\xi(N)$. Then $\mathrm{z}=\lambda N(\mathrm{z})$,for some $0<\lambda<1$. Thus for each $\mathbf{t}\in\mathcal{J}$,

$$\mathrm{z}(\mathbf{t}) = \Upsilon(0)-\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} As^{\alpha-1}\mathrm{z}(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} s^{\alpha-1}\mathrm{G}(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))})ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} A(\mathbf{t}-s)^{\alpha-1}\mathrm{z}(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} (\mathbf{t}-s)^{\alpha-1}\mathrm{G}(s,\mathrm{z}_{\varrho(s,\mathrm{z}(s))})ds$$

$$|\mathrm{z}(\mathbf{t})| = |\Upsilon(0)| + \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} s^{\alpha-1}|A\mathrm{z}(s) + \mathrm{G}(s, \mathrm{z}_{\varrho(s,\mathrm{z}(s))})|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} (\mathbf{t}-s)^{\alpha-1}|A\mathrm{z}(s) + \mathrm{G}(s, \mathrm{z}_{\varrho(s,\mathrm{z}(s))})|ds$$

This implies bz (2.1.1)-(2.1.3) that for each $\mathbf{t} \in \mathcal{J}$, we have

$$|\mathrm{z}(\mathbf{t})| \le |\Upsilon(0)| + \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} s^{\alpha-1}[\mu H||\mathrm{z}_s||_{\mathfrak{B}}B + p(s) + q(s)||\mathrm{G}_{\varrho(s,\mathrm{z}(s))}||_{\mathfrak{B}}B]ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} (\mathbf{t}-s)^{\alpha-1}[\mu H||\mathrm{z}_s||_{\mathfrak{B}} + p(s) + q(s)||\mathrm{f}_{\varrho(s,\mathrm{z}(s))}||_{\mathfrak{B}}]ds$$

Let us denote the right hand side of the above inequalitz as $V(\mathbf{t})$.
Then we have

$$|\mathrm{z}(\mathbf{t})| \le V(\mathbf{t}) \quad \forall \ \mathbf{t} \in \mathcal{J}$$

$$V(0) = \Upsilon(0)$$

$$V'(\mathbf{t}) = \frac{\mathbf{t}^{\alpha-1}}{\Gamma(\alpha)}[\mu H||\mathrm{z}_{\mathbf{t}}||_{\mathfrak{B}} + p(\mathbf{t}) + q(\mathbf{t})||\mathrm{f}_{\varrho(\mathbf{t},\mathrm{z}(\mathbf{t}))}||_{\mathfrak{B}}]$$

This shows that $\xi(N)$ is bounded. Set $X = \mathcal{C}(\mathcal{J}, \mathcal{R})$.

As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a mild solution of (1.1—1.2).

Now we present a uniqueness result for the problem (1.1-1.2). Our considerations are based on the Banach fixed point theorem.

Theorem 2.2 *Assume that f is an $Ł'$ − Caratheodory function and suppose (a) holds. In addition assume the following conditions are satisfied.*
(b) There exists a constant l such that

Lipschitz condition

$$|\mathrm{G}(s, \mathrm{z}_{\varrho(s,\mathrm{z}(s))}) - \mathrm{G}(s, \mathrm{z}^{-}_{\varrho(s,\mathrm{z}(s))})| \le Ł_f\beta \sup_{0\le s\le T}|\mathrm{z}-\mathrm{z}^{-}|$$

for each $\mathbf{t} \in \mathcal{J}$ $\forall$ $\mathrm{z}, \mathrm{z}^{-} \in \mathcal{R}$ if $\frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}[[\mu + Ł_f\beta]] \le 1$

then the initial value problem (1.1-1.2) has a unique mild solution.

Proof : Transform the problem (1.1-1.2) into a fixed point problem .

Let the operator $\mathcal{C}(\mathcal{J}, \mathcal{R}) \longrightarrow \mathcal{C}(\mathcal{J}, \mathcal{R})$ be defined as in theorem (2.2).
We will show that $\mathcal{N}$ is a contraction .

Indeed , consider $\mathrm{z}, \mathrm{z}^{-} \in \mathcal{C}(\mathcal{J}, \mathcal{R})$. Then we have , for each $\mathbf{t} \in \mathcal{J}$,

$$Nz(\mathbf{t}) - Nz^{-}(\mathbf{t}) \le \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} \left|As^{(\alpha-1)}[z(s) - z^{-}(s)]\right|ds$$

$$+\frac{1}{\Gamma\alpha}\int_0^{\mathbf{t}} \left|s^{\alpha-1}\left[G\left(s, z_{\varrho(s,z(s))}\right) - G\left(s, z^{-}_{\varrho(s,z(s))}\right)\right]\right| ds$$

$$+\frac{1}{\Gamma\alpha}\int_0^{\mathbf{t}} |(\mathbf{t}-s)^{\alpha-1}[G(s, z_{(\varrho)(s,z_{(s)})}) - G(s, z^{-}_{\varrho(s,z(s))})]|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} |A(\mathbf{t}-s)^{(\alpha-1)}|[z_(s) - z^{-}(s)]|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} |A(\mathbf{t}-s)^{(\alpha-1)}|[z_(s) - z^{-}(s)]|ds$$

$$\le \frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}\mu||z - z^{-}||$$

$$+\frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}Ł_f\beta \sup_{0\le s\le T}||z - z^{-}||$$

$$\le \{\frac{2\mathbf{t}^{\alpha}}{\Gamma(\alpha+1)}\left[\mu + Ł_f\beta\right]\}||z - z^{-}||$$

Showing that $\mathcal{N}$ is a contraction , and hence it has a unique fixed point
which is a mild solution to (1-2).

3 EXAMPLE

We present the below example for our given Riesz Caputo fractional differential equation

With state dependent delay :

$$ {}_0^{RC}D_x^{\frac{1}{2}}x(\mathbf{t}) = e^t + h(\mathbf{t}, x_{y(\mathbf{t},x_{\mathbf{t}})}) \quad (3.1)$$

$$h(\mathbf{t}, x_{y(\mathbf{t},x_{\mathbf{t}})}) = \frac{t\sin t + t\cos t}{(1+t)e^{t+2}} \quad (3.2)$$

$$\gamma(0) = 0 \quad (3.3)$$

where $t \in I = [0, 1]$ .The above choice of the system (3.1-3.3) can be written in the abstract form of the system (1.1)-(1.2).

$$ {}_0^{RC}D_x^{\alpha}z(\mathbf{t}) = {}_0^{RC}D_x^{\frac{1}{2}}x(\mathbf{t})$$

Where ${}_0^{RC}D_x^{\frac{1}{2}}x(\mathbf{t})$ denote the Riesz-Caputo fractional derivative of $\alpha = \frac{1}{2}$

$$G\left(\mathbf{t}, z_{\varrho(\mathbf{t},\mathbf{z_t})}\right) = \frac{t\sin t + t\cos t}{(1+t)e^{t+2}}$$

The Hypothesis H1 to H4 and Lipschitz conditions are satisfied.

$$|N\mathrm{x}_n(\mathbf{t}) - N\mathrm{x}(\mathbf{t})| \leq \frac{2\mathbf{t}^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}[\mu + \text{Ł}_f\beta]|\mathrm{x}_n - \mathrm{x}|$$

As $n \rightarrow \infty$. Thus N is continuous.

$$|Nx(t) \leq \frac{\mathbf{2t}^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}[\mathcal{M}_B + \mathcal{N}_B]$$

Consequence of the above results together with Arzela Ascoli theorem we can infer that

N is completely continuous.

As stated in the theorem (2.1) and (2.2), the unique solution x(t) is exists for (3.1-3.3).

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